C3. Q204 CH14B: LESSON 1 (Local Extrema and Saddles Points)

Definition:

Let *f* be a continuous function of two variables.

There is a **critical point at the coordinate** (a, b) if either:

(i) $\nabla f(a,b) = \vec{\mathbf{0}}$

OR

(ii) either [$f_x(a,b)$ DNE] or [$f_y(a,b)$ DNE]

Definitions:

Let *f* be a function of two variables that has continuous second partial derivatives.

The **Hessian Matrix (in R3)** is $\begin{bmatrix} \dfrac{\partial^2 f}{(\partial x)^2} & \dfrac{\partial^2 f}{\partial x \partial y} \\ \dfrac{\partial^2 f}{\partial x \partial y} & \dfrac{\partial^2 f}{(\partial y)^2} \end{bmatrix}$

The **Discriminant *D* of the Hessian (in R3)** is $D(x,y) = f_{xx}(x,y) f_{yy}(x,y) - \left[f_{xy}(x,y) \right]^2$

Second Derivative Test for local Extrema (in R3):

Let *f* be a function of two variables that has continuous second partial derivatives throughout an open disk R containing (a, b).

(i) *f* has a local maximum at (a, b) if $\nabla f(a,b) = \vec{\mathbf{0}}$, $D(a,b) > 0$, and $f_{xx}(a,b) < 0$.

(ii) *f* has a local minimum at (a, b) if $\nabla f(a,b) = \vec{\mathbf{0}}$, $D(a,b) > 0$, and $f_{xx}(a,b) > 0$.

(iii) the graph of *f* has a saddle point at (a, b) if $\nabla f(a,b) = \vec{\mathbf{0}}$ and $D(a,b) < 0$

(iv) the nature of *f* at (a, b) needs further investigation if $\nabla f(a,b) = \vec{\mathbf{0}}$ and $D(a,b) = 0$

Example 1: $f(x,y)=\frac{x^3}{3}+\frac{4}{3}y^3-x^2-3x-4y-3$. Find the local extrema and saddle points of *f*.

$$\nabla f = \langle x^2-2x-3,\ 4y^2-4 \rangle$$

$$x^2-2x-3=0 \qquad\qquad 4y^2-4=0$$

$$(x-3)(x+1) \qquad\qquad (y+1)(y-1)=0$$

$$x=-1,3 \qquad\qquad y=-1,1$$

$$f_{xy}=0$$

$$f_{xx}=2x-2 \qquad\qquad f_{yy}=8y$$

$$D(x,y)=(2x-2)(8y)$$

critical pt	disc	f_{xx}	conclusion
$(-1,-1)$	$32>0$	$-4<0$	$f(-1,-1)=\frac{4}{3}$ is a local max
$(-1,1)$	$-32<0$		$(-1,-1,f(-1,-1))$ is a saddle point
$(3,-1)$	$-32<0$		$(3,-1,f(3,-1))$ is a saddle point
$(3,1)$	$32>0$	$4>0$	$f(3,1)=\frac{-44}{3}$ is a local min

Example 2: $f(x,y) = x^2 - 4xy + y^3 + 4y$. Find the local extrema and saddle points of *f*.

$f_x = 2x - 4y$ $\qquad$ $f_y = 3y^2 - 4x + 4$

$f_{xy} = -4$

$f_{xx} = 2$ $\qquad$ $f_{yy} = 6y$

$f_x = 0$ $\qquad$ $f_y = 0$

$x = 2y$ $\qquad$ $x = \dfrac{3y^2+4}{4}$

$$2y = \frac{3y^2+4}{4} \Rightarrow 8y = 3y^2 + 4 = 0$$

$$(3y-2)(y-2) = 0$$

$$y = \tfrac{2}{3}, 2$$

critical points:

$\left(\frac{4}{3}, \frac{2}{3}\right)$ & $(4, 2)$

point	disc	f_{xx}	conclusion
$\left(\frac{4}{3}, \frac{2}{3}\right)$	$-8 < 0$		$\left(\frac{4}{3}, \frac{2}{3}, f\left(\frac{4}{3}, \frac{2}{3}\right)\right)$ is a non-classic saddle point
$(4,2)$	$4 > 0$	2	$f(4,2) = 0$ is a local min

hyperbolic paraboloid

Example 3: $f(x,y) = y^2 - x^2$. Find the local extrema and the saddle points of f.

$f_x = -2x$ $\qquad$ $f_y = 2y$

$f_{xx} = -2$ $\qquad$ $f_{yy} = 2$

$\therefore D(x,y) < 0 \ \forall (x,y)$

$f_x = 0$:
$-2x = 0$
$x = 0$

$f_y = 0$
$2y = 0$
$y = 0$

critical point:
$(0,0)$

disc:
$D(x,y) < 0$

$\therefore (0,0,0)$ is a classic saddle point

$D=0$

Example 4: $f(x,y) = -\left(x^2 + y^{2/3}\right)$. Find the local extrema and the saddle points of f.

$f_x = -2x$

$f_y = -\frac{2}{3} y^{-1/3}$

$f_{xy} = 0$

$f_{xx} = -2$

$f_{yy} = \frac{2}{9} y^{-4/3}$

crit pts:
all points $(x, 0)$

$f_y = 0$:
$-2/3\, y^{-1/3} = 0 \quad \emptyset$

$D(\text{critical points}) \rightarrow$ DNE

f_y DNE:
$y = 0$

for $(x,y) \neq (0,0)$ $f(x,y) < 0$

for $(x,y) = (0,0)$ $f(x,y) = 0$

$f(0,0) = 0$ is a local AND absolute max of f

C3. Q204 CH14B: LESSON 2 (ABSOLUTE EXTREMA)

Absolute Extrema on an Open Region.

Consider a continuous function f on an open region R.

If f has exactly one local extrema, then the local extrema is also the absolute extrema.

Absolute Extrema on a Closed Region (Closed Region Test)

Consider a continuous function f on a closed region R.

Now, consider all the interior local extrema values **and** all the values along the closed boundary.

The absolute maximum is the largest of these values and the absolute minimum is the smallest of these values.

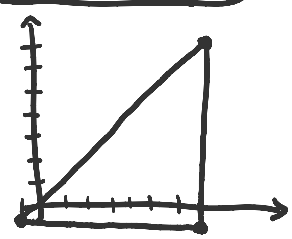
From Lesson 1 $f(4,2)=0$ is a local min

* must assess critical points inside region like L1 before all this

Absolute Extrema on a Closed Region.

Example 1: $f(x,y)=x^2-4xy+y^3+4y$. Find the extrema of f on the closed triangular region R that has vertices $(-1,-1)$, $(7,-1)$ and $(7,7)$.

restricted region



closed region test

$f(7,7)=224$

$f(7,\sqrt{8})=3.75$

$f(7,-1)=72$

$f(-1,-1)=-8$

$f(4,2)=0$

abs max

224

abs min

-8

ALONG: $x=7$: $f(7,y)=y^3-24y+49$ $\quad y\in[-1,7]$

$f_y(7,y)=3y^2-24=0$ $\quad y=\sqrt{8}$ or $\cancel{y=-\sqrt{8}}$ out of domain

closed interval

$f(7,-1)=72$ $\quad f(7,7)=224$ $\quad f(7,\sqrt{8})\approx 3.75$

ALONG: $y=-1$: $f(x,-1)=x^2+4y-5$ $\quad x\in[-1,7]$

$f_x(x,-1)=2x+4=0$ $\quad \cancel{x=-2}$ out of domain

closed interval

$f(-1,-1)=-8$ $\quad f(7,-1)=72$

ALONG: $y=x$: $f(x,x)=x^3-3x^2+4x$ $\quad x\in[-1,7]$

$f_x(x,x)=3x^2-6x+4>0$ $\therefore$ no local extrema

closed interval

$f(-1,-1)=-8$ $\quad f(7,7)=224$

Absolute Extrema on a Closed Region.

Example 2: $f(x,y) = 1 + x^2 + y^2$, with $x^2 + y^2 \le 4$. Find the extrema of *f*.

$f_x = 2x = 0 \qquad f_y = 2y = 0$

$(x,y) = (0,0)$ is a critical point

Boundary: let $x^2+y^2=4 \qquad f(x,y) = 1+4 = 5$

crt. pts.: $\{(x,y) \mid x^2+y^2=4, (x,y)=(0,0)\}$

$f(0,0) = 1$

$f(x,y) = 5$ along boundary

absolute max $= \boxed{5}$

absolute min $= \boxed{1}$

Absolute Extrema on an Open Region

Example 3: A rectangular box with no top is to be constructed to have a volume of 12ft^3. The cost per square foot of the material to be used is \$4 for the bottom, \$3 for two of the opposite sides, and \$2 for the remaining pair of opposite sides. Find the dimensions of the box that will minimize cost.

$$\text{cost} = 4xy + 3(2xz) + 2(2yz)$$

$$V = xyz = 12 \qquad z = \frac{12}{xy}$$

$$C(x,y) = 4xy + \frac{72}{y} + \frac{48}{x} \qquad x>0 \quad y>0$$

$$C_x = 4y - \frac{48}{x^2} = 0 \qquad\qquad C_y = 4x - \frac{72}{y^2} = 0$$

$$y = \frac{12}{x^2} \qquad\qquad xy^2 = 18$$

$$D: 0 < (x,y) < \infty$$

$$x\left(\frac{144}{x^4}\right) = 18 \longrightarrow 144 = 18x^3$$

$$x^3 = 8 \rightarrow x = 2 \longrightarrow y = 3$$

$$C_{xy} = 4$$

$$C_{xx} = \frac{96}{x^3} \qquad\qquad C_{yy} = \frac{144}{y^3}$$

$$D(x,y) = \frac{96}{x^3} \cdot \frac{144}{y^3} - (4)^2 \qquad D(2,3) > 0$$

$$C_{xx}(2,3) > 0$$

$\therefore$ C has a local min at $(x,y) = (2,3)$

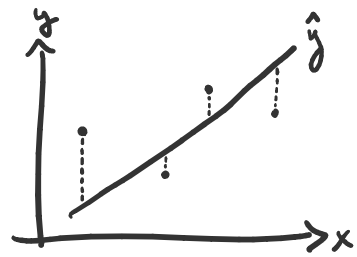
$\therefore$ C has an absolute min at $(x,y) = (2,3)$

$$x = 2\text{ft} \qquad y = 3\text{ft} \qquad z = 2\text{ft}$$

y = observed output x = input

Absolute Extrema on an Open Region

Example 4: A line $\hat{y} = b_0 + b_1 x$ that best fits a set of ordered pairs (x, y) data can be computed by finding the values of b_0 and b_1 that minimize the sum of squares between the observed y values and the fitted $\hat{y}$ values.



$\hat{y}_i = b_0 + b_1 x_i$

$e_i = y_i - \hat{y}_i = y_i - (b_0 + b_1 x_i)$

$Q = \sum (e_i)^2 = \sum (y_i - \hat{y}_i)^2$

$= \sum_{i=1}^{n} (y_i - b_0 - b_1 x_i)^2$

what values of b_0 and b_1 to minimize Q

$\boxed{1}\ \frac{\partial Q}{\partial b_0} = -2\sum(y_i - b_0 - b_1 x_i) = 0 \mapsto \sum(y_i) - n b_0 - b_1 \sum(x_i) = 0$

$\boxed{2}\ \frac{\partial Q}{\partial b_1} = -2\sum(y_i - b_0 - b_1 x_i)x_i = 0 \mapsto \sum(y_i x_i) - b_0 \sum(x_i) - b_1 \sum(x_i)^2 = 0$

solve system

$\boxed{1}\ n\bar{y} - n b_0 - n b_1 \bar{x} = 0 \qquad b_0 = \bar{y} - b_1 \bar{x}$ *

$n \cdot \boxed{2}: \ n\sum(xy) - n b_0 \sum(x) - n b_1 \sum(x^2) = 0$

$-\sum(x)\cdot\boxed{1}: -\left(\sum(x)\sum(y) - nb\sum(x) - b_1\left(\sum(x)\right)^2 = 0\right)$

$n\sum(xy) - \sum(x)\sum(y) - n b_1 \sum(x^2) + b_1\left(\sum(x)\right)^2 = 0$

$$b_1 = \frac{\sum(x)\sum(y) - \sum(xy)}{\left(\sum(x)\right)^2 - n\sum(x^2)} = \frac{\frac{\sum(x)}{n}\sum(y) - \sum(xy)}{\frac{\left(\sum(x)\right)^2}{n} - \sum(x^2)} = \frac{n\bar{x}\bar{y} - \sum(xy)}{n(\bar{x})^2 - \sum(x)^2}$$

$$= \frac{\sum(xy) - n\bar{x}\bar{y}}{\sum(x)^2 - n(\bar{x})^2} \ *$$

$$= \frac{\sum\left[(x-\bar{x})(y-\bar{y})\right]}{\sum(x-\bar{x})^2} \ *$$

C3. Q204 CH14B: LESSON 3 (LIMTS – "PATH MATH")

14.2 #8 $\lim_{(x,y)\to(1,0)} \ln\left(\frac{1+y^2}{x^2+xy}\right) = \ln\left(\frac{1}{1}\right) = 0$

#10 $\lim_{(x,y)\to(0,0)} \frac{x^2+\sin^2 y}{2x^2+y^2}$ substitution not available

path $y=0$: $\lim_{(x,0)\to(0,0)} \frac{x^2+\sin^2 0}{2x^2+0} = \lim_{x\to 0} \frac{x^2}{2x^2} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}$

path $x=0$: $\lim_{(0,y)\to(0,0)} = \frac{0^2+\sin^2 y}{2(0)^2+y^2} = \lim_{y\to 0} \frac{\sin^2 y}{y^2} = \lim_{y\to 0} \frac{2\sin y\cos y}{2y}$

$= \lim_{y\to 0} \frac{\cos y\cos y - \sin y \sin y}{1} = 1$

since two paths have limits that are not equal,

$\lim_{(x,y)\to(0,0)} \frac{x^2+\sin^2 y}{2x^2+y^2}$ DNE

#12 $\lim_{(x,y)\to(0,0)} \frac{6x^3y}{2x^4+y^4}$

path $x=0$: $\lim_{(0,y)\to(0,0)} \frac{6(0)^3y}{2(0)^4+y^4} = \lim_{y\to 0} 0 = 0$

path $x=y$ $\lim_{(x,x)\to(0,0)} \frac{6x^4}{3x^4} = \lim_{x\to 0} 2 = 2$

$\therefore \lim_{(x,y)\to(0,0)} \frac{6x^3y}{2x^4+y^4}$ DNE

#14 $\lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)(x^2-y^2)}{x^2+y^2} = \lim_{(x,y)\to(0,0)} x^2+y^2 = 0$

#18 $\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2+y^8}$

path $x=0$: $\lim_{(0,y)\to(0,0)} \frac{(0)y^4}{(0)^2+y^8} = \lim_{(0,y)\to(0,0)} = 0$

path $x=y^4$ $\lim_{(y^4,y)\to(0,0)} \frac{y^4y^4}{(y^4)^2+y^8} = \lim_{y\to0} \frac{y^8}{2y^8} = \lim_{y\to0} \frac{1}{2} = \frac{1}{2}$

$\therefore \lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2+y^8}$ DNE

SUPP $\lim_{(x,y)\to(0,0)} \frac{1-\cos(x^2+y^2)}{x^2+y^2}$

$x=r\cos\theta$
$y=r\sin\theta$

$\lim_{r\to0^+} \frac{1-\cos(r^2)}{r^2} = \lim_{r\to0^+} \frac{\sin(r^2)\cdot 2r}{2r} = \lim_{r\to0^+} \sin(r^2) = 0$

#40 $\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2)$

$x=r\cos\theta$
$y=r\sin\theta$

$\lim_{r\to0^+} r^2\ln(r^2) = \lim_{r\to0^+} \frac{\ln(r^2)}{\frac{1}{r^2}} = \lim_{r\to0^+} \frac{\frac{2r}{r^2}}{\frac{-2}{r^3}} = \lim_{r\to0^+} -r^2 = 0$

HW: Section 14.2 "pick some": 5-22, 39-41, 37

C3. Q204 CH14B: LESSON 4 (LAGRANGE MULTIPLIERS)

Suppose that f and g are functions of two variables having continuous first partial derivatives and that $\nabla g \neq \vec{\mathbf{0}}$ throughout a region of the xy-plane. If f has an extremum $f(x_0, y_0)$ subject to the constraint $g(x, y) = 0$, then there is a real number λ such that $\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$

$$\langle f_x, f_y \rangle = \lambda \langle g_x, g_y \rangle$$

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g(x,y) = 0 \end{cases}$$

Suppose that f and g are functions of two variables having continuous first partial derivatives and that $\nabla g \neq \vec{\mathbf{0}}$ throughout a region of the xyz-coordinate system. If f has an extremum $f(x_0, y_0, z_0)$ subject to the constraint $g(x, y, z) = 0$, then there is a real number λ such that $\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)$

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ f_z = \lambda g_z \\ g(x,y,z) = 0 \end{cases}$$

Example 1: Find the extrema of $\underbrace{\overset{\text{to be optimized}}{f(x,y)=xy}}_{f}$ along the boundary $\underbrace{\overset{\text{constraint}}{4x^2+y^2=4}}_{g}$.

$\nabla f = \langle y, x \rangle$

$g(x,y) = 4x^2+y^2-4$

$\nabla g = \langle 8x, 2y \rangle$

$\nabla f = \lambda \nabla g$

$$\begin{cases} y = \lambda 8x \\ x = \lambda 2y \\ 4x^2+y^2-4=0 \end{cases}$$

many ways to solve

$\lambda = \frac{y}{8x}$

$\lambda = \frac{x}{2y}$

$\frac{x}{2y} = \frac{y}{8x}$

$8x^2 = 2y^2$

$y = \pm 2x$

$4x^2+4x^2-4=0$

$8x^2=4$

$x^2=\frac{1}{2}$

$x = \pm\frac{1}{\sqrt{2}}$ $\quad y = \pm\sqrt{2}$ $\quad \lambda = \pm\frac{1}{4}$

$\sqrt{2} = \lambda \frac{8}{\sqrt{2}}$

Evaluate

$f(\frac{\sqrt{2}}{2}, \sqrt{2}) = 1$

$f(\frac{\sqrt{2}}{2}, -\sqrt{2}) = -1$

$f(-\frac{\sqrt{2}}{2}, \sqrt{2}) = -1$

$f(-\frac{\sqrt{2}}{2}, -\sqrt{2}) = 1$

absolute max of $f = \boxed{1}$

absolute min of $f = \boxed{-1}$

Example 2: Find the volume of the largest rectangular box with faces parallel to the coordinate planes that can be inscribed in the ellipsoid $16x^2 + 4y^2 + 9z^2 = 144$.

$f(x,y,z) = 8xyz$ $\qquad$ $g(x,y,z) = 16x^2 + 4y^2 + 9z^2 - 144$

$$\begin{cases} 8yz = 32x\lambda \\ 8xz = 8y\lambda \\ 8xy = 18z\lambda \\ 16x^2 + 4y^2 + 9z^2 - 144 = 0 \end{cases}$$

$\lambda = \dfrac{yz}{4x}$ $\qquad$ $\lambda = \dfrac{4xy}{9z}$

$\lambda = \dfrac{xz}{y}$

$\dfrac{yz}{4x} = \dfrac{xz}{y}$ $\qquad$ $y^2 z = 4x^2 z$

$y = \pm 2x$

$16x^2 + 16x^2 + 16x^2 = 144$ $\qquad$ $\dfrac{xz}{y} = \dfrac{4xy}{9z}$ $\qquad$ $z = \pm \dfrac{2y}{3} = \pm \dfrac{4x}{3}$

$x = \pm\sqrt{3}$ $\qquad$ $9z^2 x = 4y^2 x$

$y = \pm 2\sqrt{3}$

$z = \pm \dfrac{4}{\sqrt{3}}$ $\qquad$ $f\left(\sqrt{3}, 2\sqrt{3}, \dfrac{4}{\sqrt{3}}\right) = 8(\sqrt{3})(2\sqrt{3})\left(\dfrac{4}{\sqrt{3}}\right)$

$= \boxed{64\sqrt{3}}$

Example 3: If $T(x,y,z)=4x^2+y^2+5z^2$ represents the temperature at any point (x, y, z), find the points on the plane $2x+3y+4z=12$ at which the temperature has its smallest value.

$f(x,y,z)=4x^2+y^2+5z^2$

$g(x,y,z)=2x+3y+4z-12=0$

$$\begin{cases} 8x=2\lambda \\ 2y=3\lambda \\ 10z=4\lambda \\ 2x+3y+4z-12=0 \end{cases}$$

$x=\frac{\lambda}{4}$

$y=\frac{3\lambda}{2}$

$z=\frac{2\lambda}{5}$

$\frac{\lambda}{2}+\frac{9\lambda}{2}+\frac{8\lambda}{5}-12=0$

$5\lambda+45\lambda+16\lambda=120$

$66\lambda=120$

$\lambda=\frac{20}{11}$

$x=\frac{5}{11}$

$y=\frac{30}{11}$

$z=\frac{8}{11}$

$\boxed{\left(\frac{5}{11}, \frac{30}{11}, \frac{8}{11}\right)}$

FROM YOUR TEXTBOOK

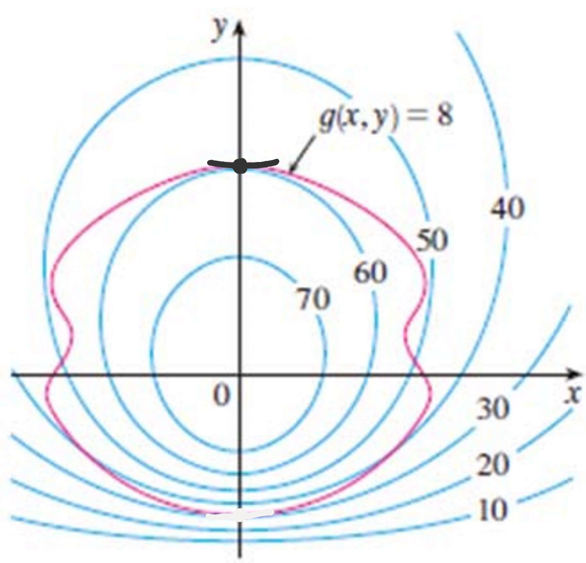
EXAMPLE 4 Find the points on the sphere $x^2 + y^2 + z^2 = 4$ that are closest to and farthest from the point $(3, 1, -1)$.

$d^2 = f(x,y,z) = (x-3)^2 + (y-1)^2 + (z+1)^2$

$g(x,y,z) = x^2+y^2+z^2-4=0$

$$\begin{cases} 2x = 2\lambda(x-3) \\ 2y = 2\lambda(y-1) \\ 2z = 2\lambda(z+1) \\ x^2+y^2+z^2-4=0 \end{cases}$$

$\boxed{1.}$ Pictured are a contour map of f and a curve with equation $g(x, y) = 8$. Estimate the maximum and minimum values of f subject to the constraint that $g(x, y) = 8$. Explain your reasoning.



max $g=60$

min $g=30$