C3. Q204 CH14B: LESSON 1 (Local Extrema and Saddles Points)

Definition:

Let *f* be a continuous function of two variables.

There is a **critical point at the coordinate** (a, b) if either:

(i) $\nabla f(a,b) = \vec{\mathbf{0}}$

OR

(ii) either [$f_x(a,b)$ DNE] or [$f_y(a,b)$ DNE]

Definitions:

Let *f* be a function of two variables that has continuous second partial derivatives.

The **Hessian Matrix (in R3)** is $\begin{bmatrix} \dfrac{\partial^2 f}{(\partial x)^2} & \dfrac{\partial^2 f}{\partial x \partial y} \\ \dfrac{\partial^2 f}{\partial x \partial y} & \dfrac{\partial^2 f}{(\partial y)^2} \end{bmatrix}$

The **Discriminant *D* of the Hessian (in R3)** is $D(x, y) = f_{xx}(x,y)\, f_{yy}(x,y) - \left[f_{xy}(x,y) \right]^2$

Second Derivative Test for local Extrema (in R3):

Let *f* be a function of two variables that has continuous second partial derivatives throughout an open disk R containing (a, b).

(i) *f* has a local maximum at (a, b) if $\nabla f(a,b) = \vec{\mathbf{0}}$, $D(a,b) > 0$, and $f_{xx}(a,b) < 0$.

(ii) *f* has a local minimum at (a, b) if $\nabla f(a,b) = \vec{\mathbf{0}}$, $D(a,b) > 0$, and $f_{xx}(a,b) > 0$.

(iii) the graph of *f* has a saddle point at (a, b) if $\nabla f(a,b) = \vec{\mathbf{0}}$ and $D(a,b) < 0$

(iv) the nature of *f* at (a, b) needs further investigation if $\nabla f(a,b) = \vec{\mathbf{0}}$ and $D(a,b) = 0$

Example 1: $f(x,y)=\frac{x^3}{3}+\frac{4}{3}y^3-x^2-3x-4y-3$. Find the local extrema and saddle points of *f*.

Example 2: $f(x, y) = x^2 - 4xy + y^3 + 4y$. Find the local extrema and saddle points of *f*.

Example 3: $f(x, y) = y^2 - x^2$. Find the local extrema and the saddle points of *f*.

Example 4: $f(x, y) = -\left(x^2 + y^{2/3}\right)$. Find the local extrema and the saddle points of *f*.

C3. Q204 CH14B: LESSON 2 (ABSOLUTE EXTREMA)

Absolute Extrema on an Open Region.

Consider a continuous function *f* on an open region R.

If *f* has exactly one local extrema, then the local extrema is also the absolute extrema.

Absolute Extrema on a Closed Region (Closed Region Test)

Consider a continuous function *f* on a closed region R.

Now, consider all the interior local extrema values **and** all the values along the closed boundary.

The absolute maximum is the largest of these values and the absolute minimum is the smallest of these values.

Absolute Extrema on a Closed Region.

Example 1: $f(x,y)=x^2-4xy+y^3+4y$. Find the extrema of *f* on the closed triangular region *R* that has vertices $(-1,-1)$, $(7,-1)$ and $(7,7)$.

Absolute Extrema on a Closed Region.

Example 2: $f(x, y) = 1 + x^2 + y^2$, with $x^2 + y^2 \leq 4$. Find the extrema of *f*.

Absolute Extrema on an Open Region

Example 3: A rectangular box with no top is to be constructed to have a volume of 12ft^3. The cost per square foot of the material to be used is \$4 for the bottom, \$3 for two of the opposite sides, and \$2 for the remaining pair of opposite sides. Find the dimensions of the box that will minimize cost.

Absolute Extrema on an Open Region

Example 4: A line $\hat{y} = b_0 + b_1 x$ that best fits a set of ordered pairs (x, y) data can be computed by finding the values of b_0 and b_1 that minimize the sum of squares between the observed y values and the fitted $\hat{y}$ values.

C3. Q204 CH14B: LESSON 3 (LIMTS – "PATH MATH”)

C3. Q204 CH14B: LESSON 4 (LAGRANGE MULTIPLIERS)

Suppose that f and g are functions of two variables having continuous first partial derivatives and that $\nabla g \neq \vec{\mathbf{0}}$ throughout a region of the xy-plane. If f has an extremum $f(x_0, y_0)$ subject to the constraint $g(x, y) = 0$, then there is a real number λ such that $\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$

Suppose that f and g are functions of two variables having continuous first partial derivatives and that $\nabla g \neq \vec{\mathbf{0}}$ throughout a region of the xyz-coordinate system. If f has an extremum $f(x_0, y_0, z_0)$ subject to the constraint $g(x, y, z) = 0$, then there is a real number λ such that $\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)$

Example 1: Find the extrema of $f(x, y) = xy$ along the boundary $4x^2 + y^2 = 4$.

Example 2: Find the volume of the largest rectangular box with faces parallel to the coordinate planes that can be inscribed in the ellipsoid $16x^2 + 4y^2 + 9z^2 = 144$.

Example 3: If $T(x,y,z)=4x^2+y^2+5z^2$ represents the temperature at any point $(x,\ y,\ z)$, find the points on the plane $2x+3y+4z=12$ at which the temperature has its smallest value.

FROM YOUR TEXTBOOK

EXAMPLE 4 Find the points on the sphere $x^2 + y^2 + z^2 = 4$ that are closest to and farthest from the point $(3, 1, -1)$.

1. Pictured are a contour map of f and a curve with equation $g(x, y) = 8$. Estimate the maximum and minimum values of f subject to the constraint that $g(x, y) = 8$. Explain your reasoning.