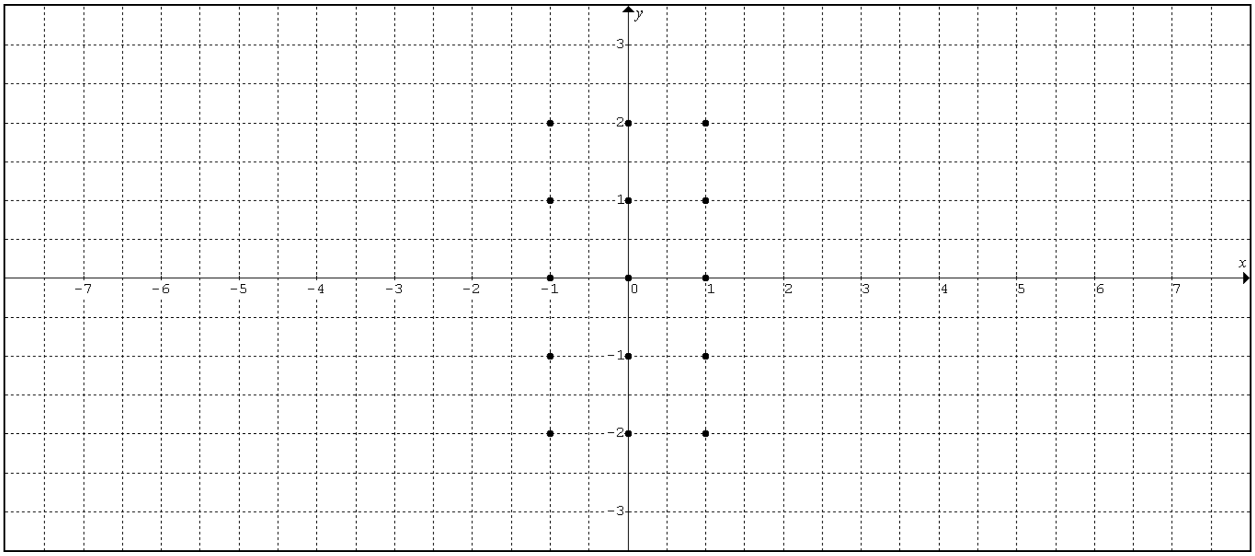
<u>C3.Q203.CH16A.LESSON1</u>

LINE INTEGRALS

VECTOR FIELD

WORK

GRAPH THE VECTOR FIELD $\mathbf{F}(x, y) = \langle x + y,\, x \rangle$ for the given 15 points shown below:



Orientation of a Non-Closed Line Integral

Orientation of a Closed Line Integral

IN CLASS PRACTICE: 16.2 #2, 4, 9, 15, 22

C3.Q203.CH16A.LESSON2

CONSERVATIVE VECTOR FIELD

DEFINITION:

Is **F** conservative? Methods to Determine:

1. Find f such that $\nabla f = \mathbf{F}$.

   A. "Magically" Pick *f*.

   B. Use "JAY'S METHOD" to find *f*.

2. Use a Theorem. **F** is conservative if …

   A. $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ in $\Re^2$

   B. $curl\mathbf{F} = \vec{\mathbf{0}}$ in $\Re^2$ and in $\Re^3$

EXAMPLES: Determine if **F** is conservative.

FOUNDAMENTAL THEOREM OF CALCULUS FOR LINE INTEGRALS

EXAMPLE:

Find $\int_C \mathbf{F}\cdot d\mathbf{r}$ for $\mathbf{F}=\left\langle 2xz+y^2,\, 2xy,\, x^2+3z^2\right\rangle$ and $C: x=t^2,\ y=t+1,\ z=2t-1,\ 0\le t\le 1$

IF AND ONLY IF … THE 99.9%

IN CLASS PRACTICE: 16.3 # 1, 22, 12, 8, 20

C3.Q203.CH16A.LESSON3

ALL ABOUT DEL

Prove: If **F** is conservative, then $curl\mathbf{F} = \vec{\mathbf{0}}$

Prove: $div\ curl\mathbf{F} = 0$

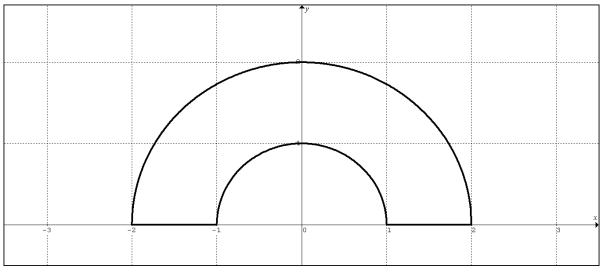
GREEN'S THEOREM (R^2)

EXAMPLES:

1. Evaluate $\oint_C 5xy\,dx + x^3\,dy$ where C is the positively oriented closed curve consisting of $y = x^2$ and $y = 2x$ between points (0, 0) and (2, 4).

2. Evaluate $\oint_C 2xy\,dx + \left(x^2 + y^2\right)dy$ where C is the positively oriented ellipse $4x^2 + 9y^2 = 36$

3. Evaluate $\oint_C y^2\,dx + 3xy\,dy$ where C is outline of the the half circular washer with the positive orientation as shown in the graph below:



EXPLAIN THIS …

VECTOR FORM OF GREEN'S THEOREM

C3.Q203.CH16A.LESSON4

2 THEOREM Let C be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \leqslant t \leqslant b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

3 THEOREM $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D if and only if $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C in D.

4 THEOREM Suppose $\mathbf{F}$ is a vector field that is continuous on an open connected region D. If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D, then $\mathbf{F}$ is a conservative vector field on D; that is, there exists a function f such that $\nabla f = \mathbf{F}$.

5 THEOREM If $\mathbf{F}(x, y) = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j}$ is a conservative vector field, where P and Q have continuous first-order partial derivatives on a domain D, then throughout D we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

6 THEOREM Let $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ be a vector field on an open simply-connected region D. Suppose that P and Q have continuous first-order derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \text{throughout } D$$

Then $\mathbf{F}$ is conservative.

4 THEOREM If $\mathbf{F}$ is a vector field defined on all of $\mathbb{R}^3$ whose component functions have continuous partial derivatives and curl $\mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is a conservative vector field.

GREEN'S THEOREM Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Curve C is simple:

Curve C is closed:

Region R is open:

Region R is closed:

Region R is connected:

Region R is simply-connected:

1. Explain why you cannot use Green's Theorem to compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = y^{1/3}\,\mathbf{i} + x^{1/3}\,\mathbf{j}$ and C is the unit circle $x^2 + y^2 = 1$. How would you compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$?

2. Explain why you cannot use Green's Theorem to compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = \frac{1}{x^2 + y^2}\mathbf{i} + y\,\mathbf{j}$ and C is the unit circle $x^2 + y^2 = 1$. How would you compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$?

3. Explain why $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for all closed curves in vector field $\mathbf{F} = 2y^{3/2}\mathbf{i} + 3x\sqrt{y}\mathbf{j}$ --- even though there are domain restrictions to **F**.

4. The Special Problem (The 0.01%)

Consider the vector field $\mathbf{F} = \frac{-y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j}$ **(Special Problem)**

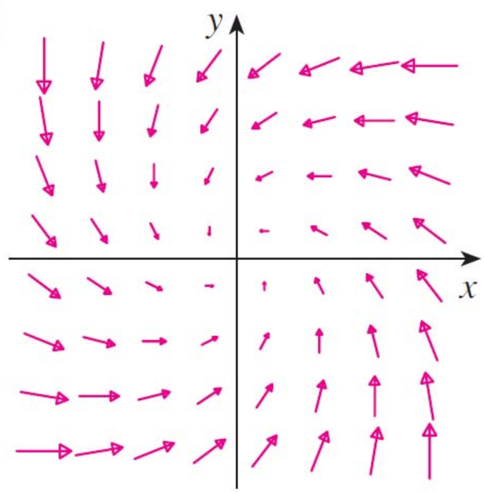
A. Compute $\oint_{\mathbf{C}} \mathbf{F} \cdot d\mathbf{r}$ where C is the unit circle $x^2+y^2=1$ that encloses the origin.

B. Compute $\oint_{\mathbf{C}} \mathbf{F} \cdot d\mathbf{r}$ where C is any curve that does not enclose the origin.

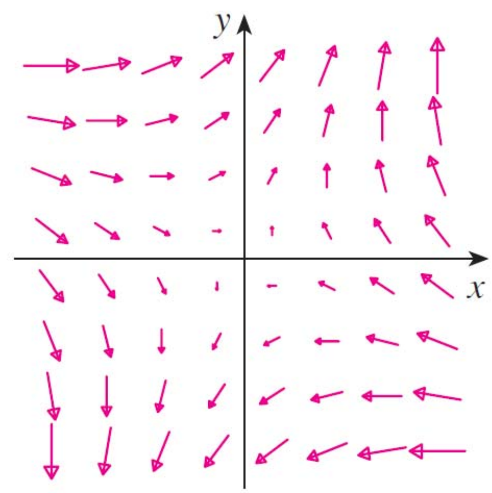
C. Prove that $\oint_{\mathbf{C}} \mathbf{F} \cdot d\mathbf{r} = 2\pi$ for any curve C that encloses the origin.

16.3 **F** CONSERVATIVE?

23.

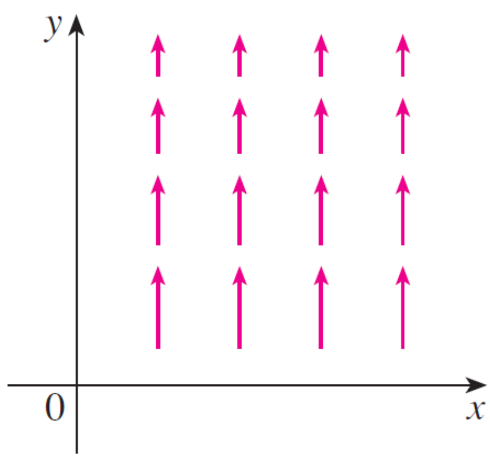


24.

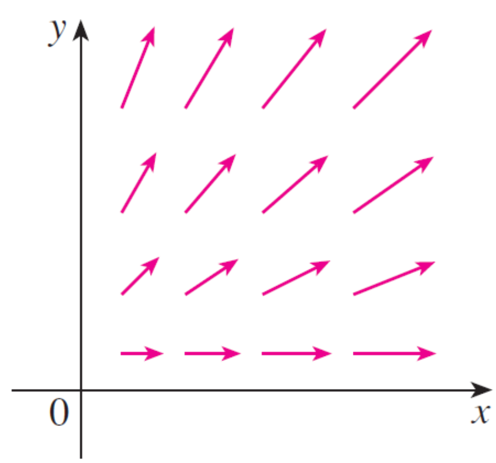


16.5 DIV**F**: POS/NEG/ZERO   CURL**F** · **k**: POS/NEG/ZERO

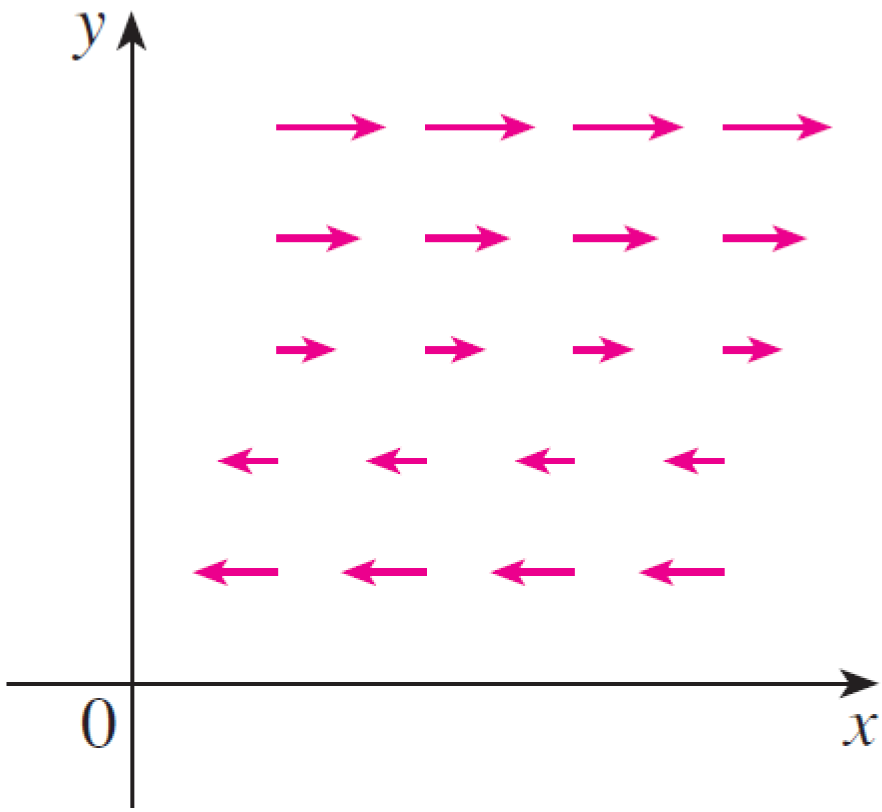
9.



10.



II.



12. Let f be a scalar field and $\mathbf{F}$ a vector field. State whether each expression is meaningful. If not, explain why. If so, state whether it is a scalar field or a vector field.

(a) curl f

(b) grad f

(c) div $\mathbf{F}$

(d) curl(grad f)

(e) grad $\mathbf{F}$

(f) grad(div $\mathbf{F}$)

(g) div(grad f)

(h) grad(div f)

(i) curl(curl $\mathbf{F}$)

(j) div(div $\mathbf{F}$)

(k) (grad f) $\times$ (div $\mathbf{F}$)

(l) div(curl(grad f))

Video Sprinkles:

- € Why is an inverse square field called as such?
- € Show that an inverse square field is conservative.
- € Prove $\oint_C xdy - ydx$ gives the area of region R enclosed by its boundary C.
- € Prove that the area of a circle with radius “a” is πa^2