C3: Q201 CH13 VECTOR VALUED FUNCTIONS

WARM UP: (AP CALCULUS REVIEW)

1. The position of a particle at time *t* is defined by the vector function:

$\mathbf{r}(t) = \langle \cos t,\ t^3,\ \ln t \rangle$ for $t > 0$.

A. Find the velocity and acceleration at time $t = 2$.

B. Find the speed of the particle at time $t = 2$.

C. Set up an expression, involving one or more integrals, used to find the total distance traveled by the particle from time $t = 1$ to time $t = 2$.

2. The velocity vector of a particle moving in R^3 space is given by $\vec{v} = \langle t,\ e^t, te^t \rangle$ for $t \geq 0$.

At $t = 0$, the particle is at the point (1, 1, 1). What is the position of the particle at $t = 3$?

C3: Q201 CH13 VECTOR VALUED FUNCTIONS   LESSON 1

Main Skills (BC REVIEW) | Language (NEW):

Consider the smooth parametrization of curve C: $x = x(t) \quad y = y(t) \quad z = z(t)$

- € Find the *vector equation* of a curve (wire) C in $\Re^3$. $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$
- € Find the *vector equation* of a line tangent to a curve C in $\Re^3$. $\mathbf{r_T}(t_0) = \mathbf{r}_0 + \mathbf{r}'(t_0)t$
- € Find the length of a curve C in $\Re^3$. $L_C = \int_C |\mathbf{r}'(t)| dt$

1. Write the vector value function for the curve C: $y = 2x + 1$

2. Write the vector value function for the curve C: $x^2 + y^2 = 4$

3. Write the vector value function for the line curve C that passes through the points $(1,-2,3)$ and $(0,5,-2)$.

4. Write the vector value function for the line segment curve C that runs from the point $(1,-2,3)$ to $(0,5,-2)$.

5. Write the vector value function for the curve C of intersection of the cylinder $x^2 + y^2 = 1$ and the plane $y + z = 2$.

6. Write the vector value function for the curve C of intersection of the cylinder $4x^2 + y^2 = 4$ and the plane $x + y + z = 2$.

7. Suppose a curve C is defined as: $\mathbf{r}(t) = \langle 3\cos(t), 3\sin(t), t \rangle$; $t \geq 0$

A. What is the shape of C?

B. Write an equation of the line tangent to C at $t = \dfrac{\pi}{2}$:

C. Find the length of C from $t = \dfrac{\pi}{2}$ to $t = 2\pi$.

8. Suppose a curve C is defined as: $\mathbf{r}(t) = \left\langle \ln t, 2\sqrt{t}, t^2 \right\rangle$

A. Write an equation of the line tangent to C at $t = 1$:

B. Find the length of C from $t = 1$ to $t = 5$.

9. Suppose a curve C is defined as the intersection of the cylinder $4x^2 + y^2 = 4$ and the plane $x + y + z = 2$.

A. Write an equation of the line tangent to C at $t = 1$:

B. Find the length of C.

C3: Q201 CH13 VECTOR VALUED FUNCTIONS   LESSON 2

1. Theorem: If $\mathbf{r}(t)$ is differentiable and $|\mathbf{r}(t)|$ is constant, then $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$ for every t in the domain of $\mathbf{r}(t)$.

Proof:

2. Language Review:

3. Prove that $\mathbf{T} \perp \mathbf{N}$

4. Parametrization with respect to arc-length s (concept and language):

5. Reparametrization $\mathbf{r}(t) \to \mathbf{r}(s)$:

A. Reparametrize $\mathbf{r}(t) = \langle 4t-3, 3t-5 \rangle$ D: $t \geq 0$ in terms of arc-length.

B. Reparametrize $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$ D: $t \geq 0$ in terms of arc-length.

C. Reparametrize $\mathbf{r}(t) = \langle e^{2t} \cos(2t), 2, e^{2t} \sin(2t) \rangle$ D: $t \geq 0$ in terms of arc-length.

12. Let C be the curve of intersection. The projection of C onto the xy-plane is the ellipse $4x^2 + y^2 = 4$ or $x^2 + y^2/4 = 1$, $z = 0$. Then we can write $x = \cos t$, $y = 2\sin t$, $0 \le t \le 2\pi$. Since C also lies on the plane $x + y + z = 2$, we have $z = 2 - x - y = 2 - \cos t - 2\sin t$. Then parametric equations for C are $x = \cos t$, $y = 2\sin t$, $z = 2 - \cos t - 2\sin t$, $0 \le t \le 2\pi$, and the corresponding vector equation is $\mathbf{r}(t) = \langle \cos t, 2\sin t, 2 - \cos t - 2\sin t\rangle$. Differentiating gives

$\mathbf{r}'(t) = \langle -\sin t, 2\cos t, \sin t - 2\cos t\rangle \quad \Rightarrow$

$|\mathbf{r}'(t)| = \sqrt{(-\sin t)^2 + (2\cos t)^2 + (\sin t - 2\cos t)^2} = \sqrt{2\sin^2 t + 8\cos^2 t - 4\sin t\,\cos t}$. The length of C is

$L = \int_0^{2\pi} |\mathbf{r}'(t)|\,dt = \int_0^{2\pi} \sqrt{2\sin^2 t + 8\cos^2 t - 4\sin t\,\cos t}\,dt \approx 13.5191$.

13. $\mathbf{r}(t) = 2t\,\mathbf{i} + (1 - 3t)\,\mathbf{j} + (5 + 4t)\,\mathbf{k} \quad \Rightarrow \quad \mathbf{r}'(t) = 2\,\mathbf{i} - 3\,\mathbf{j} + 4\,\mathbf{k}$ and $\frac{ds}{dt} = |\mathbf{r}'(t)| = \sqrt{4 + 9 + 16} = \sqrt{29}$. Then $s = s(t) = \int_0^t |\mathbf{r}'(u)|\,du = \int_0^t \sqrt{29}\,du = \sqrt{29}\,t$. Therefore, $t = \frac{1}{\sqrt{29}}s$, and substituting for t in the original equation, we have $\mathbf{r}(t(s)) = \frac{2}{\sqrt{29}}s\,\mathbf{i} + \left(1 - \frac{3}{\sqrt{29}}s\right)\mathbf{j} + \left(5 + \frac{4}{\sqrt{29}}s\right)\mathbf{k}$.

14. $\mathbf{r}(t) = e^{2t}\cos 2t\,\mathbf{i} + 2\,\mathbf{j} + e^{2t}\sin 2t\,\mathbf{k} \quad \Rightarrow \quad \mathbf{r}'(t) = 2e^{2t}(\cos 2t - \sin 2t)\,\mathbf{i} + 2e^{2t}(\cos 2t + \sin 2t)\,\mathbf{k}$,

$\frac{ds}{dt} = |\mathbf{r}'(t)| = 2e^{2t}\sqrt{(\cos 2t - \sin 2t)^2 + (\cos 2t + \sin 2t)^2} = 2e^{2t}\sqrt{2\cos^2 2t + 2\sin^2 2t} = 2\sqrt{2}\,e^{2t}$.

$s = s(t) = \int_0^t |\mathbf{r}'(u)|\,du = \int_0^t 2\sqrt{2}\,e^{2u}\,du = \sqrt{2}\,e^{2u}\Big]_0^t = \sqrt{2}\,(e^{2t} - 1) \quad \Rightarrow \quad \frac{s}{\sqrt{2}} + 1 = e^{2t} \quad \Rightarrow \quad t = \frac{1}{2}\ln\left(\frac{s}{\sqrt{2}} + 1\right)$.

Substituting, we have

$$\begin{aligned}\mathbf{r}(t(s)) &= e^{2\left(\frac{1}{2}\ln\left(\frac{s}{\sqrt{2}}+1\right)\right)}\cos 2\left(\tfrac{1}{2}\ln\left(\tfrac{s}{\sqrt{2}} + 1\right)\right)\mathbf{i} + 2\,\mathbf{j} + e^{2\left(\frac{1}{2}\ln\left(\frac{s}{\sqrt{2}}+1\right)\right)}\sin 2\left(\tfrac{1}{2}\ln\left(\tfrac{s}{\sqrt{2}} + 1\right)\right)\mathbf{k} \\ &= \left(\tfrac{s}{\sqrt{2}} + 1\right)\cos\left(\ln\left(\tfrac{s}{\sqrt{2}} + 1\right)\right)\mathbf{i} + 2\,\mathbf{j} + \left(\tfrac{s}{\sqrt{2}} + 1\right)\sin\left(\ln\left(\tfrac{s}{\sqrt{2}} + 1\right)\right)\mathbf{k}\end{aligned}$$

15. Here $\mathbf{r}(t) = \langle 3\sin t, 4t, 3\cos t\rangle$, so $\mathbf{r}'(t) = \langle 3\cos t, 4, -3\sin t\rangle$ and $|\mathbf{r}'(t)| = \sqrt{9\cos^2 t + 16 + 9\sin^2 t} = \sqrt{25} = 5$. The point $(0, 0, 3)$ corresponds to $t = 0$, so the arc length function beginning at $(0, 0, 3)$ and measuring in the positive direction is given by $s(t) = \int_0^t |\mathbf{r}'(u)|\,du = \int_0^t 5\,du = 5t$. $s(t) = 5 \quad \Rightarrow \quad 5t = 5 \quad \Rightarrow \quad t = 1$, thus your location after moving 5 units along the curve is $(3\sin 1, 4, 3\cos 1)$.

16. $\mathbf{r}(t) = \left(\frac{2}{t^2 + 1} - 1\right)\mathbf{i} + \frac{2t}{t^2 + 1}\mathbf{j} \quad \Rightarrow \quad \mathbf{r}'(t) = \frac{-4t}{(t^2 + 1)^2}\mathbf{i} + \frac{-2t^2 + 2}{(t^2 + 1)^2}\mathbf{j}$,

$$\frac{ds}{dt} = |\mathbf{r}'(t)| = \sqrt{\left[\frac{-4t}{(t^2+1)^2}\right]^2 + \left[\frac{-2t^2+2}{(t^2+1)^2}\right]^2} = \sqrt{\frac{4t^4 + 8t^2 + 4}{(t^2+1)^4}} = \sqrt{\frac{4(t^2+1)^2}{(t^2+1)^4}} = \sqrt{\frac{4}{(t^2+1)^2}} = \frac{2}{t^2+1}.$$

Since the initial point $(1, 0)$ corresponds to $t = 0$, the arc length function

$s(t) = \int_0^t |\mathbf{r}'(u)|\,du = \int_0^t \frac{2}{u^2 + 1}\,du = 2\arctan t$. Then $\arctan t = \frac{1}{2}s \quad \Rightarrow \quad t = \tan\frac{1}{2}s$. Substituting, we have

$$\begin{aligned}\mathbf{r}(t(s)) &= \left[\frac{2}{\tan^2\left(\frac{1}{2}s\right) + 1} - 1\right]\mathbf{i} + \frac{2\tan\left(\frac{1}{2}s\right)}{\tan^2\left(\frac{1}{2}s\right) + 1}\mathbf{j} = \frac{1 - \tan^2\left(\frac{1}{2}s\right)}{1 + \tan^2\left(\frac{1}{2}s\right)}\mathbf{i} + \frac{2\tan\left(\frac{1}{2}s\right)}{\sec^2\left(\frac{1}{2}s\right)}\mathbf{j} \\ &= \frac{1 - \tan^2\left(\frac{1}{2}s\right)}{\sec^2\left(\frac{1}{2}s\right)}\mathbf{i} + 2\tan\left(\tfrac{1}{2}s\right)\cos^2\left(\tfrac{1}{2}s\right)\mathbf{j} = \left[\cos^2\left(\tfrac{1}{2}s\right) - \sin^2\left(\tfrac{1}{2}s\right)\right]\mathbf{i} + 2\sin\left(\tfrac{1}{2}s\right)\cos\left(\tfrac{1}{2}s\right)\mathbf{j} = \cos s\,\mathbf{i} + \sin s\,\mathbf{j}\end{aligned}$$

With this parametrization, we recognize the function as representing the unit circle. Note here that the curve approaches, but does not include, the point $(-1, 0)$, since $\cos s = -1$ for $s = \pi + 2k\pi$ (k an integer) but then $t = \tan\left(\frac{1}{2}s\right)$ is undefined.

6. Curvature in R^2:

Concept: $k = \left| \frac{d\theta}{ds} \right|$ where θ is the angle between **T** and **i** and *s* is the arc length.

Formula when $x = x(t),\ y = y(t)$: $k(t) = \left| \frac{x'(t)y''(t) - y'(t)x''(t)}{\left[(x'(t))^2 + (y'(t))^2 \right]^{\frac{3}{2}}} \right|$

Formula when $y = f(x)$: $k(x) = \left| \frac{f''(x)}{\left[1 + (f'(x))^2 \right]^{\frac{3}{2}}} \right|$

7. Curvature in R^3:

Concept: $k = \left|\frac{d\mathbf{T}}{ds}\right|$

Formulas:

VECTOR PROPERTIES FOR CH 13 LESSON 3

① $|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$ [CH 12.3]

② $\vec{a} \cdot \vec{a} = |\vec{a}|^2$ [CH 12.3]

③ $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$ [CH 12.3]

④ $(k\vec{a}) \cdot \vec{b} = k(\vec{a} \cdot \vec{b}) = a \cdot (k\vec{b})$ [CH 12.3]
k = scalor

⑤ $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \theta$ [CH 12.3]

⑥ $\vec{a} \perp \vec{b}$ iff $\vec{a} \cdot \vec{b} = 0$ [CH 12.3]

⑦ $\text{Comp}_{\vec{a}} \vec{b} = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$ [CH 12.3]

⑧ $\text{Proj}_{\vec{a}} \vec{b} = \left(\text{Comp}_{\vec{a}} \vec{b}\right) \dfrac{\vec{a}}{|\vec{a}|} = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \vec{a}$ [CH 12.3]

⑨ $\vec{a} \times \vec{b}$ $\perp$ to both $\vec{a}$ and $\vec{b}$ [CH 12.4]

⑩ $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin \theta$ [CH 12.4]

⑪ $\vec{a} \parallel \vec{b}$ iff $\vec{a} \times \vec{b} = \vec{0}$ [CH 12.4]

⑫ A: $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$ [CH 12.4]

B: $(c\vec{a}) \times \vec{b} = c(\vec{a} \times \vec{b}) = \vec{a} \times (c\vec{b})$

C: $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

D: $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$

E: $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$

F: $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

[13.2] $\frac{d}{dt}[\vec{u} + \vec{v}] = u' + v'$

$\frac{d}{dt}[c\vec{u}] = c u'$

$\frac{d}{dt}[f(t)\vec{u}] = f'(t)\vec{u} + f(t)\vec{u}'$

$\frac{d}{dt}[\vec{u} \cdot \vec{v}] = \vec{u}' \cdot \vec{v} + \vec{u} \cdot \vec{v}'$ Product

$\frac{d}{dt}[\vec{u} \times \vec{v}] = \vec{u}' \times \vec{v} + \vec{u} \times \vec{v}'$ Product

$\frac{d}{dt}[\vec{u}(f(t))] = f'(t)\vec{u}'(f(t))$ chain

C3: Q201 CH13 VECTOR VALUED FUNCTIONS LESSON 3

1. Lesson Overview and Diagram

2. SET OF VECTOR PROPERTIES

3. Prove Property - 13B (in R^2)

$$\frac{d}{dt}[\mathbf{u} \cdot \mathbf{v}] =$$

4. Prove $\frac{d}{dt}\left[\mathbf{r}(t) \times \mathbf{r}^{/}(t)\right] = \mathbf{r}(t) \times \mathbf{r}^{//}(t)$

$$\frac{d}{dt}\left[\mathbf{r}(t) \times \mathbf{r}^{/}(t)\right] =$$

5. Prove $\frac{d}{dt}|\mathbf{r}(t)| = \frac{\mathbf{r}(t) \cdot \mathbf{r}^{/}(t)}{|\mathbf{r}(t)|}$

$$\frac{d}{dt}|\mathbf{r}(t)| =$$

6. Prove $k\mathbf{a} + m\mathbf{b} = km(\mathbf{a} \cdot \mathbf{b})$

$$k\mathbf{a} + m\mathbf{b} =$$

7. BERMEL TRUE / FALSE UNDERSTANDING

8. Prove $\mathbf{T}^{/}(s) = K\mathbf{N}(s)$

9. Prove $\mathbf{v}(t) = \mathbf{T}(s)\dfrac{ds}{dt}$

10. Prove $\mathbf{a}(t) = \dfrac{d^2 s}{dt^2}\mathbf{T}(s) + K\left(\dfrac{ds}{dt}\right)^2 \mathbf{N}(s)$ --- Hence Prove $a_{\mathbf{T}} = \dfrac{d^2 s}{dt^2}$ and $a_{\mathbf{N}} = K\left(\dfrac{ds}{dt}\right)^2$

11. Prove $a_{\mathbf{T}} = \dfrac{\mathbf{r}^{/}(t) \cdot \mathbf{r}^{//}(t)}{\left|\mathbf{r}^{/}(t)\right|}$

12. Prove $a_{\mathbf{N}} = \dfrac{\left|\mathbf{r}^{/}(t) \times \mathbf{r}^{//}(t)\right|}{\left|\mathbf{r}^{/}(t)\right|}$