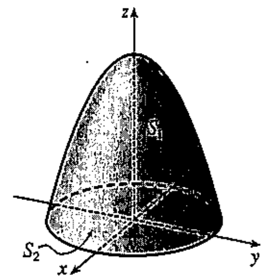
2. $\operatorname{div}\mathbf{F} = 2x + x + 1 = 3x + 1$ so



$$\iiint_E \operatorname{div}\mathbf{F}\,dV = \iiint_E (3x+1)\,dV = \int_0^{2\pi}\int_0^2\int_0^{4-r^2}(3r\cos\theta+1)\,r\,dz\,dr\,d\theta$$
$$= \int_0^2\int_0^{2\pi} r(3r\cos\theta+1)(4-r^2)\,d\theta\,dr$$
$$= \int_0^{2\pi} r(4-r^2)\big[3r\sin\theta+\theta\big]_{\theta=0}^{\theta=2\pi}\,dr$$
$$= 2\pi\int_0^2(4r-r^3)\,dr = 2\pi\big[2r^2-\tfrac14 r^4\big]_0^2$$
$$= 2\pi(8-4) = 8\pi$$

On S_1: The surface is $z = 4 - x^2 - y^2$, $x^2 + y^2 \le 4$, with upward orientation, and $\mathbf{F} = x^2\,\mathbf{i} + xy\,\mathbf{j} + (4 - x^2 - y^2)\,\mathbf{k}$. Then

$$\iint_{S_1}\mathbf{F}\cdot d\mathbf{S} = \iint_D[-(x^2)(-2x) - (xy)(-2y) + (4-x^2-y^2)]\,dA$$
$$= \iint_D\left[2x(x^2+y^2) + 4 - (x^2+y^2)\right]dA = \int_0^{2\pi}\int_0^2(2r\cos\theta\cdot r^2 + 4 - r^2)\,r\,dr\,d\theta$$
$$= \int_0^{2\pi}\left[\tfrac25 r^5\cos\theta + 2r^2 - \tfrac14 r^4\right]_{r=0}^{r=2}d\theta = \int_0^{2\pi}\left(\tfrac{64}{5}\cos\theta + 4\right)d\theta = \left[\tfrac{64}{5}\sin\theta + 4\theta\right]_0^{2\pi} = 8\pi$$

On S_2: The surface is $z = 0$ with downward orientation, so $\mathbf{F} = x^2\,\mathbf{i} + xy\,\mathbf{j}$, $\mathbf{n} = -\mathbf{k}$ and $\iint_{S_2}\mathbf{F}\cdot\mathbf{n}\,dS = \iint_{S_2}0\,dS = 0$.

Thus $\iint_S\mathbf{F}\cdot d\mathbf{S} = \iint_{S_1}\mathbf{F}\cdot d\mathbf{S} + \iint_{S_2}\mathbf{F}\cdot d\mathbf{S} = 8\pi$.

3. $\operatorname{div}\mathbf{F} = x + y + z$, so

$$\iiint_E \operatorname{div}\mathbf{F}\,dV = \int_0^{2\pi}\int_0^1\int_0^1(r\cos\theta + r\sin\theta + z)\,r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1\left(r^2\cos\theta + r^2\sin\theta + \tfrac12 r\right)dr\,d\theta$$
$$= \int_0^{2\pi}\left(\tfrac13\cos\theta + \tfrac13\sin\theta + \tfrac14\right)d\theta = \tfrac14(2\pi) = \tfrac{\pi}{2}$$

Let S_1 be the top of the cylinder, S_2 the bottom, and S_3 the vertical edge. On S_1, $z = 1$, $\mathbf{n} = \mathbf{k}$, and $\mathbf{F} = xy\,\mathbf{i} + y\,\mathbf{j} + x\,\mathbf{k}$, so

$\iint_{S_1}\mathbf{F}\cdot d\mathbf{S} = \iint_{S_1}\mathbf{F}\cdot\mathbf{n}\,dS = \iint_{S_1}x\,dS = \int_0^{2\pi}\int_0^1(r\cos\theta)\,r\,dr\,d\theta = \big[\sin\theta\big]_0^{2\pi}\left[\tfrac13 r^3\right]_0^1 = 0.$

On S_2, $z = 0$, $\mathbf{n} = -\mathbf{k}$, and $\mathbf{F} = xy\,\mathbf{i}$ so $\iint_{S_2}\mathbf{F}\cdot d\mathbf{S} = \iint_{S_2}0\,dS = 0$.

S_3 is given by $\mathbf{r}(\theta, z) = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j} + z\,\mathbf{k}$, $0 \le \theta \le 2\pi$, $0 \le z \le 1$. Then $\mathbf{r}_\theta \times \mathbf{r}_z = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$ and

$$\iint_{S_3}\mathbf{F}\cdot d\mathbf{S} = \iint_D\mathbf{F}\cdot(\mathbf{r}_\theta\times\mathbf{r}_z)\,dA = \int_0^{2\pi}\int_0^1(\cos^2\theta\sin\theta + z\sin^2\theta)\,dz\,d\theta$$
$$= \int_0^{2\pi}\left(\cos^2\theta\sin\theta + \tfrac12\sin^2\theta\right)d\theta = \left[-\tfrac13\cos^3\theta + \tfrac14\left(\theta - \tfrac12\sin 2\theta\right)\right]_0^{2\pi} = \tfrac{\pi}{2}$$

Thus $\iint_S\mathbf{F}\cdot d\mathbf{S} = 0 + 0 + \frac{\pi}{2} = \frac{\pi}{2}$.

4. $\operatorname{div}\mathbf{F} = 1 + 1 + 1 = 3$, so $\iiint_E \operatorname{div}\mathbf{F}\,dV = \iiint_E 3\,dV = 3(\text{volume of ball}) = 3\left(\frac43\pi\right) = 4\pi$. To find $\iint_S\mathbf{F}\cdot d\mathbf{S}$ we use spherical coordinates. S is the unit sphere, represented by $\mathbf{r}(\phi,\theta) = \sin\phi\,\cos\theta\,\mathbf{i} + \sin\phi\,\sin\theta\,\mathbf{j} + \cos\phi\,\mathbf{k}$, $0 \le \phi \le \pi$, $0 \le \theta \le 2\pi$. Then $\mathbf{r}_\phi \times \mathbf{r}_\theta = \sin^2\phi\,\cos\theta\,\mathbf{i} + \sin^2\phi\,\sin\theta\,\mathbf{j} + \sin\phi\,\cos\phi\,\mathbf{k}$ (see Example 17.6.10 [ET 16.6.10]) and $\mathbf{F}(\mathbf{r}(\phi,\theta)) = \sin\phi\,\cos\theta\,\mathbf{i} + \sin\phi\,\sin\theta\,\mathbf{j} + \cos\phi\,\mathbf{k}$. Thus

$$\iint_S\mathbf{F}\cdot d\mathbf{S} = \iint_D\mathbf{F}\cdot(\mathbf{r}_\phi\times\mathbf{r}_\theta)\,dA = \int_0^{2\pi}\int_0^{\pi}\left(\sin^3\phi\,\cos^2\theta + \sin^3\phi\,\sin^2\theta + \sin\phi\,\cos^2\phi\right)d\phi\,d\theta$$
$$= \int_0^{2\pi}d\theta\int_0^{\pi}\sin\phi\,d\phi = (2\pi)(2) = 4\pi$$

5. $\operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}(e^x\sin y) + \frac{\partial}{\partial y}(e^x\cos y) + \frac{\partial}{\partial z}(yz^2) = e^x\sin y - e^x\sin y + 2yz = 2yz$, so by the Divergence Theorem,

$\iint_S\mathbf{F}\cdot d\mathbf{S} = \iiint_E \operatorname{div}\mathbf{F}\,dV = \int_0^1\int_0^1\int_0^2 2yz\,dz\,dy\,dx = 2\int_0^1 dx\int_0^1 y\,dy\int_0^1 z\,dz = 2\big[x\big]_0^1\left[\tfrac12 y^2\right]_0^1\left[\tfrac12 z^2\right]_0^2 = 2.$

6. $\operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}(x^2z^3) + \frac{\partial}{\partial y}(2xyz^3) + \frac{\partial}{\partial z}(xz^4) = 2xz^3 + 2xz^3 + 4xz^3 = 8xz^3$, so by the Divergence Theorem,

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E \operatorname{div}\mathbf{F}\,dV = \int_{-1}^1\int_{-2}^2\int_{-3}^3 8xz^3\,dz\,dy\,dx = 8\int_{-1}^1 x\,dx\int_{-2}^2 dy\int_{-3}^3 z^3\,dz$$
$$= 8\left[\tfrac{1}{2}x^2\right]_{-1}^1 \left[y\right]_{-2}^2 \left[\tfrac{1}{4}z^4\right]_{-3}^3 = 0$$

7. $\operatorname{div}\mathbf{F} = 3y^2 + 0 + 3z^2$, so using cylindrical coordinates with $y = r\cos\theta$, $z = r\sin\theta$, $x = x$ we have

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E (3y^2+3z^2)\,dV = \int_0^{2\pi}\int_0^1\int_{-1}^2 (3r^2\cos^2\theta + 3r^2\sin^2\theta)\,r\,dx\,dr\,d\theta$$
$$= 3\int_0^{2\pi}d\theta\int_0^1 r^3\,dr\int_{-1}^2 dx = 3(2\pi)\left(\tfrac{1}{4}\right)(3) = \tfrac{9\pi}{2}$$

8. $\operatorname{div}\mathbf{F} = 3x^2y - 2x^2y - x^2y = 0$, so $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E 0\,dV = 0$.

9. $\operatorname{div}\mathbf{F} = y\sin z + 0 - y\sin z = 0$, so by the Divergence Theorem, $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E 0\,dV = 0$.

10. $\operatorname{div}\mathbf{F} = 2xy + 2xy + 2xy = 6xy$, so

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E 6xy\,dV = \int_0^1\int_0^{2-2y}\int_0^{2-x-2y} 6xy\,dz\,dx\,dy = \int_0^1\int_0^{2-2y} 6xy(2-x-2y)\,dx\,dy$$
$$= \int_0^1\int_0^{2-2y}(12xy - 6x^2y - 12xy^2)\,dx\,dy = \int_0^1\left[6x^2y - 2x^3y - 6x^2y^2\right]_{x=0}^{x=2-2y}dy$$
$$= \int_0^1 y(2-2y)^3\,dy = \left[-\tfrac{8}{5}y^5 + 6y^4 - 8y^3 + 4y^2\right]_0^1 = \tfrac{2}{5}$$

11. $\operatorname{div}\mathbf{F} = y^2 + 0 + x^2 = x^2 + y^2$ so

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E (x^2+y^2)\,dV = \int_0^{2\pi}\int_0^2\int_{r^2}^4 r^2\cdot r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^2 r^3(4-r^2)\,dr\,d\theta$$
$$= \int_0^{2\pi}d\theta\int_0^2(4r^3 - r^5)\,dr = 2\pi\left[r^4 - \tfrac{1}{6}r^6\right]_0^2 = \tfrac{32}{3}\pi$$

12. $\operatorname{div}\mathbf{F} = 4x^3 + 4xy^2$ so

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E 4x(x^2+y^2)\,dV = \int_0^{2\pi}\int_0^1\int_0^{r\cos\theta+2}(4r^3\cos\theta)\,r\,dz\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^1(4r^5\cos^2\theta + 8r^4\cos\theta)\,dr\,d\theta = \int_0^{2\pi}\left(\tfrac{2}{3}\cos^2\theta + \tfrac{8}{5}\cos\theta\right)d\theta = \tfrac{2}{3}\pi$$

13. $\operatorname{div}\mathbf{F} = 12x^2z + 12y^2z + 12z^3$ so

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E 12z(x^2+y^2+z^2)\,dV = \int_0^{2\pi}\int_0^{\pi}\int_0^R 12(\rho\cos\phi)(\rho^2)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$
$$= 12\int_0^{2\pi}d\theta\int_0^{\pi}\sin\phi\,\cos\phi\,d\phi\int_0^R\rho^5\,d\rho = 12(2\pi)\left[\tfrac{1}{2}\sin^2\phi\right]_0^{\pi}\left[\tfrac{1}{6}\rho^6\right]_0^R = 0$$

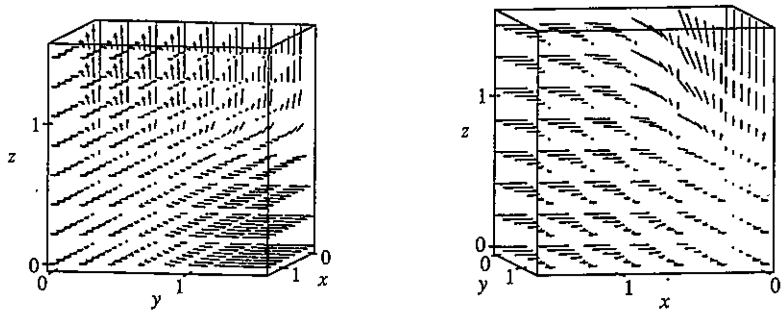
14. $\mathbf{F}(x,y,z) = \dfrac{x}{\sqrt{x^2+y^2+z^2}}\,\mathbf{i} + \dfrac{y}{\sqrt{x^2+y^2+z^2}}\,\mathbf{j} + \dfrac{z}{\sqrt{x^2+y^2+z^2}}\,\mathbf{k}$, so

$$\operatorname{div}\mathbf{F} = \frac{\sqrt{x^2+y^2+z^2} - x^2/\sqrt{x^2+y^2+z^2}}{x^2+y^2+z^2} + \frac{\sqrt{x^2+y^2+z^2} - y^2/\sqrt{x^2+y^2+z^2}}{x^2+y^2+z^2}$$
$$+ \frac{\sqrt{x^2+y^2+z^2} - z^2/\sqrt{x^2+y^2+z^2}}{x^2+y^2+z^2}$$
$$= \frac{x^2+y^2+z^2-x^2}{(x^2+y^2+z^2)^{3/2}} + \frac{x^2+y^2+z^2-y^2}{(x^2+y^2+z^2)^{3/2}} + \frac{x^2+y^2+z^2-z^2}{(x^2+y^2+z^2)^{3/2}} = \frac{2(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{3/2}} = \frac{2}{\sqrt{x^2+y^2+z^2}}.$$

Then
$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E \frac{2}{\sqrt{x^2+y^2+z^2}}\,dV = \int_0^{\pi/2}\int_0^{2\pi}\int_0^1 \frac{2}{\rho}\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$
$$= 2\int_0^{\pi/2}\sin\phi\,d\phi\int_0^{2\pi}d\theta\int_0^1\rho\,d\rho = 2\,(1)\,(2\pi)\left(\tfrac{1}{2}\right) = 2\pi$$

15. $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E \sqrt{3-x^2}\,dV = \int_{-1}^{1}\int_{-1}^{1}\int_0^{2-x^4-y^4}\sqrt{3-x^2}\,dz\,dy\,dx = \frac{341}{60}\sqrt{2}+\frac{81}{20}\sin^{-1}\left(\frac{\sqrt{3}}{3}\right)$

16.



By the Divergence Theorem, the flux of $\mathbf{F}$ across the surface of the cube is

$\iint_S \mathbf{F}\cdot d\mathbf{S} = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^{\pi/2}\left[\cos x\,\cos^2 y + 3\sin^2 y\,\cos y\,\cos^4 z + 5\sin^4 z\,\cos z\,\cos^6 x\right]dz\,dy\,dx = \frac{19}{64}\pi^2.$

17. For S_1 we have $\mathbf{n} = -\mathbf{k}$, so $\mathbf{F}\cdot\mathbf{n} = \mathbf{F}\cdot(-\mathbf{k}) = -x^2 z - y^2 = -y^2$ (since $z = 0$ on S_1). So if D is the unit disk, we get $\iint_{S_1}\mathbf{F}\cdot d\mathbf{S} = \iint_{S_1}\mathbf{F}\cdot\mathbf{n}\,dS = \iint_D(-y^2)\,dA = -\int_0^{2\pi}\int_0^1 r^2(\sin^2\theta)\,r\,dr\,d\theta = -\frac{1}{4}\pi$. Now since S_2 is closed, we can use the Divergence Theorem. Since $\operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}(z^2x) + \frac{\partial}{\partial y}\left(\frac{1}{3}y^3 + \tan z\right) + \frac{\partial}{\partial z}(x^2z + y^2) = z^2 + y^2 + x^2$, we use spherical coordinates to get $\iint_{S_2}\mathbf{F}\cdot d\mathbf{S} = \iiint_E \operatorname{div}\mathbf{F}\,dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \rho^2\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \frac{2}{5}\pi$. Finally

$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_{S_2}\mathbf{F}\cdot d\mathbf{S} - \iint_{S_1}\mathbf{F}\cdot d\mathbf{S} = \frac{2}{5}\pi - \left(-\frac{1}{4}\pi\right) = \frac{13}{20}\pi.$

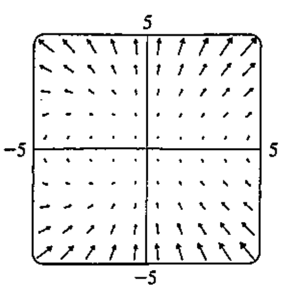
18. As in the hint to Exercise 19, we create a closed surface $S_2 = S\cup S_1$, where S is the part of the paraboloid $x^2+y^2+z=2$ that lies above the plane $z=1$, and S_1 is the disk $x^2+y^2=1$ on the plane $z=1$ oriented downward, and we then apply the Divergence Theorem. Since the disk S_1 is oriented downward, its unit normal vector is $\mathbf{n} = -\mathbf{k}$ and $\mathbf{F}\cdot(-\mathbf{k}) = -z = -1$ on S_1. So $\iint_{S_1}\mathbf{F}\cdot d\mathbf{S} = \iint_{S_1}\mathbf{F}\cdot\mathbf{n}\,dS = \iint_{S_1}(-1)\,dS = -A(S_1) = -\pi$. Let E be the region bounded by S_2. Then $\iint_{S_2}\mathbf{F}\cdot d\mathbf{S} = \iiint_E \operatorname{div}\mathbf{F}\,dV = \iiint_E 1\,dV = \int_0^1\int_0^{2\pi}\int_1^{2-r^2} r\,dz\,d\theta\,dr = \int_0^1\int_0^{2\pi}(r-r^3)\,d\theta\,dr = (2\pi)\frac{1}{4} = \frac{\pi}{2}$. Thus the flux of $\mathbf{F}$ across S is $\iint_S\mathbf{F}\cdot d\mathbf{S} = \iint_{S_2}\mathbf{F}\cdot d\mathbf{S} - \iint_{S_1}\mathbf{F}\cdot d\mathbf{S} = \frac{\pi}{2} - (-\pi) = \frac{3\pi}{2}$.

19. The vectors that end near P_1 are longer than the vectors that start near P_1, so the net flow is inward near P_1 and $\operatorname{div}\mathbf{F}(P_1)$ is negative. The vectors that end near P_2 are shorter than the vectors that start near P_2, so the net flow is outward near P_2 and $\operatorname{div}\mathbf{F}(P_2)$ is positive.

20. (a) The vectors that end near P_1 are shorter than the vectors that start near P_1, so the net flow is outward and P_1 is a source. The vectors that end near P_2 are longer than the vectors that start near P_2, so the net flow is inward and P_2 is a sink.

(b) $\mathbf{F}(x,y) = \langle x, y^2\rangle \;\Rightarrow\; \operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = 1 + 2y$. The y-value at P_1 is positive, so $\operatorname{div}\mathbf{F} = 1+2y$ is positive, thus P_1 is a source. At P_2, $y < -1$, so $\operatorname{div}\mathbf{F} = 1+2y$ is negative, and P_2 is a sink.

21.

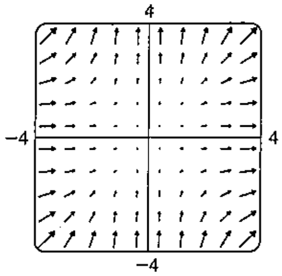


From the graph it appears that for points above the x-axis, vectors starting near a particular point are longer than vectors ending there, so divergence is positive. The opposite is true at points below the x-axis, where divergence is negative.

$\mathbf{F}(x,y) = \langle xy, x+y^2\rangle \;\Rightarrow\; \operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(x+y^2) = y + 2y = 3y.$

Thus $\operatorname{div}\mathbf{F} > 0$ for $y > 0$, and $\operatorname{div}\mathbf{F} < 0$ for $y < 0$.

22.



From the graph it appears that for points above the line $y = -x$, vectors starting near a particular point are longer than vectors ending there, so divergence is positive. The opposite is true at points below the line $y = -x$, where divergence is negative.

$\mathbf{F}(x, y) = \langle x^2, y^2 \rangle \Rightarrow \operatorname{div} \mathbf{F} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) = 2x + 2y$. Then $\operatorname{div} \mathbf{F} > 0$ for $2x + 2y > 0 \Rightarrow y > -x$, and $\operatorname{div} \mathbf{F} < 0$ for $y < -x$.

23. Since $\dfrac{\mathbf{x}}{|\mathbf{x}|^3} = \dfrac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{(x^2+y^2+z^2)^{3/2}}$ and $\dfrac{\partial}{\partial x}\left(\dfrac{x}{(x^2+y^2+z^2)^{3/2}}\right) = \dfrac{(x^2+y^2+z^2) - 3x^2}{(x^2+y^2+z^2)^{5/2}}$ with similar expressions for $\dfrac{\partial}{\partial y}\left(\dfrac{y}{(x^2+y^2+z^2)^{3/2}}\right)$ and $\dfrac{\partial}{\partial z}\left(\dfrac{z}{(x^2+y^2+z^2)^{3/2}}\right)$, we have

$\operatorname{div}\left(\dfrac{\mathbf{x}}{|\mathbf{x}|^3}\right) = \dfrac{3(x^2+y^2+z^2) - 3(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{5/2}} = 0$, except at $(0, 0, 0)$ where it is undefined.

24. We first need to find $\mathbf{F}$ so that $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_S (2x + 2y + z^2)\, dS$, so $\mathbf{F} \cdot \mathbf{n} = 2x + 2y + z^2$. But for S,

$\mathbf{n} = \dfrac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{\sqrt{x^2+y^2+z^2}} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$. Thus $\mathbf{F} = 2\,\mathbf{i} + 2\,\mathbf{j} + z\,\mathbf{k}$ and $\operatorname{div} \mathbf{F} = 1$.

If $B = \{(x, y, z) \mid x^2 + y^2 + z^2 \le 1\}$, then $\iint_S (2x + 2y + z^2)\, dS = \iiint_B dV = V(B) = \frac{4}{3}\pi(1)^3 = \frac{4}{3}\pi$.

25. $\iint_S \mathbf{a} \cdot \mathbf{n}\, dS = \iiint_E \operatorname{div} \mathbf{a}\, dV = 0$ since $\operatorname{div} \mathbf{a} = 0$.

26. $\frac{1}{3}\iint_S \mathbf{F} \cdot d\mathbf{S} = \frac{1}{3}\iiint_E \operatorname{div} \mathbf{F}\, dV = \frac{1}{3}\iiint_E 3\, dV = V(E)$

27. $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div}(\operatorname{curl} \mathbf{F})\, dV = 0$ by Theorem 17.5.11 [ET 16.5.11].

28. $\iint_S D_{\mathbf{n}} f\, dS = \iint_S (\nabla f \cdot \mathbf{n})\, dS = \iiint_E \operatorname{div}(\nabla f)\, dV = \iiint_E \nabla^2 f\, dV$

29. $\iint_S (f \nabla g) \cdot \mathbf{n}\, dS = \iiint_E \operatorname{div}(f \nabla g)\, dV = \iiint_E (f \nabla^2 g + \nabla g \cdot \nabla f)\, dV$ by Exercise 17.5.25 [ET 16.5.25].

30. $\iint_S (f \nabla g - g \nabla f) \cdot \mathbf{n}\, dS = \iiint_E \left[(f \nabla^2 g + \nabla g \cdot \nabla f) - (g \nabla^2 f + \nabla g \cdot \nabla f)\right] dV$ [by Exercise 29].

But $\nabla g \cdot \nabla f = \nabla f \cdot \nabla g$, so that $\iint_S (f \nabla g - g \nabla f) \cdot \mathbf{n}\, dS = \iiint_E (f \nabla^2 g - g \nabla^2 f)\, dV$.

31. If $\mathbf{c} = c_1\,\mathbf{i} + c_2\,\mathbf{j} + c_3\,\mathbf{k}$ is an arbitrary constant vector, we define $\mathbf{F} = f\mathbf{c} = fc_1\,\mathbf{i} + fc_2\,\mathbf{j} + fc_3\,\mathbf{k}$. Then

$\operatorname{div} \mathbf{F} = \operatorname{div} f\mathbf{c} = \dfrac{\partial f}{\partial x} c_1 + \dfrac{\partial f}{\partial y} c_2 + \dfrac{\partial f}{\partial z} c_3 = \nabla f \cdot \mathbf{c}$ and the Divergence Theorem says $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F}\, dV \Rightarrow$

$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_E \nabla f \cdot \mathbf{c}\, dV$. In particular, if $\mathbf{c} = \mathbf{i}$ then $\iint_S f\mathbf{i} \cdot \mathbf{n}\, dS = \iiint_E \nabla f \cdot \mathbf{i}\, dV \Rightarrow$

$\displaystyle\iint_S f n_1\, dS = \iiint_E \frac{\partial f}{\partial x}\, dV$ (where $\mathbf{n} = n_1\,\mathbf{i} + n_2\,\mathbf{j} + n_3\,\mathbf{k}$). Similarly, if $\mathbf{c} = \mathbf{j}$ we have $\displaystyle\iint_S f n_2\, dS = \iiint_E \frac{\partial f}{\partial y}\, dV$,

and $\mathbf{c} = \mathbf{k}$ gives $\displaystyle\iint_S f n_3\, dS = \iiint_E \frac{\partial f}{\partial z}\, dV$. Then

$$\begin{aligned}\iint_S f\mathbf{n}\, dS &= \left(\iint_S f n_1\, dS\right)\mathbf{i} + \left(\iint_S f n_2\, dS\right)\mathbf{j} + \left(\iint_S f n_3\, dS\right)\mathbf{k} \\ &= \left(\iiint_E \frac{\partial f}{\partial x}\, dV\right)\mathbf{i} + \left(\iiint_E \frac{\partial f}{\partial y}\, dV\right)\mathbf{j} + \left(\iiint_E \frac{\partial f}{\partial z}\, dV\right)\mathbf{k} = \iiint_E \left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\right) dV \\ &= \iiint_E \nabla f\, dV \quad \text{as desired.}\end{aligned}$$