41) $\vec{r}(t,\theta) = \langle 2\cos\theta, 2\sin\theta, t\rangle \quad 0\le\theta\le 2\pi \quad 0\le t\le 1$

$\vec{r}_\theta = \langle -2\sin\theta, 2\cos\theta, 0\rangle$

$\vec{r}_t = \langle 0, 0, 1\rangle$

$\vec{r}_\theta\times\vec{r}_t = \langle 2\cos\theta, 2\sin\theta, 0\rangle$

$\iint_{\theta t} \langle 2\cos\theta, 2\sin\theta, 0\rangle\cdot\langle t, 4\sin^2\theta, 4\cos^2\theta\rangle\, dA$

$= \int_0^{2\pi}\int_0^1 (2t\cos\theta + 8\sin^3\theta)\, dt\, d\theta$

$= \int_0^{2\pi} t^2\cos\theta + 8t\sin^3\theta\Big]_0^1 d\theta$

$= \int_0^{2\pi} (\cos\theta + 8\sin^3\theta)\, d\theta$

$\sin\theta + \frac{8}{3}\cos^3\theta - \frac{8}{1}\cos\theta\Big]_0^{2\pi} = 0$

$\int \sin^3\theta\, d\theta$
$u = \cos\theta$
$du = -\sin\theta\, d\theta$
$= \int -(1-u^2)\, du$
$= \int u^2 - 1\, du$
$= u^3/3 - u + C$
$= \frac{\cos^3\theta}{3} - \cos\theta + C$

45) $\vec{r}(t,\theta) = \langle t, \sqrt{6}\cos\theta, \sqrt{6}\sin\theta\rangle \quad 0\le t\le 4 \quad 0\le\theta\le 2\pi$

$\vec{r}_t = \langle 1, 0, 0\rangle$

$\vec{r}_\theta = \langle 0, -\sqrt{6}\sin\theta, \sqrt{6}\cos\theta\rangle$

$\vec{r}_t\times\vec{r}_\theta = \langle 0, -\sqrt{6}\cos\theta, -\sqrt{6}\sin\theta\rangle$

Heat Flux: $K\nabla u$

$\nabla u = \langle 0, 4y, 4z\rangle$

$-6.5\langle 0, 4y, 4z\rangle = \langle 0, -26y, -26z\rangle$

$= \langle 0, -26\sqrt{6}\cos\theta, -26\sqrt{6}\sin\theta\rangle$

$\iint_{\theta t} \langle 0, -26\sqrt{6}\cos\theta, -26\sqrt{6}\sin\theta\rangle\cdot\langle 0, -\sqrt{6}\cos\theta, -\sqrt{6}\sin\theta\rangle\, dA$

$= \iint_{\theta t} (156\cos^2\theta + 156\sin^2\theta)\, dA$

$= \iint_{\theta t} 156\, dA = \int_0^{2\pi}\int_0^4 156\, dt\, d\theta$

$= \boxed{1248\pi}$

43) $\vec{r}(\phi\theta) = \langle a\cos\theta\sin\phi, a\sin\theta\sin\phi, a\cos\phi\rangle$

S_1: $\vec{r}_\phi = \langle a\cos\theta\cos\phi, a\sin\theta\cos\phi, -a\sin\phi\rangle$

$\vec{r}_\theta = \langle -a\sin\theta\sin\phi, a\cos\theta\sin\phi, 0\rangle$

$\vec{r}_\phi\times\vec{r}_\theta = \langle a^2\cos\theta\sin^2\phi, a^2\sin\theta\sin^2\phi, \underbrace{a^2\cos^2\theta\cos\phi\sin\phi + a^2\sin^2\theta\cos\phi\sin\phi}_{a^2\cos\phi\sin\phi}\rangle$

$\iint_{\theta\phi} \langle a^2\cos\theta\sin^2\phi, a^2\sin\theta\sin^2\phi, a^2\cos\phi\sin\phi\rangle\cdot\langle a\cos\theta\sin\phi, a\sin\theta\sin\phi, 2a\cos\phi\rangle\, dA$

$= \int_0^{2\pi}\int_0^{\pi/2} a^3\cos^2\theta\sin^3\phi + a^3\sin^2\theta\sin^3\phi + 2a^3\cos^2\phi\sin\phi\, d\phi\, d\theta$

$= \int_0^{2\pi}\int_0^{\pi/2} a^3\sin\phi(1-\cos^2\phi) + 2a^3\sin\phi\cos^2\phi\, d\phi\, d\theta$

$= \int_0^{2\pi}\int_0^{\pi/2} a^3\sin\phi + a^3\cos^2\phi\sin\phi\, d\phi\, d\theta$

$= \int_0^{2\pi} -a^3\cos\phi - \frac{1}{3}a^3\cos^3\phi\Big]_0^{\pi/2} d\theta$

$= \int_0^{2\pi} a^3 + \frac{1}{3}a^3\, d\theta$

$= \frac{8}{3}\pi a^3$

S_2: $z = 0 \quad x^2+y^2\le a^2$

$\nabla g = \langle 0, 0, 1\rangle$

$-\nabla g = \langle 0, 0, -1\rangle$

$\iint_{xy} -2z\, dA$

$= \iint_{xy} -2(0)\, dA = 0$

$S_1 + S_2 = \frac{8}{3}\pi a^3 + 0 = \boxed{\frac{8}{3}\pi a^3}$

Authors Note
43) S_1 was attempted using the ∇g method, but led to a near-unsolvable integral.

7) ~~$g = z + x + y - 1$~~

~~$\nabla g = \langle 1, 1, 1\rangle \quad |\nabla g| = \sqrt{3}$~~

~~$\iint_{xy} y(1-x-y)\sqrt{3}\, dA = \boxed{\sqrt{3}\int_0^1\int_0^{1-x} (y - xy - y^2)\, dy\, dx}$~~

13) ~~$g = y - x^2 - z^2$~~

~~$\nabla g = \langle -2x, 1, -2z\rangle \quad |\nabla g| = \sqrt{4x^2+4z^2+1}$~~

~~$\iint_{xz} (x^2+z^2)\sqrt{4x^2+4z^2+1}\, dA = \boxed{\int_0^{2\pi}\int_0^4 r^3\sqrt{4r^2+1}\, dr\, d\theta}$~~

9) ~~$\vec{r}(u,v) = \langle u^2, u\sin v, u\cos v\rangle \quad 0\le u\le 1 \quad 0\le v\le \pi/2$~~

~~$\vec{r}_u = \langle 2u, \sin v, \cos v\rangle$~~

~~$\vec{r}_v = \langle 0, u\cos v, -u\sin v\rangle$~~

~~$\vec{r}_u\times\vec{r}_v = \langle -u\sin^2 v - u\cos^2 v, 2u^2\sin v, 2u^2\cos v\rangle$~~

~~$|\vec{r}_u\times\vec{r}_v| = \sqrt{u^2 + 4u^4(\sin^2 v+\cos^2 v)} = u\sqrt{1+4u^2}$~~

~~$\iint_{uv} u^3\sin v\cos v\sqrt{1+4u^2}\, dA$~~

~~$= \int_0^1\int_0^{\pi/2} u^3\sqrt{1+4u^2}\sin v\cos v\, dv\, du$~~

~~$= \int_0^1 u^3\sqrt{1+4u^2}\cdot -\frac{1}{2}\cos v\Big]_0^{\pi/2} du$~~

~~$= \int_0^1 \frac{1}{2}u^3\sqrt{1+4u^2}\, du$~~

~~$= \int_1^5 \frac{1}{2}\cdot\frac{1}{8}\left(\frac{w-1}{4}\right)\sqrt{w}\, dw$~~

~~$= \frac{1}{2}\cdot\frac{1}{8}\left(\frac{2}{5}\cdot\frac{w^{5/2}}{4} - \frac{2}{3}\cdot\frac{w^{3/2}}{4}\right)\Big]_1^5 = \boxed{\frac{1}{16}(3.752)}$~~

~~$\int \sin v\cos v\, dv = -\frac{1}{2}\cos v + C$~~

~~$\int u^3\sqrt{1+4u^2}\, du$~~
~~T: $w = 1+4u^2 \rightarrow u^2 = \frac{w-1}{4}$~~
~~$dw = 8u\, du$~~
~~$du = dw/8u$~~
~~$\frac{1}{8}\left(\frac{w-1}{4}\right)\sqrt{w} + C$~~

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19) ☆ $g = z - 4 + x^2 + y^2$ $\quad 0 \le x \le 1$, $0 \le y \le 1$

$\nabla g = \langle 2x, 2y, 1 \rangle$

$$\iint_{xy} \langle 2x, 2y, 1\rangle \cdot \langle xy, yz, zx\rangle \, dA$$

$$= \int_0^1\int_0^1 \left(2x^2y + 2y^2(4-x^2-y^2) + x(4-x^2-y^2)\right) dy\,dx$$

$$= \int_0^1\int_0^1 \left(2x^2y + 8y^2 - 2x^2y^2 - 2y^4 + 4x - x^3 - xy^2\right) dy\,dx$$

$$= \int_0^1 x^2y^2 + \frac{8}{3}y^3 - \frac{2}{3}y^3x^2 - \frac{2}{5}y^5 + 4xy - x^3y - \frac{1}{3}xy^3 \Big]_0^1 dx$$

$$= \int_0^1 \left(x^2 + \frac{8}{3} - \frac{2}{3}x^2 - \frac{2}{5} + 4x - x^3 - \frac{1}{3}x\right) dx$$

$$= \frac{1}{3}x^3 + \frac{8}{3}x - \frac{2}{9}x^3 - \frac{2}{5}x + 2x^2 - \frac{1}{4}x^4 - \frac{1}{6}x^2 \Big]_0^1$$

$$= \frac{1}{3} + \frac{8}{3} - \frac{2}{9} - \frac{2}{5} + 2 - \frac{1}{4} - \frac{1}{6} = \boxed{\frac{713}{180}}$$

23) ☆ $z = \sqrt{4 - x^2 - y^2}$

$g = z - \sqrt{4-x^2-y^2}$

$$\nabla g = \left\langle \frac{x}{\sqrt{4-x^2-y^2}}, \frac{y}{\sqrt{4-x^2-y^2}}, 1 \right\rangle$$

$$-\iint_{xy}\left(\frac{x^2}{\sqrt{4-x^2-y^2}} - \frac{zy}{\sqrt{4-x^2-y^2}} + y\right) dA$$

$$= -\iint_{xy}\left(\frac{x^2}{\sqrt{4-x^2-y^2}} - \frac{\sqrt{4-x^2-y^2}\,y}{\sqrt{4-x^2-y^2}} + y\right) dA$$

$$= -\iint_{xy}\left(\frac{x^2}{\sqrt{4-x^2-y^2}} - y + y\right) dA$$

$$= -\iint_{xy} \frac{x^2}{\sqrt{4-x^2-y^2}}\, dA$$

Polar: $x = r\cos\theta$, $y = r\sin\theta$, $|J| = r$

u-sub: $4 - r^2 = u$, $-2r\,dr = du$, $dr = du/{-2r}$, $r^2 = 4 - u$

$$= -\int_0^{\pi/2}\int_0^2 \frac{r^2\cos^2\theta}{\sqrt{4-r^2}}\, r\,dr\,d\theta$$

$$= -\int_0^{\pi/2}\int_0^2 \frac{(4-u)\cos^2\theta}{\sqrt{u}} \cdot \left(-\frac{1}{2}\right) du\,d\theta$$

$$= -\int_0^{\pi/2}\int_0^2 \frac{4\cos^2\theta}{2\sqrt{u}} + \frac{\sqrt{u}\cos^2\theta}{2}\Big) du\,d\theta$$

$$= -\int_0^{\pi/2} -4\sqrt{u}\cos^2\theta + \frac{1}{3}u^{3/2}\cos^2\theta \Big]_{r=0}^{r=2} d\theta$$

$$= -\int_0^{\pi/2} -4\sqrt{4-r^2}\cos^2\theta + \frac{1}{3}(4-r^2)^{3/2}\cos^2\theta \Big]_0^2 d\theta$$

$$= -\int_0^{\pi/2}\left(8\cos^2\theta - \frac{8}{3}\cos^2\theta\right) d\theta$$

$$-\int_0^{\pi/2} \frac{16}{3}\cos^2\theta\, d\theta = \boxed{-\frac{4}{3}\pi}$$

$$\int_0^{\pi/2} \cos^2\theta\, d\theta = \pi/4$$

21) ☆ $g = z + x + y - 1$

$\nabla g = \langle 1, 1, 1\rangle$

$-\nabla g = \langle -1, -1, -1\rangle$

$$\iint_{xy} \langle -1,-1,-1\rangle \cdot \langle xze^y, xze^y, z\rangle\, dA$$

$$= \int_0^1\int_0^{1-x} \left(-xze^y + xze^y - z\right) dy\,dx$$

$$= \int_0^1\int_0^{1-x} (-1 + x + y)\, dy\,dx$$

$$= \int_0^1 -y + xy + \frac{1}{2}y^2 \Big]_0^{1-x} dx$$

$$= \int_0^1 \left(-(1-x) + x(1-x) + \frac{1}{2}(1-x)^2\right) dx$$

$$= \int_0^1 \left(-\frac{1}{2} + x - \frac{1}{2}x^2\right) dx$$

$$= -\frac{1}{2}x + \frac{1}{2}x^2 - \frac{1}{6}x^3 \Big]_0^1$$

$$= -\frac{1}{2} + \frac{1}{2} - \frac{1}{6} = \boxed{-\frac{1}{6}}$$

25) ☆ S_1 $\quad g = y - x^2 - z^2$

$\nabla g = \langle -2x, 1, -2z\rangle$

$-\nabla g = \langle 2x, -1, 2z\rangle$

$$\iint_{xz} \left(-(x^2+z^2) - 2z^2\right) dA$$

Polar: $x = r\cos\theta$, $z = r\sin\theta$, $|J| = r$

$$= \int_0^{2\pi}\int_0^1 \left(-r^3 - 2r^3\cos^2\theta\right) dr\,d\theta$$

$$= \int_0^{2\pi} -\frac{1}{4}r^4 - \frac{1}{2}r^4\cos^2\theta \Big]_0^1 d\theta$$

$$= \int_0^{2\pi} -\frac{1}{4} - \frac{1}{2}\cos^2\theta\, d\theta$$

$$= -\pi/2 - \pi/2 = -\pi$$

S_2 $\quad g = y - 1$ $\quad x^2 + z^2 \le 1$

$\nabla g = \langle 0, 1, 0\rangle$

$$\iint_{xz} y\, dA = \iint_{xz} (1)\, dA = \pi$$

$S_1 + S_2 = -\pi + \pi = \boxed{0}$

Q204 - Lesson 2 HW Solutions