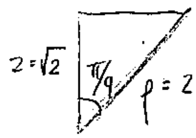
$\boxed{16.6 \ \#23, 25, 26}$ to completion

23. Parametric representation for sphere $(x^2+y^2+z^2=4)$ ⋯ ①
*
that lies above cone $\left(z=\sqrt{x^2+y^2}\right)$ ⋯ ②



$z^2 = \underbrace{x^2+y^2}_{\text{substitute into ①}} \longrightarrow (z^2)+z^2=4$

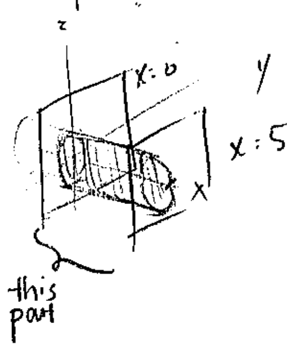
$2z^2 = 4$

$z = \sqrt{2}$

can be determined intuitively

$\boxed{\vec{r}(\theta,\phi) = \langle 2\cos\theta\sin\phi,\ 2\sin\theta\sin\phi,\ 2\cos\phi\rangle}$ with $\boxed{0 \le \theta \le 2\pi \text{ and } 0 \le \phi \le \pi/4}$

25. Cylinder $y^2+z^2=16$ between $x=0$ and $x=5$
*



$\boxed{\vec{r}(x,t) = \langle x, 4\cos t, 4\sin t\rangle}$ with $\boxed{0 \le x \le 5 \text{ and } 0 \le t \le 2\pi}$

26. Part of plane $z = x+3$ that lies inside $x^2+y^2=1$
*

$\boxed{\vec{r}(x,y) = \langle x, y, x+3\rangle}$ with $\boxed{x^2+y^2 \le 1}$

$\boxed{16.6\ \#35}$ to completion

35. (extra) $\vec{r}(u,v) = \langle u^2, 2u\sin v, u\cos v \rangle$

Find tangent plane at $(u,v) = (1,0)$

$$\begin{array}{l} \vec{r}_u = \\ \vec{r}_v = \end{array} \left| \begin{array}{ccc} i & j & k \\ \langle 2u, & 2\sin v, & \cos v \rangle \\ \langle 0, & 2u\cos v, & -u\sin v \rangle \end{array} \right|$$

$$\vec{r}_u \times \vec{r}_v = \langle -2u \underbrace{\sin^2 v - 2u\cos^2 v}_{\text{Pythagorean}},\ 2u^2 \sin v,\ 4u^2 \cos v \rangle$$

$$= \langle -2u,\ 2u^2 \sin v,\ 4u^2 \cos v \rangle$$

$$\vec{n}_{(1,0)} = \langle -2(1),\ \cancel{2(1)^2 \sin(0)}^{\to 0},\ 4(1)^2 \cancel{\cos(0)}^{1} \rangle = \langle -2, 0, 4 \rangle$$

$$\vec{r}(1,0) = \langle 1, 0, 1 \rangle$$

Equation of a plane: $-2(x-1) + 0(y-0) + 4(z-1) = 0$

$$-2x + 2 + 4z - 4 = 0$$

$$\boxed{-x + 2z = 1}$$

$\boxed{16.6\ \#37, 41, 43}$ Setup only

or grind out dy dx from here

area in the first quadrant

37. * $A = \iint |\nabla g|\, dA = \overbrace{\iint_D \sqrt{14}\, dA} = \sqrt{14} \iint_D dA$

$3x + 2y + z = 6$ (in first octant)

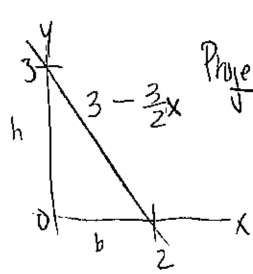
$3x + 2y + z - 6 = g$

$\nabla g = \langle 3, 2, 1 \rangle$

$|\nabla g| = \sqrt{9 + 4 + 1} = \sqrt{14}$

$\longrightarrow$ so $\boxed{A = \sqrt{14} \cdot A(D)} = \sqrt{14} \cdot \left(\tfrac{1}{2} bh\right) = \boxed{3\sqrt{14}}$

$\longrightarrow$ or $\boxed{A = \sqrt{14} \int_0^2 \int_0^{(3 - \frac{3}{2}x)} dy\, dx}$



Projection onto xy

a function: $f(x,y) = xy$

41. Part of surface $z = xy$ lying within $x^2+y^2=1$

*

$$A = \iint |\nabla g|\, dA = \iint_D \sqrt{1+x^2+y^2}\, dx\, dy = \int_0^{2\pi}\int_0^1 \sqrt{1+r^2}\, r\, dr\, d\theta$$

Find $|\nabla g|$

$$\begin{cases} z - xy = 0 \Rightarrow g = z - xy \\ \nabla g = \langle -y, -x, 1\rangle \\ |\nabla g| = \sqrt{1+x^2+y^2} \end{cases}$$

$$\therefore \boxed{A = \int_0^{2\pi}\int_0^1 \sqrt{1+r^2}\, r\, dr\, d\theta}$$

For kicks: $u = 1+r^2 \quad du = 2r\, dr$

NOTE: u-bounds:

$r=1 \to u = 1+1^2 = 2$

$r=0 \to u = 1+0^2 = 1$

$$A = \int_0^{2\pi} d\theta \int_0^1 r\sqrt{1+r^2}\, dr$$

$$= 2\pi \cdot \frac{1}{2}\int \sqrt{u}\, du = \pi \cdot \frac{2}{3}\left[u^{3/2}\right]_1^2$$

$$= \boxed{\frac{2}{3}\pi \cdot \left(2\sqrt{2}-1\right)}$$

43. Part of hyperbolic paraboloid $z = y^2 - x^2$ between cylinders $x^2+y^2=1$ and $x^2+y^2=4$

*

Let $g = z - y^2 + x^2$

$\nabla g = \langle 2x, -2y, 1\rangle$

$|\nabla g| = \sqrt{1+4x^2+4y^2}$

$$A = \iint |\nabla g|\, dA = \iint_D \sqrt{1+4(x^2+y^2)}\, dA$$

$$= \boxed{\int_0^{2\pi}\int_1^2 \sqrt{1+4r^2}\, r\, dr\, d\theta}$$

For kicks: $u = 1+4r^2 \quad du = 8r\, dr$

NOTE: u-bounds

$r=1 \to u = 1+4\cdot 1^2 = 5$

$r=2 \to u = 1+4\cdot 2^2 = 17$

$$A = \int_0^{2\pi} d\theta \int_1^2 \frac{1}{8}\sqrt{u}\, du = \cancel{2}\pi \cdot \frac{1}{\cancel{8}_{4}} \cdot \frac{\cancel{2}}{3} \cdot \left[u^{3/2}\right]_5^{17}$$

$$\boxed{A = \frac{\pi}{6}\left(17^{3/2} - 5^{3/2}\right)}$$

5. $\iint_S x^2yz\,dS$

☆

S is part of plane $z = 1 + 2x + 3y$ above $[0,3] \times [0,2]$

Let $g = z - (1 + 2x + 3y)$

$\nabla g = \langle -2, -3, 1 \rangle \rightarrow |\nabla g| = \sqrt{14}$

$= \iint_S x^2yz \cdot \sqrt{14}\,dA = \boxed{\int_0^3\int_0^2 x^2y\,(1+2x+3y)\sqrt{14}\,dy\,dx}$

To completion: $= \sqrt{14}\int_0^3\int_0^2 (x^2y + 2x^3y + 3x^2y^2)\,dy\,dx$

$= \sqrt{14}\int_0^3 \left[\frac{1}{2}x^2y^2 + x^3y^2 + x^2y^3\right]_0^2 dx$ $\quad \boxed{171\sqrt{14}}$

$= \sqrt{14}\int_0^3 (2x^2 + 4x^3 + 8x^2)\,dx = \sqrt{14}\int_0^3 (10x^2 + 4x^3)\,dx = \sqrt{14}\left[\frac{10}{3}x^3 + x^4\right]_0^3$

(page skipped unintentionally)

11. ☆ $\iint_S x^2 z^2 \, dS$ where S is part of cone between $z=1$ and $z=3$

$\vec{r}(x,y) = \langle x, y, \sqrt{x^2+y^2} \rangle \rightarrow \quad \vec{r}_x = \langle 1, 0, \frac{x}{\sqrt{x^2+y^2}} \rangle$

$\vec{r}_y = \langle 0, 1, \frac{y}{\sqrt{x^2+y^2}} \rangle$

$\vec{r}_x \times \vec{r}_y = \left\langle -\frac{x}{\sqrt{x^2+y^2}}, -\frac{y}{\sqrt{x^2+y^2}}, 1 \right\rangle$

$|\vec{r}_x \times \vec{r}_y| = \sqrt{\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} + 1} = \sqrt{\cancel{\frac{x^2+y^2}{x^2+y^2}}^{\,1} + 1} = \sqrt{2}$

$\boxed{z=r}$

$\rightarrow \iint_S x^2 z^2 \, dS = \iint_D x^2 z^2 \sqrt{2} \, dx\,dy = \iint r^2\cos^2\theta \cdot r^2 \cdot \sqrt{2} \cdot dr\,d\theta \cdot r = \boxed{\sqrt{2} \int_0^{2\pi}\int_1^3 r^5 \cos^2\theta \, dr\,d\theta}$

alt. solution (Dan's method)

OR Alternate parametrization:

let $x = r\cos\theta$ $\quad \vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, r \rangle$

$y = r\sin\theta$ $\quad \vec{r}_r = \langle \cos\theta, \sin\theta, 1 \rangle$

$z = r$ $\quad \vec{r}_\theta = \langle -r\sin\theta, r\cos\theta, 0 \rangle$

$\vec{r}_r \times \vec{r}_\theta = \langle -r\cos\theta, -r\sin\theta, r \rangle$

$|\vec{r}_r \times \vec{r}_\theta| = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta + r^2} = \sqrt{2}\, r$

then substitution proceeds <u>same</u>.

<u>To completion</u>: $\sqrt{2} \int_0^{2\pi} \cos^2\theta \, d\theta \int_1^3 r^5 \, dr$

$= \sqrt{2} \cdot \pi \cdot \frac{1}{6} \left[r^6 \right]_1^3 = \sqrt{2}\pi \cdot \frac{1}{6} \cdot (729-1) = \boxed{\frac{364}{3}\sqrt{2}\pi}$

15. $\iint_S (x^2 z + y^2 z)\, dS$ where S is top hemisphere $(z \geq 0)$ of $x^2 + y^2 + z^2 = 4$

$\ast$

$$\vec{r}(\theta, \phi) = \langle 2\cos\theta \sin\phi,\ 2\sin\theta \sin\phi,\ 2\cos\phi \rangle$$

with $0 \leq \theta \leq 2\pi$

$0 \leq \phi \leq \pi/2$

$$\vec{r}_\theta = \langle -2\sin\theta \sin\phi,\ 2\cos\theta \sin\phi,\ 0 \rangle$$

$$\vec{r}_\phi = \langle 2\cos\theta \cos\phi,\ 2\sin\phi \cos\phi,\ -2\sin\phi \rangle$$

$$|\vec{r}_\theta \times \vec{r}_\phi| = 4\sin\phi \quad [\text{see notes}]$$

$$\iint_S (x^2 z + y^2 z)\, dS = \int_0^{2\pi}\int_0^{\pi/2} \left(4\cos^2\theta \sin^2\phi \cdot 2\cos\phi + 4\sin^2\theta \sin^2\phi \cdot 2\cos\phi\right)(4\sin\phi)\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2} (4\sin^2\phi)(2\cos\phi)(4\sin\phi)\, d\phi\, d\theta = \boxed{64\pi \cdot \int_0^{\pi/2} \sin^3\phi \cos\phi\, d\phi}$$

To completion:

$$= 64\pi \int_0^{\pi/2} \sin\phi\,(1 - \cos^2\phi)\cos\phi\, d\phi$$

BC Q501 Powers of trig functions.

$\to [\ u = \cos\phi \quad du = -\sin\phi\, d\phi$

$\ast$ u-bounds:

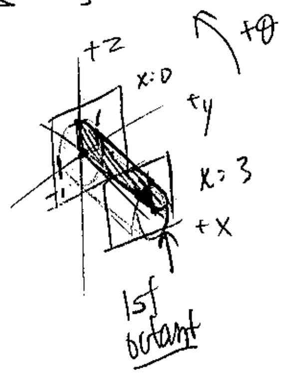
$\phi = 0 \to u = 1$

$\phi = \pi/2 \to u = 0$

$$= -64\pi \int_1^0 (1 - u^2)\, u\, du = 64\pi \int_0^1 (u - u^3)\, du$$

$$= 64\pi \left[\tfrac{1}{2}u^2 - \tfrac{1}{4}u^4\right]_0^1 = 64\pi \cdot \left(\tfrac{1}{2} - \tfrac{1}{4}\right) = \boxed{16\pi}$$

17. $\iint_S (z + x^2 y)\, dS$ where S is part of cylinder $y^2 + z^2 = 1$ between $x = 0$ and $x = 3$ / 1st octant



$\vec{r}(x,\theta) = \langle x, \cos\theta, \sin\theta \rangle \qquad 0 \le x \le 3$ and $0 \le \theta \le \pi/2$

$\vec{r}_x = \langle 1, 0, 0 \rangle$

$\vec{r}_\theta = \langle 0, -\sin\theta, \cos\theta \rangle$

$\vec{r}_x \times \vec{r}_\theta = \langle 0, -\cos\theta, -\sin\theta \rangle$

$|\vec{r}_x \times \vec{r}_\theta| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$

$$\iint_S (z + x^2 y)\, dS = \iint_S \left(\sin\theta + x^2\cos\theta\right)(1)\, dA = \boxed{\int_0^{\pi/2}\int_0^3 \left(\sin\theta + x^2\cos\theta\right) dx\, d\theta}$$

To completion:

$$= \int_0^{\pi/2} \left[x\sin\theta + \tfrac{1}{3}x^3\cos\theta\right]_0^3 d\theta$$

$$= \int_0^{\pi/2} \left(3\sin\theta + 9\cos\theta\right) d\theta = \left[-3\cos\theta + 9\sin\theta\right]_0^{\pi/2}$$

$$= \left(\cancel{-3\cos\tfrac{\pi}{2}} + 9\underbrace{\sin\tfrac{\pi}{2}}_{1} + 3\underbrace{\cos 0}_{1} - \cancel{9\sin 0}\right) = \boxed{12}$$