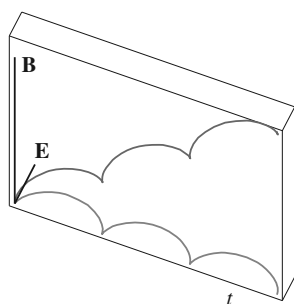
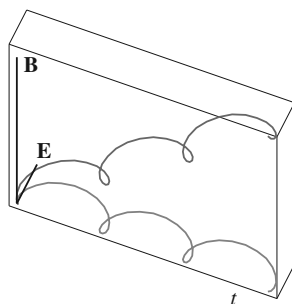


■ Some computer algebra systems provide us with a clearer picture of a space curve by enclosing it in a tube. Such a plot enables us to see whether one part of a curve passes in front of or behind another part of the curve. For example, Figure 13 shows the curve of Figure 12(b) as rendered by the `tubeplot` command in Maple.



$$(a) \mathbf{r}(t) = \langle t - \sin t, 1 - \cos t, t \rangle$$



$$(b) \mathbf{r}(t) = \left\langle t - \frac{3}{2} \sin t, 1 - \frac{3}{2} \cos t, t \right\rangle$$

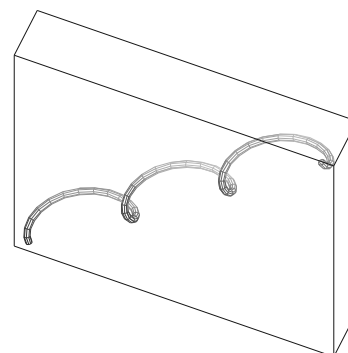


FIGURE 13

FIGURE 12

Motion of a charged particle in orthogonally oriented electric and magnetic fields

For further details concerning the physics involved and animations of the trajectories of the particles, see the following websites:

- [www.phy.ntnu.edu.tw/java/emField/emField.html](http://www.phy.ntnu.edu.tw/java/emField/emField.html)
- [www.physics.ucla.edu/plasma-exp/Beam/](http://www.physics.ucla.edu/plasma-exp/Beam/)

### 13.1 EXERCISES

1–2 Find the domain of the vector function.

$$1. \mathbf{r}(t) = \langle \sqrt{4 - t^2}, e^{-3t}, \ln(t + 1) \rangle$$

$$2. \mathbf{r}(t) = \frac{t - 2}{t + 2} \mathbf{i} + \sin t \mathbf{j} + \ln(9 - t^2) \mathbf{k}$$

3–6 Find the limit.

$$3. \lim_{t \rightarrow 0^+} \langle \cos t, \sin t, t \ln t \rangle$$

$$4. \lim_{t \rightarrow 0} \left\langle \frac{e^t - 1}{t}, \frac{\sqrt{1 + t} - 1}{t}, \frac{3}{1 + t} \right\rangle$$

$$5. \lim_{t \rightarrow 0} \left( e^{-3t} \mathbf{i} + \frac{t^2}{\sin^2 t} \mathbf{j} + \cos 2t \mathbf{k} \right)$$

$$6. \lim_{t \rightarrow \infty} \left\langle \arctan t, e^{-2t}, \frac{\ln t}{t} \right\rangle$$

7–14 Sketch the curve with the given vector equation. Indicate with an arrow the direction in which  $t$  increases.

$$7. \mathbf{r}(t) = \langle \sin t, t \rangle$$

$$8. \mathbf{r}(t) = \langle t^3, t^2 \rangle$$

$$9. \mathbf{r}(t) = \langle t, \cos 2t, \sin 2t \rangle$$

$$10. \mathbf{r}(t) = \langle 1 + t, 3t, -t \rangle$$

$$11. \mathbf{r}(t) = \langle 1, \cos t, 2 \sin t \rangle$$

$$12. \mathbf{r}(t) = t^2 \mathbf{i} + t \mathbf{j} + 2 \mathbf{k}$$

$$13. \mathbf{r}(t) = t^2 \mathbf{i} + t^4 \mathbf{j} + t^6 \mathbf{k}$$

$$14. \mathbf{r}(t) = \cos t \mathbf{i} - \cos t \mathbf{j} + \sin t \mathbf{k}$$

15–18 Find a vector equation and parametric equations for the line segment that joins  $P$  to  $Q$ .

$$15. P(0, 0, 0), \quad Q(1, 2, 3)$$

$$16. P(1, 0, 1), \quad Q(2, 3, 1)$$

$$17. P(1, -1, 2), \quad Q(4, 1, 7)$$

$$18. P(-2, 4, 0), \quad Q(6, -1, 2)$$

19–24 Match the parametric equations with the graphs (labeled I–VI). Give reasons for your choices.

$$19. x = \cos 4t, \quad y = t, \quad z = \sin 4t$$

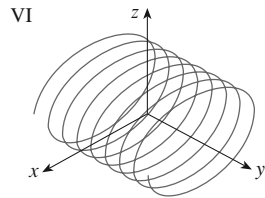
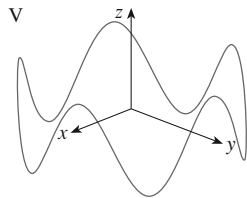
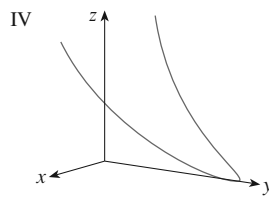
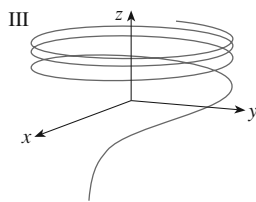
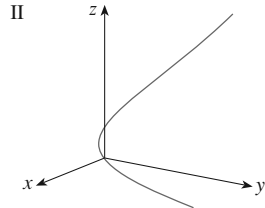
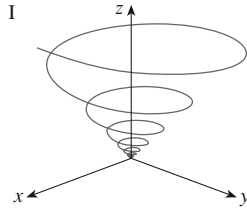
$$20. x = t, \quad y = t^2, \quad z = e^{-t}$$

21.  $x = t, y = 1/(1 + t^2), z = t^2$

22.  $x = e^{-t} \cos 10t, y = e^{-t} \sin 10t, z = e^{-t}$

23.  $x = \cos t, y = \sin t, z = \sin 5t$

24.  $x = \cos t, y = \sin t, z = \ln t$



25. Show that the curve with parametric equations  $x = t \cos t, y = t \sin t, z = t$  lies on the cone  $z^2 = x^2 + y^2$ , and use this fact to help sketch the curve.

26. Show that the curve with parametric equations  $x = \sin t, y = \cos t, z = \sin^2 t$  is the curve of intersection of the surfaces  $z = x^2$  and  $x^2 + y^2 = 1$ . Use this fact to help sketch the curve.

27. At what points does the curve  $\mathbf{r}(t) = t\mathbf{i} + (2t - t^2)\mathbf{k}$  intersect the paraboloid  $z = x^2 + y^2$ ?

28. At what points does the helix  $\mathbf{r}(t) = \langle \sin t, \cos t, t \rangle$  intersect the sphere  $x^2 + y^2 + z^2 = 5$ ?

29–32 Use a computer to graph the curve with the given vector equation. Make sure you choose a parameter domain and viewpoints that reveal the true nature of the curve.

29.  $\mathbf{r}(t) = \langle \cos t \sin 2t, \sin t \sin 2t, \cos 2t \rangle$

30.  $\mathbf{r}(t) = \langle t^2, \ln t, t \rangle$

31.  $\mathbf{r}(t) = \langle t, t \sin t, t \cos t \rangle$

32.  $\mathbf{r}(t) = \langle t, e^t, \cos t \rangle$

33. Graph the curve with parametric equations  $x = (1 + \cos 16t) \cos t, y = (1 + \cos 16t) \sin t, z = 1 + \cos 16t$ . Explain the appearance of the graph by showing that it lies on a cone.

34. Graph the curve with parametric equations

$$x = \sqrt{1 - 0.25 \cos^2 10t} \cos t$$

$$y = \sqrt{1 - 0.25 \cos^2 10t} \sin t$$

$$z = 0.5 \cos 10t$$

Explain the appearance of the graph by showing that it lies on a sphere.

35. Show that the curve with parametric equations  $x = t^2, y = 1 - 3t, z = 1 + t^3$  passes through the points  $(1, 4, 0)$  and  $(9, -8, 28)$  but not through the point  $(4, 7, -6)$ .

36–38 Find a vector function that represents the curve of intersection of the two surfaces.

36. The cylinder  $x^2 + y^2 = 4$  and the surface  $z = xy$

37. The cone  $z = \sqrt{x^2 + y^2}$  and the plane  $z = 1 + y$

38. The paraboloid  $z = 4x^2 + y^2$  and the parabolic cylinder  $y = x^2$

39. Try to sketch by hand the curve of intersection of the circular cylinder  $x^2 + y^2 = 4$  and the parabolic cylinder  $z = x^2$ . Then find parametric equations for this curve and use these equations and a computer to graph the curve.

40. Try to sketch by hand the curve of intersection of the parabolic cylinder  $y = x^2$  and the top half of the ellipsoid  $x^2 + 4y^2 + 4z^2 = 16$ . Then find parametric equations for this curve and use these equations and a computer to graph the curve.

41. If two objects travel through space along two different curves, it's often important to know whether they will collide. (Will a missile hit its moving target? Will two aircraft collide?) The curves might intersect, but we need to know whether the objects are in the same position *at the same time*. Suppose the trajectories of two particles are given by the vector functions

$$\mathbf{r}_1(t) = \langle t^2, 7t - 12, t^2 \rangle \quad \mathbf{r}_2(t) = \langle 4t - 3, t^2, 5t - 6 \rangle$$

for  $t \geq 0$ . Do the particles collide?

42. Two particles travel along the space curves

$$\mathbf{r}_1(t) = \langle t, t^2, t^3 \rangle \quad \mathbf{r}_2(t) = \langle 1 + 2t, 1 + 6t, 1 + 14t \rangle$$

Do the particles collide? Do their paths intersect?

43. Suppose  $\mathbf{u}$  and  $\mathbf{v}$  are vector functions that possess limits as  $t \rightarrow a$  and let  $c$  be a constant. Prove the following properties of limits.

$$(a) \lim_{t \rightarrow a} [\mathbf{u}(t) + \mathbf{v}(t)] = \lim_{t \rightarrow a} \mathbf{u}(t) + \lim_{t \rightarrow a} \mathbf{v}(t)$$

- (b)  $\lim_{t \rightarrow a} c\mathbf{u}(t) = c \lim_{t \rightarrow a} \mathbf{u}(t)$   
 (c)  $\lim_{t \rightarrow a} [\mathbf{u}(t) \cdot \mathbf{v}(t)] = \lim_{t \rightarrow a} \mathbf{u}(t) \cdot \lim_{t \rightarrow a} \mathbf{v}(t)$   
 (d)  $\lim_{t \rightarrow a} [\mathbf{u}(t) \times \mathbf{v}(t)] = \lim_{t \rightarrow a} \mathbf{u}(t) \times \lim_{t \rightarrow a} \mathbf{v}(t)$

44. The view of the trefoil knot shown in Figure 8 is accurate, but it doesn't reveal the whole story. Use the parametric equations


$$x = (2 + \cos 1.5t) \cos t$$

$$y = (2 + \cos 1.5t) \sin t$$

$$z = \sin 1.5t$$

to sketch the curve by hand as viewed from above, with gaps indicating where the curve passes over itself. Start by showing

that the projection of the curve onto the  $xy$ -plane has polar coordinates  $r = 2 + \cos 1.5t$  and  $\theta = t$ , so  $r$  varies between 1 and 3. Then show that  $z$  has maximum and minimum values when the projection is halfway between  $r = 1$  and  $r = 3$ .

 When you have finished your sketch, use a computer to draw the curve with viewpoint directly above and compare with your sketch. Then use the computer to draw the curve from several other viewpoints. You can get a better impression of the curve if you plot a tube with radius 0.2 around the curve. (Use the `tubeplot` command in Maple.)

45. Show that  $\lim_{t \rightarrow a} \mathbf{r}(t) = \mathbf{b}$  if and only if for every  $\varepsilon > 0$  there is a number  $\delta > 0$  such that

$$\text{if } 0 < |t - a| < \delta \text{ then } |\mathbf{r}(t) - \mathbf{b}| < \varepsilon$$

## 13.2 DERIVATIVES AND INTEGRALS OF VECTOR FUNCTIONS

Later in this chapter we are going to use vector functions to describe the motion of planets and other objects through space. Here we prepare the way by developing the calculus of vector functions.

### DERIVATIVES

The **derivative**  $\mathbf{r}'$  of a vector function  $\mathbf{r}$  is defined in much the same way as for real-valued functions:

**1**

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h \rightarrow 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

if this limit exists. The geometric significance of this definition is shown in Figure 1. If the points  $P$  and  $Q$  have position vectors  $\mathbf{r}(t)$  and  $\mathbf{r}(t+h)$ , then  $\overrightarrow{PQ}$  represents the vector  $\mathbf{r}(t+h) - \mathbf{r}(t)$ , which can therefore be regarded as a secant vector. If  $h > 0$ , the scalar multiple  $(1/h)(\mathbf{r}(t+h) - \mathbf{r}(t))$  has the same direction as  $\mathbf{r}(t+h) - \mathbf{r}(t)$ . As  $h \rightarrow 0$ , it appears that this vector approaches a vector that lies on the tangent line. For this reason, the vector  $\mathbf{r}'(t)$  is called the **tangent vector** to the curve defined by  $\mathbf{r}$  at the point  $P$ , provided that  $\mathbf{r}'(t)$  exists and  $\mathbf{r}'(t) \neq \mathbf{0}$ . The **tangent line** to  $C$  at  $P$  is defined to be the line through  $P$  parallel to the tangent vector  $\mathbf{r}'(t)$ . We will also have occasion to consider the **unit tangent vector**, which is

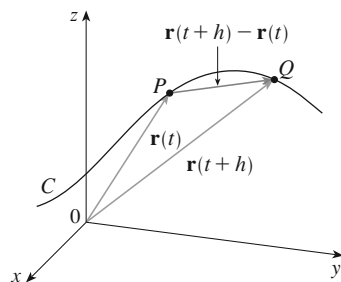
$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

The following theorem gives us a convenient method for computing the derivative of a vector function  $\mathbf{r}$ : just differentiate each component of  $\mathbf{r}$ .

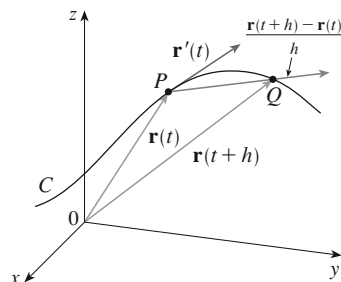
**2 THEOREM** If  $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ , where  $f$ ,  $g$ , and  $h$  are differentiable functions, then

$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$$

**TEC** Visual 13.2 shows an animation of Figure 1.



(a) The secant vector



(b) The tangent vector

**FIGURE 1**

This means that we can evaluate an integral of a vector function by integrating each component function.

We can extend the Fundamental Theorem of Calculus to continuous vector functions as follows:

$$\int_a^b \mathbf{r}(t) dt = \mathbf{R}(t) \Big|_a^b = \mathbf{R}(b) - \mathbf{R}(a)$$

where  $\mathbf{R}$  is an antiderivative of  $\mathbf{r}$ , that is,  $\mathbf{R}'(t) = \mathbf{r}(t)$ . We use the notation  $\int \mathbf{r}(t) dt$  for indefinite integrals (antiderivatives).

**EXAMPLE 5** If  $\mathbf{r}(t) = 2 \cos t \mathbf{i} + \sin t \mathbf{j} + 2t \mathbf{k}$ , then

$$\begin{aligned} \int \mathbf{r}(t) dt &= \left( \int 2 \cos t dt \right) \mathbf{i} + \left( \int \sin t dt \right) \mathbf{j} + \left( \int 2t dt \right) \mathbf{k} \\ &= 2 \sin t \mathbf{i} - \cos t \mathbf{j} + t^2 \mathbf{k} + \mathbf{C} \end{aligned}$$

where  $\mathbf{C}$  is a vector constant of integration, and

$$\int_0^{\pi/2} \mathbf{r}(t) dt = [2 \sin t \mathbf{i} - \cos t \mathbf{j} + t^2 \mathbf{k}]_0^{\pi/2} = 2 \mathbf{i} + \mathbf{j} + \frac{\pi^2}{4} \mathbf{k} \quad \blacksquare$$

## 13.2 EXERCISES

**1.** The figure shows a curve  $C$  given by a vector function  $\mathbf{r}(t)$ .

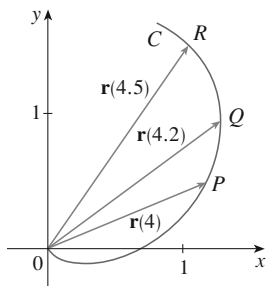
(a) Draw the vectors  $\mathbf{r}(4.5) - \mathbf{r}(4)$  and  $\mathbf{r}(4.2) - \mathbf{r}(4)$ .

(b) Draw the vectors

$$\frac{\mathbf{r}(4.5) - \mathbf{r}(4)}{0.5} \quad \text{and} \quad \frac{\mathbf{r}(4.2) - \mathbf{r}(4)}{0.2}$$

(c) Write expressions for  $\mathbf{r}'(4)$  and the unit tangent vector  $\mathbf{T}(4)$ .

(d) Draw the vector  $\mathbf{T}(4)$ .



- 2.** (a) Make a large sketch of the curve described by the vector function  $\mathbf{r}(t) = \langle t^2, t \rangle$ ,  $0 \leq t \leq 2$ , and draw the vectors  $\mathbf{r}(1)$ ,  $\mathbf{r}(1.1)$ , and  $\mathbf{r}(1.1) - \mathbf{r}(1)$ .

(b) Draw the vector  $\mathbf{r}'(1)$  starting at  $(1, 1)$  and compare it with the vector

$$\frac{\mathbf{r}(1.1) - \mathbf{r}(1)}{0.1}$$

Explain why these vectors are so close to each other in length and direction.

**3–8**

- (a) Sketch the plane curve with the given vector equation.  
 (b) Find  $\mathbf{r}'(t)$ .  
 (c) Sketch the position vector  $\mathbf{r}(t)$  and the tangent vector  $\mathbf{r}'(t)$  for the given value of  $t$ .

**3.**  $\mathbf{r}(t) = \langle t - 2, t^2 + 1 \rangle$ ,  $t = -1$

**4.**  $\mathbf{r}(t) = \langle 1 + t, \sqrt{t} \rangle$ ,  $t = 1$

**5.**  $\mathbf{r}(t) = \sin t \mathbf{i} + 2 \cos t \mathbf{j}$ ,  $t = \pi/4$

**6.**  $\mathbf{r}(t) = e^t \mathbf{i} + e^{-t} \mathbf{j}$ ,  $t = 0$

**7.**  $\mathbf{r}(t) = e^t \mathbf{i} + e^{3t} \mathbf{j}$ ,  $t = 0$

**8.**  $\mathbf{r}(t) = (1 + \cos t) \mathbf{i} + (2 + \sin t) \mathbf{j}$ ,  $t = \pi/6$

**9–16** Find the derivative of the vector function.

**9.**  $\mathbf{r}(t) = \langle t \sin t, t^2, t \cos 2t \rangle$

10.  $\mathbf{r}(t) = \langle \tan t, \sec t, 1/t^2 \rangle$

11.  $\mathbf{r}(t) = \mathbf{i} - \mathbf{j} + e^{4t} \mathbf{k}$

12.  $\mathbf{r}(t) = \sin^{-1}t \mathbf{i} + \sqrt{1-t^2} \mathbf{j} + \mathbf{k}$

13.  $\mathbf{r}(t) = e^{t^2} \mathbf{i} - \mathbf{j} + \ln(1+3t) \mathbf{k}$

14.  $\mathbf{r}(t) = at \cos 3t \mathbf{i} + b \sin^3 t \mathbf{j} + c \cos^3 t \mathbf{k}$

15.  $\mathbf{r}(t) = \mathbf{a} + t \mathbf{b} + t^2 \mathbf{c}$

16.  $\mathbf{r}(t) = t \mathbf{a} \times (\mathbf{b} + t \mathbf{c})$

17–20 Find the unit tangent vector  $\mathbf{T}(t)$  at the point with the given value of the parameter  $t$ .

17.  $\mathbf{r}(t) = \langle te^{-t}, 2 \arctan t, 2e^t \rangle, \quad t = 0$

18.  $\mathbf{r}(t) = 4\sqrt{t} \mathbf{i} + t^2 \mathbf{j} + t \mathbf{k}, \quad t = 1$

19.  $\mathbf{r}(t) = \cos t \mathbf{i} + 3t \mathbf{j} + 2 \sin 2t \mathbf{k}, \quad t = 0$

20.  $\mathbf{r}(t) = 2 \sin t \mathbf{i} + 2 \cos t \mathbf{j} + \tan t \mathbf{k}, \quad t = \pi/4$

21. If  $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$ , find  $\mathbf{r}'(t)$ ,  $\mathbf{T}(1)$ ,  $\mathbf{r}''(t)$ , and  $\mathbf{r}'(t) \times \mathbf{r}''(t)$ .

22. If  $\mathbf{r}(t) = \langle e^{2t}, e^{-2t}, te^{2t} \rangle$ , find  $\mathbf{T}(0)$ ,  $\mathbf{r}''(0)$ , and  $\mathbf{r}'(t) \cdot \mathbf{r}''(t)$ .


23–26 Find parametric equations for the tangent line to the curve with the given parametric equations at the specified point.

23.  $x = 1 + 2\sqrt{t}, \quad y = t^3 - t, \quad z = t^3 + t; \quad (3, 0, 2)$

24.  $x = e^t, \quad y = te^t, \quad z = te^{t^2}; \quad (1, 0, 0)$

25.  $x = e^{-t} \cos t, \quad y = e^{-t} \sin t, \quad z = e^{-t}; \quad (1, 0, 1)$

26.  $x = \ln t, \quad y = 2\sqrt{t}, \quad z = t^2; \quad (0, 2, 1)$


 27–29 Find parametric equations for the tangent line to the curve with the given parametric equations at the specified point. Illustrate by graphing both the curve and the tangent line on a common screen.

27.  $x = t, \quad y = e^{-t}, \quad z = 2t - t^2; \quad (0, 1, 0)$

28.  $x = 2 \cos t, \quad y = 2 \sin t, \quad z = 4 \cos 2t; \quad (\sqrt{3}, 1, 2)$

29.  $x = t \cos t, \quad y = t, \quad z = t \sin t; \quad (-\pi, \pi, 0)$

30. (a) Find the point of intersection of the tangent lines to the curve  $\mathbf{r}(t) = \langle \sin \pi t, 2 \sin \pi t, \cos \pi t \rangle$  at the points where  $t = 0$  and  $t = 0.5$ .

 (b) Illustrate by graphing the curve and both tangent lines.

31. The curves  $\mathbf{r}_1(t) = \langle t, t^2, t^3 \rangle$  and  $\mathbf{r}_2(t) = \langle \sin t, \sin 2t, t \rangle$  intersect at the origin. Find their angle of intersection correct to the nearest degree.

32. At what point do the curves  $\mathbf{r}_1(t) = \langle t, 1 - t, 3 + t^2 \rangle$  and  $\mathbf{r}_2(s) = \langle 3 - s, s - 2, s^2 \rangle$  intersect? Find their angle of intersection correct to the nearest degree.

33–38 Evaluate the integral.

33.  $\int_0^1 (16t^3 \mathbf{i} - 9t^2 \mathbf{j} + 25t^4 \mathbf{k}) dt$

34.  $\int_0^1 \left( \frac{4}{1+t^2} \mathbf{j} + \frac{2t}{1+t^2} \mathbf{k} \right) dt$

35.  $\int_0^{\pi/2} (3 \sin^2 t \cos t \mathbf{i} + 3 \sin t \cos^2 t \mathbf{j} + 2 \sin t \cos t \mathbf{k}) dt$

36.  $\int_1^2 (t^2 \mathbf{i} + t\sqrt{t-1} \mathbf{j} + t \sin \pi t \mathbf{k}) dt$

37.  $\int (e^t \mathbf{i} + 2t \mathbf{j} + \ln t \mathbf{k}) dt$

38.  $\int (\cos \pi t \mathbf{i} + \sin \pi t \mathbf{j} + t \mathbf{k}) dt$

39. Find  $\mathbf{r}(t)$  if  $\mathbf{r}'(t) = 2t \mathbf{i} + 3t^2 \mathbf{j} + \sqrt{t} \mathbf{k}$  and  $\mathbf{r}(1) = \mathbf{i} + \mathbf{j}$ .

40. Find  $\mathbf{r}(t)$  if  $\mathbf{r}'(t) = t \mathbf{i} + e^t \mathbf{j} + te^t \mathbf{k}$  and  $\mathbf{r}(0) = \mathbf{i} + \mathbf{j} + \mathbf{k}$ .

41. Prove Formula 1 of Theorem 3.

42. Prove Formula 3 of Theorem 3.

43. Prove Formula 5 of Theorem 3.

44. Prove Formula 6 of Theorem 3.

45. If  $\mathbf{u}(t) = \langle \sin t, \cos t, t \rangle$  and  $\mathbf{v}(t) = \langle t, \cos t, \sin t \rangle$ , use Formula 4 of Theorem 3 to find

$$\frac{d}{dt} [\mathbf{u}(t) \cdot \mathbf{v}(t)]$$

46. If  $\mathbf{u}$  and  $\mathbf{v}$  are the vector functions in Exercise 45, use Formula 5 of Theorem 3 to find

$$\frac{d}{dt} [\mathbf{u}(t) \times \mathbf{v}(t)]$$

47. Show that if  $\mathbf{r}$  is a vector function such that  $\mathbf{r}''$  exists, then

$$\frac{d}{dt} [\mathbf{r}(t) \times \mathbf{r}'(t)] = \mathbf{r}(t) \times \mathbf{r}''(t)$$

48. Find an expression for  $\frac{d}{dt} [\mathbf{u}(t) \cdot (\mathbf{v}(t) \times \mathbf{w}(t))]$ .

49. If  $\mathbf{r}(t) \neq \mathbf{0}$ , show that  $\frac{d}{dt} |\mathbf{r}(t)| = \frac{1}{|\mathbf{r}(t)|} \mathbf{r}(t) \cdot \mathbf{r}'(t)$ .

$$[\text{Hint: } |\mathbf{r}(t)|^2 = \mathbf{r}(t) \cdot \mathbf{r}(t)]$$

50. If a curve has the property that the position vector  $\mathbf{r}(t)$  is always perpendicular to the tangent vector  $\mathbf{r}'(t)$ , show that the curve lies on a sphere with center the origin.

51. If  $\mathbf{u}(t) = \mathbf{r}(t) \cdot [\mathbf{r}'(t) \times \mathbf{r}''(t)]$ , show that

$$\mathbf{u}'(t) = \mathbf{r}(t) \cdot [\mathbf{r}'(t) \times \mathbf{r}'''(t)]$$

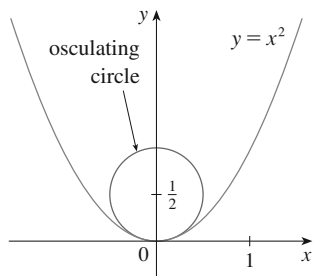


FIGURE 9

**TEC** Visual 13.3C shows how the osculating circle changes as a point moves along a curve.

**EXAMPLE 8** Find and graph the osculating circle of the parabola  $y = x^2$  at the origin.

**SOLUTION** From Example 5 the curvature of the parabola at the origin is  $\kappa(0) = 2$ . So the radius of the osculating circle at the origin is  $1/\kappa = \frac{1}{2}$  and its center is  $(0, \frac{1}{2})$ . Its equation is therefore

$$x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}$$

For the graph in Figure 9 we use parametric equations of this circle:

$$x = \frac{1}{2} \cos t \quad y = \frac{1}{2} + \frac{1}{2} \sin t$$

We summarize here the formulas for unit tangent, unit normal and binormal vectors, and curvature.

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \quad \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} \quad \mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$$

$$\kappa = \left| \frac{d\mathbf{T}}{ds} \right| = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

### 13.3 EXERCISES

1–6 Find the length of the curve.

1.  $\mathbf{r}(t) = \langle 2 \sin t, 5t, 2 \cos t \rangle, \quad -10 \leq t \leq 10$

2.  $\mathbf{r}(t) = \langle 2t, t^2, \frac{1}{3}t^3 \rangle, \quad 0 \leq t \leq 1$

3.  $\mathbf{r}(t) = \sqrt{2}t \mathbf{i} + e^t \mathbf{j} + e^{-t} \mathbf{k}, \quad 0 \leq t \leq 1$

4.  $\mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + \ln \cos t \mathbf{k}, \quad 0 \leq t \leq \pi/4$

5.  $\mathbf{r}(t) = \mathbf{i} + t^2 \mathbf{j} + t^3 \mathbf{k}, \quad 0 \leq t \leq 1$


6.  $\mathbf{r}(t) = 12t \mathbf{i} + 8t^{3/2} \mathbf{j} + 3t^2 \mathbf{k}, \quad 0 \leq t \leq 1$

7–9 Find the length of the curve correct to four decimal places. (Use your calculator to approximate the integral.)

7.  $\mathbf{r}(t) = \langle \sqrt{t}, t, t^2 \rangle, \quad 1 \leq t \leq 4$

8.  $\mathbf{r}(t) = \langle t, \ln t, t \ln t \rangle, \quad 1 \leq t \leq 2$

9.  $\mathbf{r}(t) = \langle \sin t, \cos t, \tan t \rangle, \quad 0 \leq t \leq \pi/4$

 10. Graph the curve with parametric equations  $x = \sin t$ ,  $y = \sin 2t$ ,  $z = \sin 3t$ . Find the total length of this curve correct to four decimal places.

11. Let  $C$  be the curve of intersection of the parabolic cylinder  $x^2 = 2y$  and the surface  $3z = xy$ . Find the exact length of  $C$  from the origin to the point  $(6, 18, 36)$ .

12. Find, correct to four decimal places, the length of the curve of intersection of the cylinder  $4x^2 + y^2 = 4$  and the plane  $x + y + z = 2$ .

13–14 Reparametrize the curve with respect to arc length measured from the point where  $t = 0$  in the direction of increasing  $t$ .

13.  $\mathbf{r}(t) = 2t \mathbf{i} + (1 - 3t) \mathbf{j} + (5 + 4t) \mathbf{k}$

14.  $\mathbf{r}(t) = e^{2t} \cos 2t \mathbf{i} + 2 \mathbf{j} + e^{2t} \sin 2t \mathbf{k}$

15. Suppose you start at the point  $(0, 0, 3)$  and move 5 units along the curve  $x = 3 \sin t$ ,  $y = 4t$ ,  $z = 3 \cos t$  in the positive direction. Where are you now?

16. Reparametrize the curve

$$\mathbf{r}(t) = \left( \frac{2}{t^2 + 1} - 1 \right) \mathbf{i} + \frac{2t}{t^2 + 1} \mathbf{j}$$

with respect to arc length measured from the point  $(1, 0)$  in the direction of increasing  $t$ . Express the reparametrization in its simplest form. What can you conclude about the curve?

17–20

(a) Find the unit tangent and unit normal vectors  $\mathbf{T}(t)$  and  $\mathbf{N}(t)$ .  
(b) Use Formula 9 to find the curvature.

17.  $\mathbf{r}(t) = \langle 2 \sin t, 5t, 2 \cos t \rangle$

18.  $\mathbf{r}(t) = \langle t^2, \sin t - t \cos t, \cos t + t \sin t \rangle, \quad t > 0$

19.  $\mathbf{r}(t) = \langle \sqrt{2}t, e^t, e^{-t} \rangle$

20.  $\mathbf{r}(t) = \langle t, \frac{1}{2}t^2, t^2 \rangle$

21–23 Use Theorem 10 to find the curvature.

21.  $\mathbf{r}(t) = t^2 \mathbf{i} + t \mathbf{k}$

22.  $\mathbf{r}(t) = t \mathbf{i} + t \mathbf{j} + (1 + t^2) \mathbf{k}$

23.  $\mathbf{r}(t) = 3t \mathbf{i} + 4 \sin t \mathbf{j} + 4 \cos t \mathbf{k}$

24. Find the curvature of  $\mathbf{r}(t) = \langle e^t \cos t, e^t \sin t, t \rangle$  at the point  $(1, 0, 0)$ .25. Find the curvature of  $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$  at the point  $(1, 1, 1)$ .

26. Graph the curve with parametric equations

$$x = t \quad y = 4t^{3/2} \quad z = -t^2$$

and find the curvature at the point  $(1, 4, -1)$ .

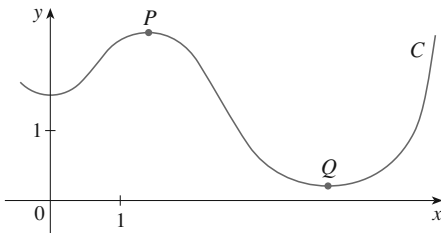
27–29 Use Formula 11 to find the curvature.

27.  $y = 2x - x^2$       28.  $y = \cos x$       29.  $y = 4x^{5/2}$

30–31 At what point does the curve have maximum curvature? What happens to the curvature as  $x \rightarrow \infty$ ?

30.  $y = \ln x$       31.  $y = e^x$

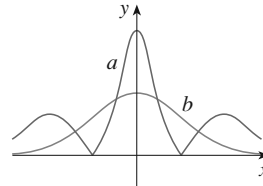
32. Find an equation of a parabola that has curvature 4 at the origin.

33. (a) Is the curvature of the curve  $C$  shown in the figure greater at  $P$  or at  $Q$ ? Explain.(b) Estimate the curvature at  $P$  and at  $Q$  by sketching the osculating circles at those points.34–35 Use a graphing calculator or computer to graph both the curve and its curvature function  $\kappa(x)$  on the same screen. Is the graph of  $\kappa$  what you would expect?

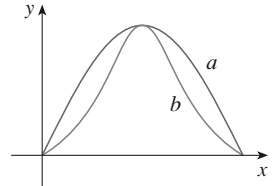
34.  $y = x^4 - 2x^2$       35.  $y = x^{-2}$

36–37 Two graphs,  $a$  and  $b$ , are shown. One is a curve  $y = f(x)$  and the other is the graph of its curvature function  $y = \kappa(x)$ . Identify each curve and explain your choices.

36.



37.

38. (a) Graph the curve  $\mathbf{r}(t) = \langle \sin 3t, \sin 2t, \sin 3t \rangle$ . At how many points on the curve does it appear that the curvature has a local or absolute maximum?

(b) Use a CAS to find and graph the curvature function. Does this graph confirm your conclusion from part (a)?

39. The graph of  $\mathbf{r}(t) = \langle t - \frac{3}{2} \sin t, 1 - \frac{3}{2} \cos t, t \rangle$  is shown in Figure 12(b) in Section 13.1. Where do you think the curvature is largest? Use a CAS to find and graph the curvature function. For which values of  $t$  is the curvature largest?40. Use Theorem 10 to show that the curvature of a plane parametric curve  $x = f(t)$ ,  $y = g(t)$  is

$$\kappa = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{[\dot{x}^2 + \dot{y}^2]^{3/2}}$$

where the dots indicate derivatives with respect to  $t$ .

41–42 Use the formula in Exercise 40 to find the curvature.

41.  $x = e^t \cos t, \quad y = e^t \sin t$

42.  $x = 1 + t^3, \quad y = t + t^2$

43–44 Find the vectors  $\mathbf{T}$ ,  $\mathbf{N}$ , and  $\mathbf{B}$  at the given point.

43.  $\mathbf{r}(t) = \langle t^2, \frac{2}{3}t^3, t \rangle, \quad (1, \frac{2}{3}, 1)$

44.  $\mathbf{r}(t) = \langle \cos t, \sin t, \ln \cos t \rangle, \quad (1, 0, 0)$

45–46 Find equations of the normal plane and osculating plane of the curve at the given point.

45.  $x = 2 \sin 3t, \quad y = t, \quad z = 2 \cos 3t; \quad (0, \pi, -2)$

46.  $x = t, \quad y = t^2, \quad z = t^3; \quad (1, 1, 1)$

47. Find equations of the osculating circles of the ellipse  $9x^2 + 4y^2 = 36$  at the points  $(2, 0)$  and  $(0, 3)$ . Use a graphing calculator or computer to graph the ellipse and both osculating circles on the same screen.

48. Find equations of the osculating circles of the parabola  $y = \frac{1}{2}x^2$  at the points  $(0, 0)$  and  $(1, \frac{1}{2})$ . Graph both osculating circles and the parabola on the same screen.

49. At what point on the curve  $x = t^3, y = 3t, z = t^4$  is the normal plane parallel to the plane  $6x + 6y - 8z = 1$ ?

50. Is there a point on the curve in Exercise 49 where the osculating plane is parallel to the plane  $x + y + z = 1$ ? [Note: You will need a CAS for differentiating, for simplifying, and for computing a cross product.]

51. Show that the curvature  $\kappa$  is related to the tangent and normal vectors by the equation

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}$$

52. Show that the curvature of a plane curve is  $\kappa = |d\phi/ds|$ , where  $\phi$  is the angle between  $\mathbf{T}$  and  $\mathbf{i}$ ; that is,  $\phi$  is the angle of inclination of the tangent line. (This shows that the definition of curvature is consistent with the definition for plane curves given in Exercise 69 in Section 10.2.)

53. (a) Show that  $d\mathbf{B}/ds$  is perpendicular to  $\mathbf{B}$ .  
 (b) Show that  $d\mathbf{B}/ds$  is perpendicular to  $\mathbf{T}$ .  
 (c) Deduce from parts (a) and (b) that  $d\mathbf{B}/ds = -\tau(s)\mathbf{N}$  for some number  $\tau(s)$  called the **torsion** of the curve. (The torsion measures the degree of twisting of a curve.)  
 (d) Show that for a plane curve the torsion is  $\tau(s) = 0$ .

54. The following formulas, called the **Frenet-Serret formulas**, are of fundamental importance in differential geometry:

1.  $d\mathbf{T}/ds = \kappa\mathbf{N}$
2.  $d\mathbf{N}/ds = -\kappa\mathbf{T} + \tau\mathbf{B}$
3.  $d\mathbf{B}/ds = -\tau\mathbf{N}$

(Formula 1 comes from Exercise 51 and Formula 3 comes from Exercise 53.) Use the fact that  $\mathbf{N} = \mathbf{B} \times \mathbf{T}$  to deduce Formula 2 from Formulas 1 and 3.

55. Use the Frenet-Serret formulas to prove each of the following. (Primes denote derivatives with respect to  $t$ . Start as in the proof of Theorem 10.)

- (a)  $\mathbf{r}'' = s''\mathbf{T} + \kappa(s')^2\mathbf{N}$       (b)  $\mathbf{r}' \times \mathbf{r}'' = \kappa(s')^3\mathbf{B}$   
 (c)  $\mathbf{r}''' = [s''' - \kappa^2(s')^3]\mathbf{T} + [3\kappa s's'' + \kappa'(s')^2]\mathbf{N} + \kappa\tau(s')^3\mathbf{B}$   
 (d)  $\tau = \frac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2}$

56. Show that the circular helix  $\mathbf{r}(t) = \langle a \cos t, a \sin t, bt \rangle$ , where  $a$  and  $b$  are positive constants, has constant curvature and constant torsion. [Use the result of Exercise 55(d).]

57. Use the formula in Exercise 55(d) to find the torsion of the curve  $\mathbf{r}(t) = \langle t, \frac{1}{2}t^2, \frac{1}{3}t^3 \rangle$ .

58. Find the curvature and torsion of the curve  $x = \sinh t, y = \cosh t, z = t$  at the point  $(0, 1, 0)$ .

59. The DNA molecule has the shape of a double helix (see Figure 3 on page 819). The radius of each helix is about 10 angstroms ( $1 \text{ \AA} = 10^{-8} \text{ cm}$ ). Each helix rises about  $34 \text{ \AA}$  during each complete turn, and there are about  $2.9 \times 10^8$  complete turns. Estimate the length of each helix.

60. Let's consider the problem of designing a railroad track to make a smooth transition between sections of straight track. Existing track along the negative  $x$ -axis is to be joined smoothly to a track along the line  $y = 1$  for  $x \geq 1$ .  
 (a) Find a polynomial  $P = P(x)$  of degree 5 such that the function  $F$  defined by

$$F(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ P(x) & \text{if } 0 < x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$

is continuous and has continuous slope and continuous curvature.

(b) Use a graphing calculator or computer to draw the graph of  $F$ .

## 13.4 MOTION IN SPACE: VELOCITY AND ACCELERATION

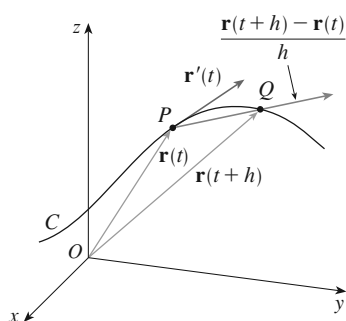


FIGURE 1

In this section we show how the ideas of tangent and normal vectors and curvature can be used in physics to study the motion of an object, including its velocity and acceleration, along a space curve. In particular, we follow in the footsteps of Newton by using these methods to derive Kepler's First Law of planetary motion.

Suppose a particle moves through space so that its position vector at time  $t$  is  $\mathbf{r}(t)$ . Notice from Figure 1 that, for small values of  $h$ , the vector

(1)

$$\frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

approximates the direction of the particle moving along the curve  $\mathbf{r}(t)$ . Its magnitude measures the size of the displacement vector per unit time. The vector (1) gives the average



where  $c = |\mathbf{c}|$ . Then

$$r = \frac{\mathbf{r} \cdot (\mathbf{v} \times \mathbf{h})}{GM + c \cos \theta} = \frac{1}{GM} \frac{\mathbf{r} \cdot (\mathbf{v} \times \mathbf{h})}{1 + e \cos \theta}$$

where  $e = c/(GM)$ . But

$$\mathbf{r} \cdot (\mathbf{v} \times \mathbf{h}) = (\mathbf{r} \times \mathbf{v}) \cdot \mathbf{h} = \mathbf{h} \cdot \mathbf{h} = |\mathbf{h}|^2 = h^2$$

where  $h = |\mathbf{h}|$ . So

$$r = \frac{h^2/(GM)}{1 + e \cos \theta} = \frac{eh^2/c}{1 + e \cos \theta}$$

Writing  $d = h^2/c$ , we obtain the equation

$$\boxed{12} \quad r = \frac{ed}{1 + e \cos \theta}$$

Comparing with Theorem 10.6.6, we see that Equation 12 is the polar equation of a conic section with focus at the origin and eccentricity  $e$ . We know that the orbit of a planet is a closed curve and so the conic must be an ellipse.

This completes the derivation of Kepler's First Law. We will guide you through the derivation of the Second and Third Laws in the Applied Project on page 848. The proofs of these three laws show that the methods of this chapter provide a powerful tool for describing some of the laws of nature.

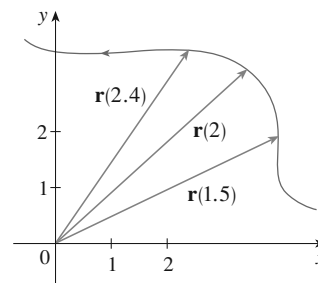
### 13.4 EXERCISES

1. The table gives coordinates of a particle moving through space along a smooth curve.
- Find the average velocities over the time intervals  $[0, 1]$ ,  $[0.5, 1]$ ,  $[1, 2]$ , and  $[1, 1.5]$ .
  - Estimate the velocity and speed of the particle at  $t = 1$ .

$t$	$x$	$y$	$z$
0	2.7	9.8	3.7
0.5	3.5	7.2	3.3
1.0	4.5	6.0	3.0
1.5	5.9	6.4	2.8
2.0	7.3	7.8	2.7

2. The figure shows the path of a particle that moves with position vector  $\mathbf{r}(t)$  at time  $t$ .
- Draw a vector that represents the average velocity of the particle over the time interval  $2 \leq t \leq 2.4$ .
  - Draw a vector that represents the average velocity over the time interval  $1.5 \leq t \leq 2$ .
  - Write an expression for the velocity vector  $\mathbf{v}(2)$ .

- (d) Draw an approximation to the vector  $\mathbf{v}(2)$  and estimate the speed of the particle at  $t = 2$ .



- 3–8 Find the velocity, acceleration, and speed of a particle with the given position function. Sketch the path of the particle and draw the velocity and acceleration vectors for the specified value of  $t$ .
- $\mathbf{r}(t) = \langle -\frac{1}{2}t^2, t \rangle$ ,  $t = 2$
  - $\mathbf{r}(t) = \langle 2 - t, 4\sqrt{t} \rangle$ ,  $t = 1$

5.  $\mathbf{r}(t) = 3 \cos t \mathbf{i} + 2 \sin t \mathbf{j}, \quad t = \pi/3$

6.  $\mathbf{r}(t) = e^t \mathbf{i} + e^{2t} \mathbf{j}, \quad t = 0$

7.  $\mathbf{r}(t) = t \mathbf{i} + t^2 \mathbf{j} + 2 \mathbf{k}, \quad t = 1$

8.  $\mathbf{r}(t) = t \mathbf{i} + 2 \cos t \mathbf{j} + \sin t \mathbf{k}, \quad t = 0$

9–14 Find the velocity, acceleration, and speed of a particle with the given position function.

9.  $\mathbf{r}(t) = \langle t^2 + 1, t^3, t^2 - 1 \rangle$

10.  $\mathbf{r}(t) = \langle 2 \cos t, 3t, 2 \sin t \rangle$

11.  $\mathbf{r}(t) = \sqrt{2} t \mathbf{i} + e^t \mathbf{j} + e^{-t} \mathbf{k}$

12.  $\mathbf{r}(t) = t^2 \mathbf{i} + \ln t \mathbf{j} + t \mathbf{k}$

13.  $\mathbf{r}(t) = e^t(\cos t \mathbf{i} + \sin t \mathbf{j} + t \mathbf{k})$

14.  $\mathbf{r}(t) = t \sin t \mathbf{i} + t \cos t \mathbf{j} + t^2 \mathbf{k}$


15–16 Find the velocity and position vectors of a particle that has the given acceleration and the given initial velocity and position.

15.  $\mathbf{a}(t) = \mathbf{i} + 2 \mathbf{j}, \quad \mathbf{v}(0) = \mathbf{k}, \quad \mathbf{r}(0) = \mathbf{i}$

16.  $\mathbf{a}(t) = 2 \mathbf{i} + 6t \mathbf{j} + 12t^2 \mathbf{k}, \quad \mathbf{v}(0) = \mathbf{i}, \quad \mathbf{r}(0) = \mathbf{j} - \mathbf{k}$

17–18

(a) Find the position vector of a particle that has the given acceleration and the specified initial velocity and position.

 (b) Use a computer to graph the path of the particle.

17.  $\mathbf{a}(t) = 2t \mathbf{i} + \sin t \mathbf{j} + \cos 2t \mathbf{k}, \quad \mathbf{v}(0) = \mathbf{i}, \quad \mathbf{r}(0) = \mathbf{j}$

18.  $\mathbf{a}(t) = t \mathbf{i} + e^t \mathbf{j} + e^{-t} \mathbf{k}, \quad \mathbf{v}(0) = \mathbf{k}, \quad \mathbf{r}(0) = \mathbf{j} + \mathbf{k}$

19. The position function of a particle is given by  $\mathbf{r}(t) = \langle t^2, 5t, t^2 - 16t \rangle$ . When is the speed a minimum?

20. What force is required so that a particle of mass  $m$  has the position function  $\mathbf{r}(t) = t^3 \mathbf{i} + t^2 \mathbf{j} + t^3 \mathbf{k}$ ?

21. A force with magnitude 20 N acts directly upward from the  $xy$ -plane on an object with mass 4 kg. The object starts at the origin with initial velocity  $\mathbf{v}(0) = \mathbf{i} - \mathbf{j}$ . Find its position function and its speed at time  $t$ .

22. Show that if a particle moves with constant speed, then the velocity and acceleration vectors are orthogonal.

23. A projectile is fired with an initial speed of 500 m/s and angle of elevation  $30^\circ$ . Find (a) the range of the projectile, (b) the maximum height reached, and (c) the speed at impact.

24. Rework Exercise 23 if the projectile is fired from a position 200 m above the ground.

25. A ball is thrown at an angle of  $45^\circ$  to the ground. If the ball lands 90 m away, what was the initial speed of the ball?


26. A gun is fired with angle of elevation  $30^\circ$ . What is the muzzle speed if the maximum height of the shell is 500 m?

27. A gun has muzzle speed 150 m/s. Find two angles of elevation that can be used to hit a target 800 m away.

28. A batter hits a baseball 3 ft above the ground toward the center field fence, which is 10 ft high and 400 ft from home plate. The ball leaves the bat with speed 115 ft/s at an angle  $50^\circ$  above the horizontal. Is it a home run? (In other words, does the ball clear the fence?)

29. A medieval city has the shape of a square and is protected by walls with length 500 m and height 15 m. You are the commander of an attacking army and the closest you can get to the wall is 100 m. Your plan is to set fire to the city by catapulting heated rocks over the wall (with an initial speed of 80 m/s). At what range of angles should you tell your men to set the catapult? (Assume the path of the rocks is perpendicular to the wall.)

30. A ball with mass 0.8 kg is thrown southward into the air with a speed of 30 m/s at an angle of  $30^\circ$  to the ground. A west wind applies a steady force of 4 N to the ball in an easterly direction. Where does the ball land and with what speed?

 31. Water traveling along a straight portion of a river normally flows fastest in the middle, and the speed slows to almost zero at the banks. Consider a long straight stretch of river flowing north, with parallel banks 40 m apart. If the maximum water speed is 3 m/s, we can use a quadratic function as a basic model for the rate of water flow  $x$  units from the west bank:  $f(x) = \frac{3}{400}x(40 - x)$ .

(a) A boat proceeds at a constant speed of 5 m/s from a point  $A$  on the west bank while maintaining a heading perpendicular to the bank. How far down the river on the opposite bank will the boat touch shore? Graph the path of the boat.

(b) Suppose we would like to pilot the boat to land at the point  $B$  on the east bank directly opposite  $A$ . If we maintain a constant speed of 5 m/s and a constant heading, find the angle at which the boat should head. Then graph the actual path the boat follows. Does the path seem realistic?

32. Another reasonable model for the water speed of the river in Exercise 31 is a sine function:  $f(x) = 3 \sin(\pi x/40)$ . If a boater would like to cross the river from  $A$  to  $B$  with constant heading and a constant speed of 5 m/s, determine the angle at which the boat should head.

33–38 Find the tangential and normal components of the acceleration vector.

33.  $\mathbf{r}(t) = (3t - t^3) \mathbf{i} + 3t^2 \mathbf{j}$

34.  $\mathbf{r}(t) = (1 + t) \mathbf{i} + (t^2 - 2t) \mathbf{j}$

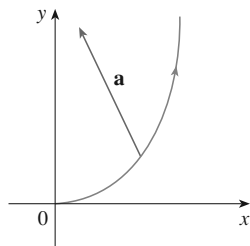
35.  $\mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + t \mathbf{k}$

36.  $\mathbf{r}(t) = t \mathbf{i} + t^2 \mathbf{j} + 3t \mathbf{k}$

37.  $\mathbf{r}(t) = e^t \mathbf{i} + \sqrt{2}t \mathbf{j} + e^{-t} \mathbf{k}$

38.  $\mathbf{r}(t) = t \mathbf{i} + \cos^2 t \mathbf{j} + \sin^2 t \mathbf{k}$

39. The magnitude of the acceleration vector
- $\mathbf{a}$
- is
- $10 \text{ cm/s}^2$
- . Use the figure to estimate the tangential and normal components of
- $\mathbf{a}$
- .



40. If a particle with mass
- $m$
- moves with position vector
- $\mathbf{r}(t)$
- , then its
- angular momentum**
- is defined as
- $\mathbf{L}(t) = m\mathbf{r}(t) \times \mathbf{v}(t)$
- and its
- torque**
- as
- $\boldsymbol{\tau}(t) = m\mathbf{r}(t) \times \mathbf{a}(t)$
- . Show that
- $\mathbf{L}'(t) = \boldsymbol{\tau}(t)$
- . Deduce that if
- $\boldsymbol{\tau}(t) = \mathbf{0}$
- for all
- $t$
- , then
- $\mathbf{L}(t)$
- is constant. (This is the
- law of conservation of angular momentum*
- .)

41. The position function of a spaceship is

$$\mathbf{r}(t) = (3 + t) \mathbf{i} + (2 + \ln t) \mathbf{j} + \left(7 - \frac{4}{t^2 + 1}\right) \mathbf{k}$$

and the coordinates of a space station are  $(6, 4, 9)$ . The captain wants the spaceship to coast into the space station. When should the engines be turned off?

42. A rocket burning its onboard fuel while moving through space has velocity
- $\mathbf{v}(t)$
- and mass
- $m(t)$
- at time
- $t$
- . If the exhaust gases escape with velocity
- $\mathbf{v}_e$
- relative to the rocket, it can be deduced from Newton's Second Law of Motion that

$$m \frac{d\mathbf{v}}{dt} = \frac{dm}{dt} \mathbf{v}_e$$

- (a) Show that  $\mathbf{v}(t) = \mathbf{v}(0) - \ln \frac{m(0)}{m(t)} \mathbf{v}_e$ .
- (b) For the rocket to accelerate in a straight line from rest to twice the speed of its own exhaust gases, what fraction of its initial mass would the rocket have to burn as fuel?

### APPLIED PROJECT

#### KEPLER'S LAWS

Johannes Kepler stated the following three laws of planetary motion on the basis of masses of data on the positions of the planets at various times.

##### KEPLER'S LAWS

1. A planet revolves around the sun in an elliptical orbit with the sun at one focus.
2. The line joining the sun to a planet sweeps out equal areas in equal times.
3. The square of the period of revolution of a planet is proportional to the cube of the length of the major axis of its orbit.

Kepler formulated these laws because they fitted the astronomical data. He wasn't able to see why they were true or how they related to each other. But Sir Isaac Newton, in his *Principia Mathematica* of 1687, showed how to deduce Kepler's three laws from two of Newton's own laws, the Second Law of Motion and the Law of Universal Gravitation. In Section 13.4 we proved Kepler's First Law using the calculus of vector functions. In this project we guide you through the proofs of Kepler's Second and Third Laws and explore some of their consequences.

1. Use the following steps to prove Kepler's Second Law. The notation is the same as in the proof of the First Law in Section 13.4. In particular, use polar coordinates so that  $\mathbf{r} = (r \cos \theta) \mathbf{i} + (r \sin \theta) \mathbf{j}$ .

(a) Show that  $\mathbf{h} = r^2 \frac{d\theta}{dt} \mathbf{k}$ .

(b) Deduce that  $r^2 \frac{d\theta}{dt} = h$ .

- (c) If  $A = A(t)$  is the area swept out by the radius vector  $\mathbf{r} = \mathbf{r}(t)$  in the time interval  $[t_0, t]$  as in the figure, show that

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt}$$

