It's very difficult to visualize a function $f$ of three variables by its graph, since that would lie in a four-dimensional space. However, we do gain some insight into $f$ by examining its **level surfaces**, which are the surfaces with equations $f(x, y, z) = k$, where $k$ is a constant. If the point $(x, y, z)$ moves along a level surface, the value of $f(x, y, z)$ remains fixed.

**EXAMPLE 15** Find the level surfaces of the function

$$f(x, y, z) = x^2 + y^2 + z^2$$

SOLUTION The level surfaces are $x^2 + y^2 + z^2 = k$, where $k \geqslant 0$. These form a family of concentric spheres with radius $\sqrt{k}$. (See Figure 20.) Thus, as $(x, y, z)$ varies over any sphere with center $O$, the value of $f(x, y, z)$ remains fixed. ■

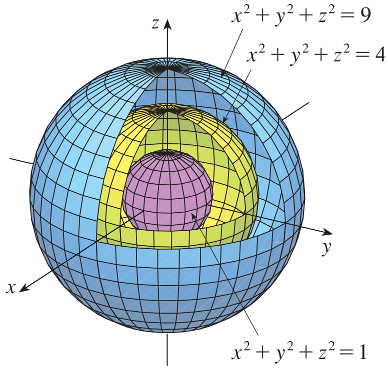



FIGURE 20

Functions of any number of variables can be considered. A **function of *n* variables** is a rule that assigns a number $z = f(x_1, x_2, \ldots, x_n)$ to an $n$-tuple $(x_1, x_2, \ldots, x_n)$ of real numbers. We denote by $\mathbb{R}^n$ the set of all such $n$-tuples. For example, if a company uses $n$ different ingredients in making a food product, $c_i$ is the cost per unit of the $i$th ingredient, and $x_i$ units of the $i$th ingredient are used, then the total cost $C$ of the ingredients is a function of the $n$ variables $x_1, x_2, \ldots, x_n$:

$$C = f(x_1, x_2, \ldots, x_n) = c_1x_1 + c_2x_2 + \cdots + c_nx_n \tag{3}$$

The function $f$ is a real-valued function whose domain is a subset of $\mathbb{R}^n$. Sometimes we will use vector notation to write such functions more compactly: If $\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle$, we often write $f(\mathbf{x})$ in place of $f(x_1, x_2, \ldots, x_n)$. With this notation we can rewrite the function defined in Equation 3 as

$$f(\mathbf{x}) = \mathbf{c} \cdot \mathbf{x}$$

where $\mathbf{c} = \langle c_1, c_2, \ldots, c_n \rangle$ and $\mathbf{c} \cdot \mathbf{x}$ denotes the dot product of the vectors $\mathbf{c}$ and $\mathbf{x}$ in $V_n$.

In view of the one-to-one correspondence between points $(x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$ and their position vectors $\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle$ in $V_n$, we have three ways of looking at a function $f$ defined on a subset of $\mathbb{R}^n$:

1. As a function of $n$ real variables $x_1, x_2, \ldots, x_n$
2. As a function of a single point variable $(x_1, x_2, \ldots, x_n)$
3. As a function of a single vector variable $\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle$

We will see that all three points of view are useful.

## 14.1 EXERCISES

1. In Example 2 we considered the function $W = f(T, v)$, where $W$ is the wind-chill index, $T$ is the actual temperature, and $v$ is the wind speed. A numerical representation is given in Table 1.
   (a) What is the value of $f(-15, 40)$? What is its meaning?
   (b) Describe in words the meaning of the question "For what value of $v$ is $f(-20, v) = -30$?" Then answer the question.
   (c) Describe in words the meaning of the question "For what value of $T$ is $f(T, 20) = -49$?" Then answer the question.
   (d) What is the meaning of the function $W = f(-5, v)$? Describe the behavior of this function.
   (e) What is the meaning of the function $W = f(T, 50)$? Describe the behavior of this function.

**2.** The *temperature-humidity index* $I$ (or humidex, for short) is the perceived air temperature when the actual temperature is $T$ and the relative humidity is $h$, so we can write $I = f(T, h)$. The following table of values of $I$ is an excerpt from a table compiled by the National Oceanic & Atmospheric Administration.

**TABLE 3** Apparent temperature as a function of temperature and humidity

Relative humidity (%)

| Actual temperature (°F) $T$ \ $h$ | 20 | 30 | 40 | 50 | 60 | 70 |
|---|---|---|---|---|---|---|
| 80 | 77 | 78 | 79 | 81 | 82 | 83 |
| 85 | 82 | 84 | 86 | 88 | 90 | 93 |
| 90 | 87 | 90 | 93 | 96 | 100 | 106 |
| 95 | 93 | 96 | 101 | 107 | 114 | 124 |
| 100 | 99 | 104 | 110 | 120 | 132 | 144 |

(a) What is the value of $f(95, 70)$? What is its meaning?
(b) For what value of $h$ is $f(90, h) = 100$?
(c) For what value of $T$ is $f(T, 50) = 88$?
(d) What are the meanings of the functions $I = f(80, h)$ and $I = f(100, h)$? Compare the behavior of these two functions of $h$.

**3.** Verify for the Cobb-Douglas production function

$$P(L, K) = 1.01L^{0.75}K^{0.25}$$

discussed in Example 3 that the production will be doubled if both the amount of labor and the amount of capital are doubled. Determine whether this is also true for the general production function

$$P(L, K) = bL^{\alpha}K^{1-\alpha}$$

**4.** The wind-chill index $W$ discussed in Example 2 has been modeled by the following function:

$$W(T, v) = 13.12 + 0.6215T - 11.37v^{0.16} + 0.3965Tv^{0.16}$$

Check to see how closely this model agrees with the values in Table 1 for a few values of $T$ and $v$.

**5.** The wave heights $h$ in the open sea depend on the speed $v$ of the wind and the length of time $t$ that the wind has been blowing at that speed. Values of the function $h = f(v, t)$ are recorded in feet in Table 4.
(a) What is the value of $f(40, 15)$? What is its meaning?
(b) What is the meaning of the function $h = f(30, t)$? Describe the behavior of this function.
(c) What is the meaning of the function $h = f(v, 30)$? Describe the behavior of this function.

**TABLE 4**

Duration (hours)

| Wind speed (knots) $v$ \ $t$ | 5 | 10 | 15 | 20 | 30 | 40 | 50 |
|---|---|---|---|---|---|---|---|
| 10 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| 15 | 4 | 4 | 5 | 5 | 5 | 5 | 5 |
| 20 | 5 | 7 | 8 | 8 | 9 | 9 | 9 |
| 30 | 9 | 13 | 16 | 17 | 18 | 19 | 19 |
| 40 | 14 | 21 | 25 | 28 | 31 | 33 | 33 |
| 50 | 19 | 29 | 36 | 40 | 45 | 48 | 50 |
| 60 | 24 | 37 | 47 | 54 | 62 | 67 | 69 |

**6.** Let $f(x, y) = \ln(x + y - 1)$.
(a) Evaluate $f(1, 1)$. (b) Evaluate $f(e, 1)$.
(c) Find and sketch the domain of $f$.
(d) Find the range of $f$.

**7.** Let $f(x, y) = x^2e^{3xy}$.
(a) Evaluate $f(2, 0)$. (b) Find the domain of $f$.
(c) Find the range of $f$.

**8.** Find and sketch the domain of the function $f(x, y) = \sqrt{1 + x - y^2}$. What is the range of $f$?

**9.** Let $f(x, y, z) = e^{\sqrt{z-x^2-y^2}}$.
(a) Evaluate $f(2, -1, 6)$. (b) Find the domain of $f$.
(c) Find the range of $f$.

**10.** Let $g(x, y, z) = \ln(25 - x^2 - y^2 - z^2)$.
(a) Evaluate $g(2, -2, 4)$. (b) Find the domain of $g$.
(c) Find the range of $g$.

**11–20** Find and sketch the domain of the function.

**11.** $f(x, y) = \sqrt{x + y}$

**12.** $f(x, y) = \sqrt{xy}$

**13.** $f(x, y) = \ln(9 - x^2 - 9y^2)$

**14.** $f(x, y) = \sqrt{y - x}\,\ln(y + x)$

**15.** $f(x, y) = \sqrt{1 - x^2} - \sqrt{1 - y^2}$

**16.** $f(x, y) = \sqrt{y} + \sqrt{25 - x^2 - y^2}$

**17.** $f(x, y) = \dfrac{\sqrt{y - x^2}}{1 - x^2}$

**18.** $f(x, y) = \arcsin(x^2 + y^2 - 2)$

**19.** $f(x, y, z) = \sqrt{1 - x^2 - y^2 - z^2}$

**20.** $f(x, y, z) = \ln(16 - 4x^2 - 4y^2 - z^2)$

**21–29** Sketch the graph of the function.

**21.** $f(x, y) = 3$

**22.** $f(x, y) = y$

**23.** $f(x, y) = 10 - 4x - 5y$

**24.** $f(x, y) = \cos x$

**25.** $f(x, y) = y^2 + 1$

**26.** $f(x, y) = 3 - x^2 - y^2$

**27.** $f(x, y) = 4x^2 + y^2 + 1$

**28.** $f(x, y) = \sqrt{16 - x^2 - 16y^2}$

**29.** $f(x, y) = \sqrt{x^2 + y^2}$

**30.** Match the function with its graph (labeled I–VI).Give reasons for your choices.

(a) $f(x, y) = |x| + |y|$

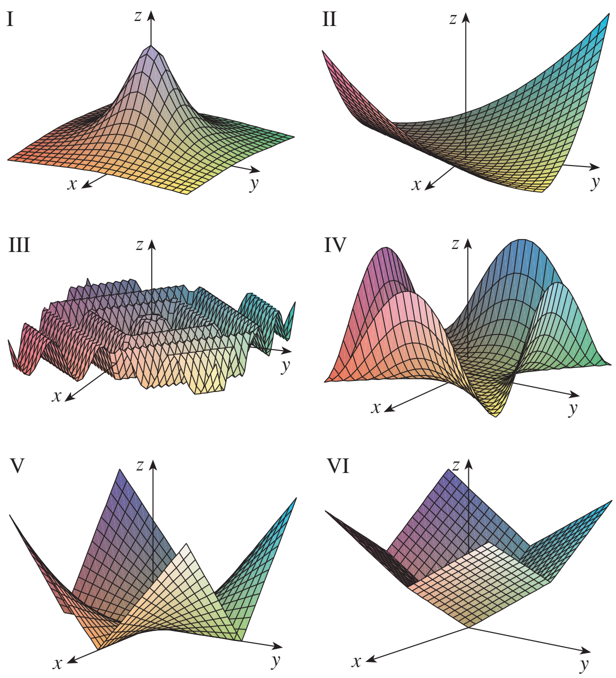
(b) $f(x, y) = |xy|$

(c) $f(x, y) = \dfrac{1}{1 + x^2 + y^2}$

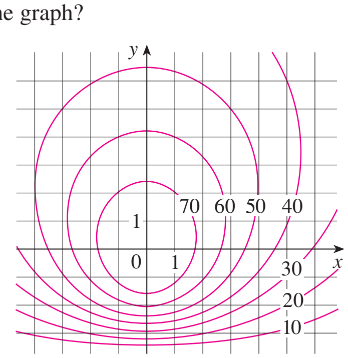
(d) $f(x, y) = (x^2 - y^2)^2$

(e) $f(x, y) = (x - y)^2$

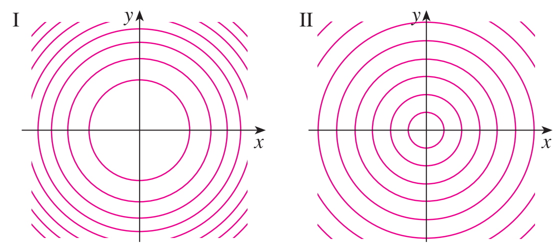
(f) $f(x, y) = \sin(|x| + |y|)$




**31.** A contour map for a function $f$ is shown. Use it to estimate the values of $f(-3, 3)$ and $f(3, -2)$. What can you say about the shape of the graph?

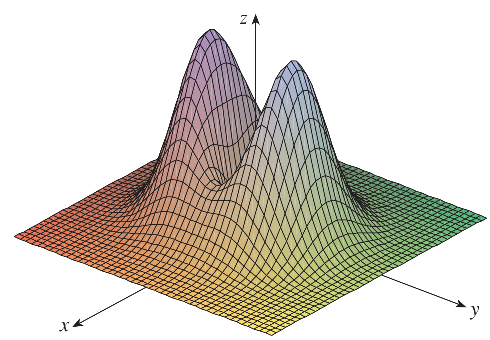



**32.** Two contour maps are shown. One is for a function $f$ whose graph is a cone. The other is for a function $g$ whose graph is a paraboloid. Which is which, and why?



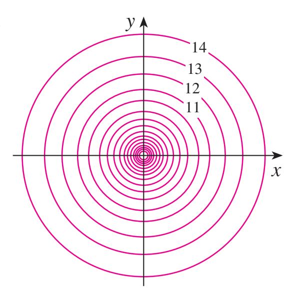

**33.** Locate the points $A$ and $B$ in the map of Lonesome Mountain (Figure 12). How would you describe the terrain near $A$? Near $B$?

**34.** Make a rough sketch of a contour map for the function whose graph is shown.

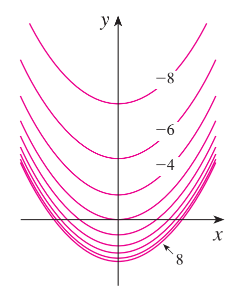


**35–38** A contour map of a function is shown. Use it to make a rough sketch of the graph of $f$.

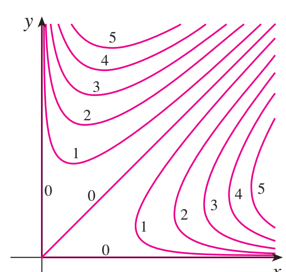
**35.**



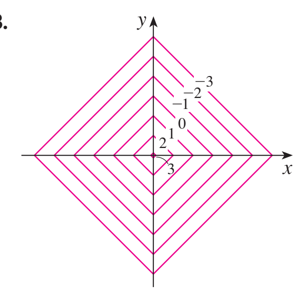

**36.**




**37.**




**38.**



**39–46** Draw a contour map of the function showing several level curves.

**39.** $f(x, y) = (y - 2x)^2$

**40.** $f(x, y) = x^3 - y$

**41.** $f(x, y) = y - \ln x$

**42.** $f(x, y) = e^{y/x}$

**43.** $f(x, y) = ye^x$

**44.** $f(x, y) = y \sec x$

**45.** $f(x, y) = \sqrt{y^2 - x^2}$

**46.** $f(x, y) = y/(x^2 + y^2)$

**47–48** Sketch both a contour map and a graph of the function and compare them.

**47.** $f(x, y) = x^2 + 9y^2$

**48.** $f(x, y) = \sqrt{36 - 9x^2 - 4y^2}$

**49.** A thin metal plate, located in the $xy$-plane, has temperature $T(x, y)$ at the point $(x, y)$. The level curves of $T$ are called *isothermals* because at all points on an isothermal the temperature is the same. Sketch some isothermals if the temperature function is given by

$$T(x, y) = 100/(1 + x^2 + 2y^2)$$

**50.** If $V(x, y)$ is the electric potential at a point $(x, y)$ in the $xy$-plane, then the level curves of $V$ are called *equipotential curves* because at all points on such a curve the electric potential is the same. Sketch some equipotential curves if $V(x, y) = c/\sqrt{r^2 - x^2 - y^2}$, where $c$ is a positive constant.

**51–54** Use a computer to graph the function using various domains and viewpoints. Get a printout of one that, in your opinion, gives a good view. If your software also produces level curves, then plot some contour lines of the same function and compare with the graph.

**51.** $f(x, y) = e^{-x^2} + e^{-2y^2}$

**52.** $f(x, y) = (1 - 3x^2 + y^2)e^{1 - x^2 - y^2}$

**53.** $f(x, y) = xy^2 - x^3$ (monkey saddle)

**54.** $f(x, y) = xy^3 - yx^3$ (dog saddle)

**55–60** Match the function (a) with its graph (labeled A–F on page 869) and (b) with its contour map (labeled I–VI). Give reasons for your choices.

**55.** $z = \sin(xy)$

**56.** $z = e^x \cos y$

**57.** $z = \sin(x - y)$

**58.** $z = \sin x - \sin y$

**59.** $z = (1 - x^2)(1 - y^2)$

**60.** $z = \dfrac{x - y}{1 + x^2 + y^2}$

**61–64** Describe the level surfaces of the function.

**61.** $f(x, y, z) = x + 3y + 5z$

**62.** $f(x, y, z) = x^2 + 3y^2 + 5z^2$

**63.** $f(x, y, z) = x^2 - y^2 + z^2$

**64.** $f(x, y, z) = x^2 - y^2$

**65–66** Describe how the graph of $g$ is obtained from the graph of $f$.

**65.** (a) $g(x, y) = f(x, y) + 2$ (b) $g(x, y) = 2f(x, y)$
(c) $g(x, y) = -f(x, y)$ (d) $g(x, y) = 2 - f(x, y)$

**66.** (a) $g(x, y) = f(x - 2, y)$ (b) $g(x, y) = f(x, y + 2)$
(c) $g(x, y) = f(x + 3, y - 4)$

**67–68** Use a computer to graph the function using various domains and viewpoints. Get a printout that gives a good view of the "peaks and valleys." Would you say the function has a maximum value? Can you identify any points on the graph that you might consider to be "local maximum points"? What about "local minimum points"?

**67.** $f(x, y) = 3x - x^4 - 4y^2 - 10xy$

**68.** $f(x, y) = xye^{-x^2 - y^2}$

**69–70** Use a computer to graph the function using various domains and viewpoints. Comment on the limiting behavior of the function. What happens as both $x$ and $y$ become large? What happens as $(x, y)$ approaches the origin?

**69.** $f(x, y) = \dfrac{x + y}{x^2 + y^2}$

**70.** $f(x, y) = \dfrac{xy}{x^2 + y^2}$

**71.** Use a computer to investigate the family of functions $f(x, y) = e^{cx^2 + y^2}$. How does the shape of the graph depend on $c$?

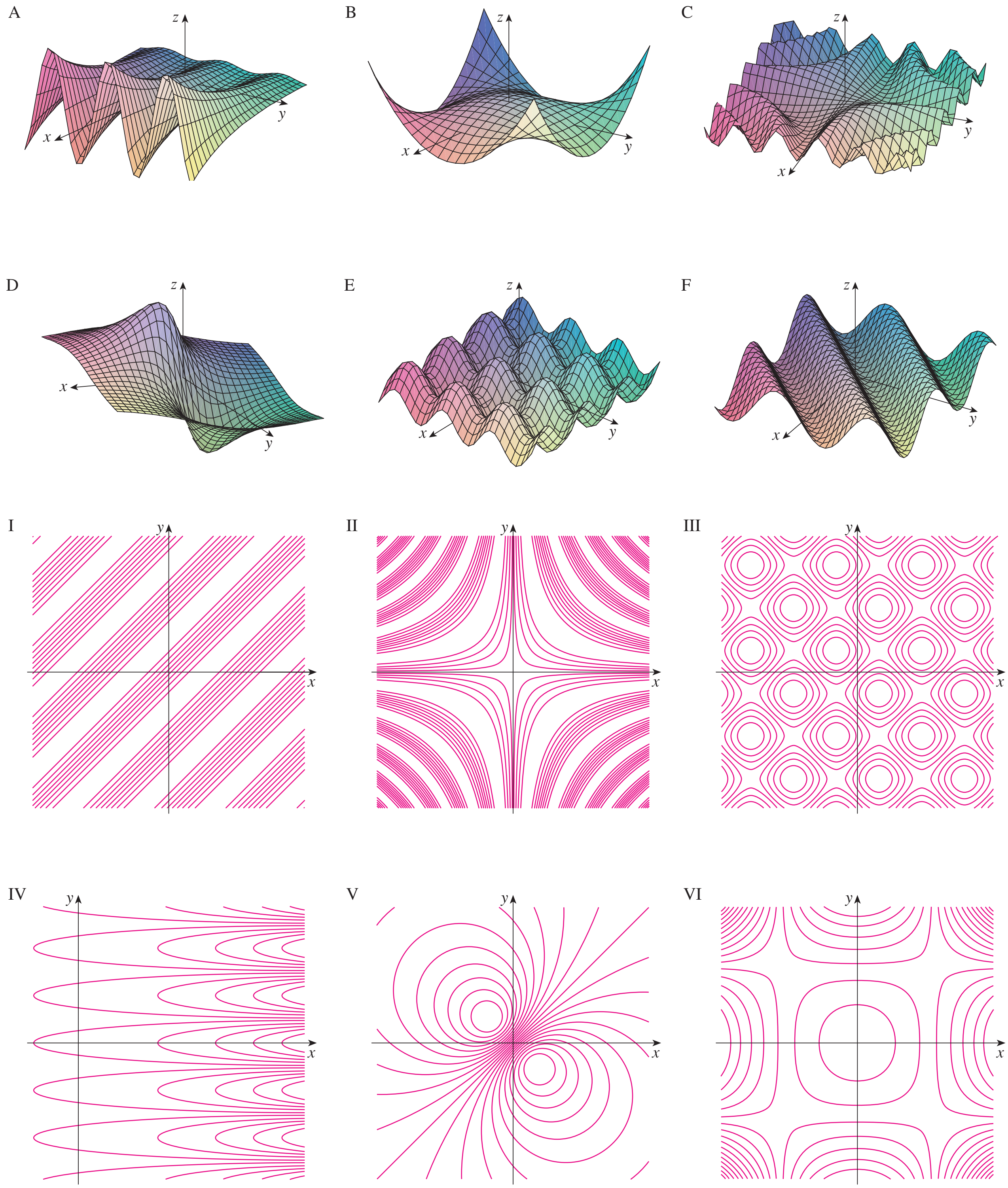
**72.** Use a computer to investigate the family of surfaces

$$z = (ax^2 + by^2)e^{-x^2 - y^2}$$

How does the shape of the graph depend on the numbers $a$ and $b$?

**73.** Use a computer to investigate the family of surfaces $z = x^2 + y^2 + cxy$. In particular, you should determine the transitional values of $c$ for which the surface changes from one type of quadric surface to another.

**Graphs and Contour Maps for Exercises 55–60**



where $b$ is a constant that is independent of both $L$ and $K$. Assumption (i) shows that $\alpha > 0$ and $\beta > 0$.

Notice from Equation 8 that if labor and capital are both increased by a factor $m$, then

$$P(mL, mK) = b(mL)^\alpha(mK)^\beta = m^{\alpha+\beta}bL^\alpha K^\beta = m^{\alpha+\beta}P(L, K)$$

If $\alpha + \beta = 1$, then $P(mL, mK) = mP(L, K)$, which means that production is also increased by a factor of $m$. That is why Cobb and Douglas assumed that $\alpha + \beta = 1$ and therefore

$$P(L, K) = bL^\alpha K^{1-\alpha}$$

This is the Cobb-Douglas production function that we discussed in Section 14.1.

## 14.3 EXERCISES

**1.** The temperature $T$ at a location in the Northern Hemisphere depends on the longitude $x$, latitude $y$, and time $t$, so we can write $T = f(x, y, t)$. Let's measure time in hours from the beginning of January.

(a) What are the meanings of the partial derivatives $\partial T/\partial x$, $\partial T/\partial y$, and $\partial T/\partial t$?

(b) Honolulu has longitude 158° W and latitude 21° N. Suppose that at 9:00 AM on January 1 the wind is blowing hot air to the northeast, so the air to the west and south is warm and the air to the north and east is cooler. Would you expect $f_x(158, 21, 9)$, $f_y(158, 21, 9)$, and $f_t(158, 21, 9)$ to be positive or negative? Explain.

**2.** At the beginning of this section we discussed the function $I = f(T, H)$, where $I$ is the heat index, $T$ is the temperature, and $H$ is the relative humidity. Use Table 1 to estimate $f_T(92, 60)$ and $f_H(92, 60)$. What are the practical interpretations of these values?

**3.** The wind-chill index $W$ is the perceived temperature when the actual temperature is $T$ and the wind speed is $v$, so we can write $W = f(T, v)$. The following table of values is an excerpt from Table 1 in Section 14.1.

Wind speed (km/h)

| Actual temperature (°C): $T$ \ $v$ | 20 | 30 | 40 | 50 | 60 | 70 |
|---|---|---|---|---|---|---|
| −10 | −18 | −20 | −21 | −22 | −23 | −23 |
| −15 | −24 | −26 | −27 | −29 | −30 | −30 |
| −20 | −30 | −33 | −34 | −35 | −36 | −37 |
| −25 | −37 | −39 | −41 | −42 | −43 | −44 |

(a) Estimate the values of $f_T(-15, 30)$ and $f_v(-15, 30)$. What are the practical interpretations of these values?

(b) In general, what can you say about the signs of $\partial W/\partial T$ and $\partial W/\partial v$?

(c) What appears to be the value of the following limit?

$$\lim_{v\to\infty} \frac{\partial W}{\partial v}$$

**4.** The wave heights $h$ in the open sea depend on the speed $v$ of the wind and the length of time $t$ that the wind has been blowing at that speed. Values of the function $h = f(v, t)$ are recorded in feet in the following table.

Duration (hours)

| Wind speed (knots): $v$ \ $t$ | 5 | 10 | 15 | 20 | 30 | 40 | 50 |
|---|---|---|---|---|---|---|---|
| 10 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| 15 | 4 | 4 | 5 | 5 | 5 | 5 | 5 |
| 20 | 5 | 7 | 8 | 8 | 9 | 9 | 9 |
| 30 | 9 | 13 | 16 | 17 | 18 | 19 | 19 |
| 40 | 14 | 21 | 25 | 28 | 31 | 33 | 33 |
| 50 | 19 | 29 | 36 | 40 | 45 | 48 | 50 |
| 60 | 24 | 37 | 47 | 54 | 62 | 67 | 69 |

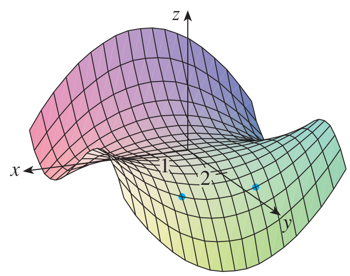
(a) What are the meanings of the partial derivatives $\partial h/\partial v$ and $\partial h/\partial t$?

(b) Estimate the values of $f_v(40, 15)$ and $f_t(40, 15)$. What are the practical interpretations of these values?

(c) What appears to be the value of the following limit?

$$\lim_{t\to\infty} \frac{\partial h}{\partial t}$$

**5–8** Determine the signs of the partial derivatives for the function $f$ whose graph is shown.



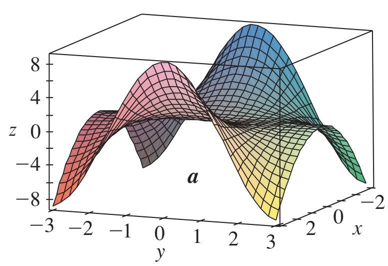

**5.** (a) $f_x(1, 2)$ (b) $f_y(1, 2)$

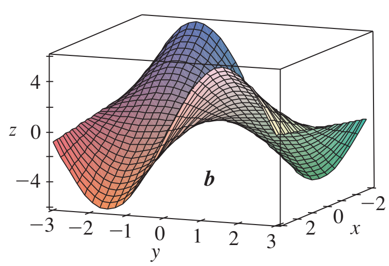
**6.** (a) $f_x(-1, 2)$ (b) $f_y(-1, 2)$

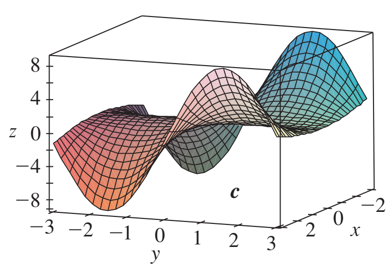
**7.** (a) $f_{xx}(-1, 2)$ (b) $f_{yy}(-1, 2)$

**8.** (a) $f_{xy}(1, 2)$ (b) $f_{xy}(-1, 2)$

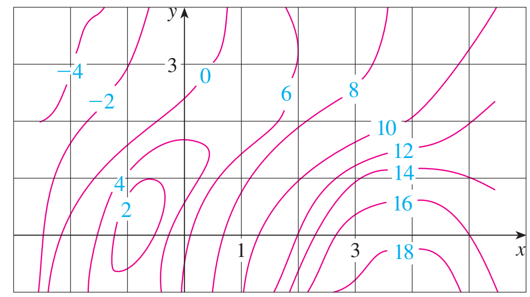
**9.** The following surfaces, labeled $a$, $b$, and $c$, are graphs of a function $f$ and its partial derivatives $f_x$ and $f_y$. Identify each surface and give reasons for your choices.




**10.** A contour map is given for a function $f$. Use it to estimate $f_x(2, 1)$ and $f_y(2, 1)$.




**11.** If $f(x, y) = 16 - 4x^2 - y^2$, find $f_x(1, 2)$ and $f_y(1, 2)$ and interpret these numbers as slopes. Illustrate with either hand-drawn sketches or computer plots.

**12.** If $f(x, y) = \sqrt{4 - x^2 - 4y^2}$, find $f_x(1, 0)$ and $f_y(1, 0)$ and interpret these numbers as slopes. Illustrate with either hand-drawn sketches or computer plots.

**13–14** Find $f_x$ and $f_y$ and graph $f$, $f_x$, and $f_y$ with domains and viewpoints that enable you to see the relationships between them.

**13.** $f(x, y) = x^2 + y^2 + x^2 y$ **14.** $f(x, y) = xe^{-x^2-y^2}$

**15–38** Find the first partial derivatives of the function.

**15.** $f(x, y) = y^5 - 3xy$

**16.** $f(x, y) = x^4 y^3 + 8x^2 y$

**17.** $f(x, t) = e^{-t} \cos \pi x$

**18.** $f(x, t) = \sqrt{x} \ln t$

**19.** $z = (2x + 3y)^{10}$

**20.** $z = \tan xy$

**21.** $f(x, y) = \dfrac{x - y}{x + y}$

**22.** $f(x, y) = x^y$

**23.** $w = \sin \alpha \cos \beta$

**24.** $w = e^v/(u + v^2)$

**25.** $f(r, s) = r \ln(r^2 + s^2)$

**26.** $f(x, t) = \arctan\left(x\sqrt{t}\right)$

**27.** $u = te^{w/t}$

**28.** $f(x, y) = \displaystyle\int_y^x \cos(t^2)\, dt$

**29.** $f(x, y, z) = xz - 5x^2 y^3 z^4$

**30.** $f(x, y, z) = x \sin(y - z)$

**31.** $w = \ln(x + 2y + 3z)$

**32.** $w = ze^{xyz}$

**33.** $u = xy \sin^{-1}(yz)$

**34.** $u = x^{y/z}$

**35.** $f(x, y, z, t) = xyz^2 \tan(yt)$

**36.** $f(x, y, z, t) = \dfrac{xy^2}{t + 2z}$

**37.** $u = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$

**38.** $u = \sin(x_1 + 2x_2 + \cdots + nx_n)$

**39–42** Find the indicated partial derivatives.

**39.** $f(x, y) = \ln\left(x + \sqrt{x^2 + y^2}\right)$; $f_x(3, 4)$

**40.** $f(x, y) = \arctan(y/x)$; $f_x(2, 3)$

**41.** $f(x, y, z) = \dfrac{y}{x + y + z}$; $f_y(2, 1, -1)$

**42.** $f(x, y, z) = \sqrt{\sin^2 x + \sin^2 y + \sin^2 z}$; $f_z(0, 0, \pi/4)$

**43–44** Use the definition of partial derivatives as limits (4) to find $f_x(x, y)$ and $f_y(x, y)$.

**43.** $f(x, y) = xy^2 - x^3 y$

**44.** $f(x, y) = \dfrac{x}{x + y^2}$

**45–48** Use implicit differentiation to find $\partial z/\partial x$ and $\partial z/\partial y$.

**45.** $x^2 + y^2 + z^2 = 3xyz$

**46.** $yz = \ln(x + z)$

**47.** $x - z = \arctan(yz)$

**48.** $\sin(xyz) = x + 2y + 3z$

**49–50** Find $\partial z/\partial x$ and $\partial z/\partial y$.

**49.** (a) $z = f(x) + g(y)$ (b) $z = f(x + y)$

**50.** (a) $z = f(x)g(y)$ (b) $z = f(xy)$
(c) $z = f(x/y)$

**51–56** Find all the second partial derivatives.

**51.** $f(x, y) = x^3 y^5 + 2x^4 y$

**52.** $f(x, y) = \sin^2(mx + ny)$

**53.** $w = \sqrt{u^2 + v^2}$

**54.** $v = \dfrac{xy}{x - y}$

**55.** $z = \arctan \dfrac{x + y}{1 - xy}$

**56.** $v = e^{xe^y}$

**57–60** Verify that the conclusion of Clairaut's Theorem holds, that is, $u_{xy} = u_{yx}$.

**57.** $u = x \sin(x + 2y)$

**58.** $u = x^4 y^2 - 2xy^5$

**59.** $u = \ln \sqrt{x^2 + y^2}$

**60.** $u = xye^y$

**61–68** Find the indicated partial derivative.

**61.** $f(x, y) = 3xy^4 + x^3 y^2$; $f_{xxy}$, $f_{yyy}$

**62.** $f(x, t) = x^2 e^{-ct}$; $f_{ttt}$, $f_{txx}$

**63.** $f(x, y, z) = \cos(4x + 3y + 2z)$; $f_{xyz}$, $f_{yzz}$

**64.** $f(r, s, t) = r \ln(rs^2 t^3)$; $f_{rss}$, $f_{rst}$

**65.** $u = e^{r\theta} \sin \theta$; $\dfrac{\partial^3 u}{\partial r^2 \partial \theta}$

**66.** $z = u\sqrt{v - w}$; $\dfrac{\partial^3 z}{\partial u\, \partial v\, \partial w}$

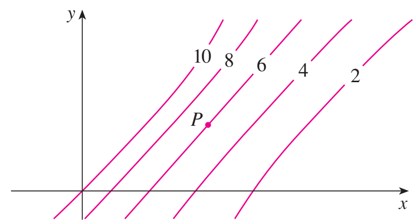
**67.** $w = \dfrac{x}{y + 2z}$; $\dfrac{\partial^3 w}{\partial z\, \partial y\, \partial x}$, $\dfrac{\partial^3 w}{\partial x^2\, \partial y}$

**68.** $u = x^a y^b z^c$; $\dfrac{\partial^6 u}{\partial x\, \partial y^2\, \partial z^3}$

**69.** Use the table of values of $f(x, y)$ to estimate the values of $f_x(3, 2)$, $f_x(3, 2.2)$, and $f_{xy}(3, 2)$.

| $x \backslash y$ | 1.8 | 2.0 | 2.2 |
|---|---|---|---|
| 2.5 | 12.5 | 10.2 | 9.3 |
| 3.0 | 18.1 | 17.5 | 15.9 |
| 3.5 | 20.0 | 22.4 | 26.1 |

**70.** Level curves are shown for a function $f$. Determine whether the following partial derivatives are positive or negative at the point $P$.
(a) $f_x$ (b) $f_y$ (c) $f_{xx}$
(d) $f_{xy}$ (e) $f_{yy}$




**71.** Verify that the function $u = e^{-\alpha^2 k^2 t} \sin kx$ is a solution of the *heat conduction equation* $u_t = \alpha^2 u_{xx}$.

**72.** Determine whether each of the following functions is a solution of Laplace's equation $u_{xx} + u_{yy} = 0$.
(a) $u = x^2 + y^2$ (b) $u = x^2 - y^2$
(c) $u = x^3 + 3xy^2$ (d) $u = \ln \sqrt{x^2 + y^2}$
(e) $u = \sin x \cosh y + \cos x \sinh y$
(f) $u = e^{-x} \cos y - e^{-y} \cos x$

**73.** Verify that the function $u = 1/\sqrt{x^2 + y^2 + z^2}$ is a solution of the three-dimensional Laplace equation $u_{xx} + u_{yy} + u_{zz} = 0$.

**74.** Show that each of the following functions is a solution of the wave equation $u_{tt} = a^2 u_{xx}$.
(a) $u = \sin(kx)\sin(akt)$ (b) $u = t/(a^2 t^2 - x^2)$
(c) $u = (x - at)^6 + (x + at)^6$
(d) $u = \sin(x - at) + \ln(x + at)$

**75.** If $f$ and $g$ are twice differentiable functions of a single variable, show that the function

$$u(x, t) = f(x + at) + g(x - at)$$

is a solution of the wave equation given in Exercise 74.

**76.** If $u = e^{a_1 x_1 + a_2 x_2 + \cdots + a_n x_n}$, where $a_1^2 + a_2^2 + \cdots + a_n^2 = 1$, show that

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} = u$$

**77.** Verify that the function $z = \ln(e^x + e^y)$ is a solution of the differential equations

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 1$$

and
$$\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x\,\partial y}\right)^2 = 0$$

**78.** Show that the Cobb-Douglas production function $P = bL^{\alpha}K^{\beta}$ satisfies the equation
$$L\frac{\partial P}{\partial L} + K\frac{\partial P}{\partial K} = (\alpha + \beta)P$$

**79.** Show that the Cobb-Douglas production function satisfies $P(L, K_0) = C_1(K_0)L^{\alpha}$ by solving the differential equation
$$\frac{dP}{dL} = \alpha\frac{P}{L}$$
(See Equation 5.)

**80.** The temperature at a point $(x, y)$ on a flat metal plate is given by $T(x, y) = 60/(1 + x^2 + y^2)$, where $T$ is measured in °C and $x$, $y$ in meters. Find the rate of change of temperature with respect to distance at the point $(2, 1)$ in (a) the $x$-direction and (b) the $y$-direction.

**81.** The total resistance $R$ produced by three conductors with resistances $R_1$, $R_2$, $R_3$ connected in a parallel electrical circuit is given by the formula
$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$
Find $\partial R/\partial R_1$.

**82.** The gas law for a fixed mass $m$ of an ideal gas at absolute temperature $T$, pressure $P$, and volume $V$ is $PV = mRT$, where $R$ is the gas constant. Show that
$$\frac{\partial P}{\partial V}\frac{\partial V}{\partial T}\frac{\partial T}{\partial P} = -1$$

**83.** For the ideal gas of Exercise 82, show that
$$T\frac{\partial P}{\partial T}\frac{\partial V}{\partial T} = mR$$

**84.** The wind-chill index is modeled by the function
$$W = 13.12 + 0.6215T - 11.37v^{0.16} + 0.3965Tv^{0.16}$$
where $T$ is the temperature (°C) and $v$ is the wind speed (km/h). When $T = -15$°C and $v = 30$ km/h, by how much would you expect the apparent temperature $W$ to drop if the actual temperature decreases by 1°C? What if the wind speed increases by 1 km/h?

**85.** The kinetic energy of a body with mass $m$ and velocity $v$ is $K = \frac{1}{2}mv^2$. Show that
$$\frac{\partial K}{\partial m}\frac{\partial^2 K}{\partial v^2} = K$$

**86.** If $a$, $b$, $c$ are the sides of a triangle and $A$, $B$, $C$ are the opposite angles, find $\partial A/\partial a$, $\partial A/\partial b$, $\partial A/\partial c$ by implicit differentiation of the Law of Cosines.

**87.** You are told that there is a function $f$ whose partial derivatives are $f_x(x, y) = x + 4y$ and $f_y(x, y) = 3x - y$. Should you believe it?

**88.** The paraboloid $z = 6 - x - x^2 - 2y^2$ intersects the plane $x = 1$ in a parabola. Find parametric equations for the tangent line to this parabola at the point $(1, 2, -4)$. Use a computer to graph the paraboloid, the parabola, and the tangent line on the same screen.

**89.** The ellipsoid $4x^2 + 2y^2 + z^2 = 16$ intersects the plane $y = 2$ in an ellipse. Find parametric equations for the tangent line to this ellipse at the point $(1, 2, 2)$.

**90.** In a study of frost penetration it was found that the temperature $T$ at time $t$ (measured in days) at a depth $x$ (measured in feet) can be modeled by the function
$$T(x, t) = T_0 + T_1 e^{-\lambda x}\sin(\omega t - \lambda x)$$
where $\omega = 2\pi/365$ and $\lambda$ is a positive constant.

(a) Find $\partial T/\partial x$. What is its physical significance?
(b) Find $\partial T/\partial t$. What is its physical significance?
(c) Show that $T$ satisfies the heat equation $T_t = kT_{xx}$ for a certain constant $k$.
(d) If $\lambda = 0.2$, $T_0 = 0$, and $T_1 = 10$, use a computer to graph $T(x, t)$.
(e) What is the physical significance of the term $-\lambda x$ in the expression $\sin(\omega t - \lambda x)$?

**91.** Use Clairaut's Theorem to show that if the third-order partial derivatives of $f$ are continuous, then
$$f_{xyy} = f_{yxy} = f_{yyx}$$

**92.** (a) How many $n$th-order partial derivatives does a function of two variables have?
(b) If these partial derivatives are all continuous, how many of them can be distinct?
(c) Answer the question in part (a) for a function of three variables.

**93.** If $f(x, y) = x(x^2 + y^2)^{-3/2}e^{\sin(x^2 y)}$, find $f_x(1, 0)$. [*Hint:* Instead of finding $f_x(x, y)$ first, note that it's easier to use Equation 1 or Equation 2.]

**94.** If $f(x, y) = \sqrt[3]{x^3 + y^3}$, find $f_x(0, 0)$.

**95.** Let
$$f(x, y) = \begin{cases} \dfrac{x^3y - xy^3}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

(a) Use a computer to graph $f$.
(b) Find $f_x(x, y)$ and $f_y(x, y)$ when $(x, y) \neq (0, 0)$.
(c) Find $f_x(0, 0)$ and $f_y(0, 0)$ using Equations 2 and 3.
(d) Show that $f_{xy}(0, 0) = -1$ and $f_{yx}(0, 0) = 1$.
(e) CAS Does the result of part (d) contradict Clairaut's Theorem? Use graphs of $f_{xy}$ and $f_{yx}$ to illustrate your answer.

## 14.4 EXERCISES

**1–6** Find an equation of the tangent plane to the given surface at the specified point.

**1.** $z = 4x^2 - y^2 + 2y, \quad (-1, 2, 4)$

**2.** $z = 3(x - 1)^2 + 2(y + 3)^2 + 7, \quad (2, -2, 12)$

**3.** $z = \sqrt{xy}, \quad (1, 1, 1)$

**4.** $z = y \ln x, \quad (1, 4, 0)$

**5.** $z = y \cos(x - y), \quad (2, 2, 2)$

**6.** $z = e^{x^2 - y^2}, \quad (1, -1, 1)$

**7–8** Graph the surface and the tangent plane at the given point. (Choose the domain and viewpoint so that you get a good view of both the surface and the tangent plane.) Then zoom in until the surface and the tangent plane become indistinguishable.

**7.** $z = x^2 + xy + 3y^2, \quad (1, 1, 5)$

**8.** $z = \arctan(xy^2), \quad (1, 1, \pi/4)$

CAS **9–10** Draw the graph of $f$ and its tangent plane at the given point. (Use your computer algebra system both to compute the partial derivatives and to graph the surface and its tangent plane.) Then zoom in until the surface and the tangent plane become indistinguishable.

**9.** $f(x, y) = \dfrac{xy \sin(x - y)}{1 + x^2 + y^2}, \quad (1, 1, 0)$

**10.** $f(x, y) = e^{-xy/10}(\sqrt{x} + \sqrt{y} + \sqrt{xy}), \quad (1, 1, 3e^{-0.1})$

**11–16** Explain why the function is differentiable at the given point. Then find the linearization $L(x, y)$ of the function at that point.

**11.** $f(x, y) = x\sqrt{y}, \quad (1, 4)$

**12.** $f(x, y) = x^3 y^4, \quad (1, 1)$

**13.** $f(x, y) = \dfrac{x}{x + y}, \quad (2, 1)$

**14.** $f(x, y) = \sqrt{x + e^{4y}}, \quad (3, 0)$

**15.** $f(x, y) = e^{-xy} \cos y, \quad (\pi, 0)$

**16.** $f(x, y) = \sin(2x + 3y), \quad (-3, 2)$

**17–18** Verify the linear approximation at (0, 0).

**17.** $\dfrac{2x + 3}{4y + 1} \approx 3 + 2x - 12y$

**18.** $\sqrt{y + \cos^2 x} \approx 1 + \frac{1}{2}y$

**19.** Find the linear approximation of the function $f(x, y) = \sqrt{20 - x^2 - 7y^2}$ at (2, 1) and use it to approximate $f(1.95, 1.08)$.

**20.** Find the linear approximation of the function $f(x, y) = \ln(x - 3y)$ at (7, 2) and use it to approximate $f(6.9, 2.06)$. Illustrate by graphing $f$ and the tangent plane.

**21.** Find the linear approximation of the function $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ at (3, 2, 6) and use it to approximate the number $\sqrt{(3.02)^2 + (1.97)^2 + (5.99)^2}$.

**22.** The wave heights $h$ in the open sea depend on the speed $v$ of the wind and the length of time $t$ that the wind has been blowing at that speed. Values of the function $h = f(v, t)$ are recorded in feet in the following table.

Duration (hours)

| Wind speed (knots) $v$ \ $t$ | 5 | 10 | 15 | 20 | 30 | 40 | 50 |
|---|---|---|---|---|---|---|---|
| 20 | 5 | 7 | 8 | 8 | 9 | 9 | 9 |
| 30 | 9 | 13 | 16 | 17 | 18 | 19 | 19 |
| 40 | 14 | 21 | 25 | 28 | 31 | 33 | 33 |
| 50 | 19 | 29 | 36 | 40 | 45 | 48 | 50 |
| 60 | 24 | 37 | 47 | 54 | 62 | 67 | 69 |

Use the table to find a linear approximation to the wave height function when $v$ is near 40 knots and $t$ is near 20 hours. Then estimate the wave heights when the wind has been blowing for 24 hours at 43 knots.

**23.** Use the table in Example 3 to find a linear approximation to the heat index function when the temperature is near 94°F and the relative humidity is near 80%. Then estimate the heat index when the temperature is 95°F and the relative humidity is 78%.

**24.** The wind-chill index $W$ is the perceived temperature when the actual temperature is $T$ and the wind speed is $v$, so we can write $W = f(T, v)$. The following table of values is an excerpt from Table 1 in Section 14.1.

Wind speed (km/h)

| Actual temperature (°C) $T$ \ $v$ | 20 | 30 | 40 | 50 | 60 | 70 |
|---|---|---|---|---|---|---|
| −10 | −18 | −20 | −21 | −22 | −23 | −23 |
| −15 | −24 | −26 | −27 | −29 | −30 | −30 |
| −20 | −30 | −33 | −34 | −35 | −36 | −37 |
| −25 | −37 | −39 | −41 | −42 | −43 | −44 |

Use the table to find a linear approximation to the wind-chill index function when $T$ is near $-15°C$ and $v$ is near 50 km/h. Then estimate the wind-chill index when the temperature is $-17°C$ and the wind speed is 55 km/h.

**25–30** Find the differential of the function.

**25.** $z = x^3 \ln(y^2)$

**26.** $v = y \cos xy$

**27.** $m = p^5 q^3$

**28.** $T = \dfrac{v}{1 + uvw}$

**29.** $R = \alpha\beta^2 \cos \gamma$

**30.** $w = xye^{xz}$

**31.** If $z = 5x^2 + y^2$ and $(x, y)$ changes from $(1, 2)$ to $(1.05, 2.1)$, compare the values of $\Delta z$ and $dz$.

**32.** If $z = x^2 - xy + 3y^2$ and $(x, y)$ changes from $(3, -1)$ to $(2.96, -0.95)$, compare the values of $\Delta z$ and $dz$.

**33.** The length and width of a rectangle are measured as 30 cm and 24 cm, respectively, with an error in measurement of at most 0.1 cm in each. Use differentials to estimate the maximum error in the calculated area of the rectangle.

**34.** The dimensions of a closed rectangular box are measured as 80 cm, 60 cm, and 50 cm, respectively, with a possible error of 0.2 cm in each dimension. Use differentials to estimate the maximum error in calculating the surface area of the box.

**35.** Use differentials to estimate the amount of tin in a closed tin can with diameter 8 cm and height 12 cm if the tin is 0.04 cm thick.

**36.** Use differentials to estimate the amount of metal in a closed cylindrical can that is 10 cm high and 4 cm in diameter if the metal in the top and bottom is 0.1 cm thick and the metal in the sides is 0.05 cm thick.

**37.** A boundary stripe 3 in. wide is painted around a rectangle whose dimensions are 100 ft by 200 ft. Use differentials to approximate the number of square feet of paint in the stripe.

**38.** The pressure, volume, and temperature of a mole of an ideal gas are related by the equation $PV = 8.31T$, where $P$ is measured in kilopascals, $V$ in liters, and $T$ in kelvins. Use differentials to find the approximate change in the pressure if the volume increases from 12 L to 12.3 L and the temperature decreases from 310 K to 305 K.

**39.** If $R$ is the total resistance of three resistors, connected in parallel, with resistances $R_1, R_2, R_3$, then

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

If the resistances are measured in ohms as $R_1 = 25\ \Omega$, $R_2 = 40\ \Omega$, and $R_3 = 50\ \Omega$, with a possible error of 0.5% in each case, estimate the maximum error in the calculated value of $R$.

**40.** Four positive numbers, each less than 50, are rounded to the first decimal place and then multiplied together. Use differentials to estimate the maximum possible error in the computed product that might result from the rounding.

**41.** A model for the surface area of a human body is given by $S = 0.1091w^{0.425}h^{0.725}$, where $w$ is the weight (in pounds), $h$ is the height (in inches), and $S$ is measured in square feet. If the errors in measurement of $w$ and $h$ are at most 2%, use differentials to estimate the maximum percentage error in the calculated surface area.

**42.** Suppose you need to know an equation of the tangent plane to a surface $S$ at the point $P(2, 1, 3)$. You don't have an equation for $S$ but you know that the curves

$$\mathbf{r}_1(t) = \langle 2 + 3t,\ 1 - t^2,\ 3 - 4t + t^2 \rangle$$

$$\mathbf{r}_2(u) = \langle 1 + u^2,\ 2u^3 - 1,\ 2u + 1 \rangle$$

both lie on $S$. Find an equation of the tangent plane at $P$.

**43–44** Show that the function is differentiable by finding values of $\varepsilon_1$ and $\varepsilon_2$ that satisfy Definition 7.

**43.** $f(x, y) = x^2 + y^2$

**44.** $f(x, y) = xy - 5y^2$

**45.** Prove that if $f$ is a function of two variables that is differentiable at $(a, b)$, then $f$ is continuous at $(a, b)$.
*Hint:* Show that

$$\lim_{(\Delta x, \Delta y) \to (0, 0)} f(a + \Delta x, b + \Delta y) = f(a, b)$$

**46.** (a) The function

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

was graphed in Figure 4. Show that $f_x(0, 0)$ and $f_y(0, 0)$ both exist but $f$ is not differentiable at $(0, 0)$. [*Hint:* Use the result of Exercise 45.]

(b) Explain why $f_x$ and $f_y$ are not continuous at $(0, 0)$.

**EXAMPLE 9** Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ if $x^3 + y^3 + z^3 + 6xyz = 1$.

SOLUTION Let $F(x, y, z) = x^3 + y^3 + z^3 + 6xyz - 1$. Then, from Equations 7, we have

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{3x^2 + 6yz}{3z^2 + 6xy} = -\frac{x^2 + 2yz}{z^2 + 2xy}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{3y^2 + 6xz}{3z^2 + 6xy} = -\frac{y^2 + 2xz}{z^2 + 2xy}$$

■ The solution to Example 9 should be compared to the one in Example 4 in Section 14.3.

## 14.5 EXERCISES

**1–6** Use the Chain Rule to find $dz/dt$ or $dw/dt$.

**1.** $z = x^2 + y^2 + xy, \quad x = \sin t, \quad y = e^t$

**2.** $z = \cos(x + 4y), \quad x = 5t^4, \quad y = 1/t$

**3.** $z = \sqrt{1 + x^2 + y^2}, \quad x = \ln t, \quad y = \cos t$

**4.** $z = \tan^{-1}(y/x), \quad x = e^t, \quad y = 1 - e^{-t}$

**5.** $w = xe^{y/z}, \quad x = t^2, \quad y = 1 - t, \quad z = 1 + 2t$

**6.** $w = \ln\sqrt{x^2 + y^2 + z^2}, \quad x = \sin t, \quad y = \cos t, \quad z = \tan t$

**7–12** Use the Chain Rule to find $\partial z/\partial s$ and $\partial z/\partial t$.

**7.** $z = x^2y^3, \quad x = s\cos t, \quad y = s\sin t$

**8.** $z = \arcsin(x - y), \quad x = s^2 + t^2, \quad y = 1 - 2st$

**9.** $z = \sin\theta\cos\phi, \quad \theta = st^2, \quad \phi = s^2t$

**10.** $z = e^{x+2y}, \quad x = s/t, \quad y = t/s$

**11.** $z = e^r\cos\theta, \quad r = st, \quad \theta = \sqrt{s^2 + t^2}$

**12.** $z = \tan(u/v), \quad u = 2s + 3t, \quad v = 3s - 2t$

**13.** If $z = f(x, y)$, where $f$ is differentiable, and

$$\begin{array}{ll} x = g(t) & y = h(t) \\ g(3) = 2 & h(3) = 7 \\ g'(3) = 5 & h'(3) = -4 \\ f_x(2, 7) = 6 & f_y(2, 7) = -8 \end{array}$$

find $dz/dt$ when $t = 3$.

**14.** Let $W(s, t) = F(u(s, t), v(s, t))$, where $F$, $u$, and $v$ are differentiable, and

$$\begin{array}{ll} u(1, 0) = 2 & v(1, 0) = 3 \\ u_s(1, 0) = -2 & v_s(1, 0) = 5 \\ u_t(1, 0) = 6 & v_t(1, 0) = 4 \\ F_u(2, 3) = -1 & F_v(2, 3) = 10 \end{array}$$

Find $W_s(1, 0)$ and $W_t(1, 0)$.

**15.** Suppose $f$ is a differentiable function of $x$ and $y$, and $g(u, v) = f(e^u + \sin v, e^u + \cos v)$. Use the table of values to calculate $g_u(0, 0)$ and $g_v(0, 0)$.

| | $f$ | $g$ | $f_x$ | $f_y$ |
|---|---|---|---|---|
| (0, 0) | 3 | 6 | 4 | 8 |
| (1, 2) | 6 | 3 | 2 | 5 |

**16.** Suppose $f$ is a differentiable function of $x$ and $y$, and $g(r, s) = f(2r - s, s^2 - 4r)$. Use the table of values in Exercise 15 to calculate $g_r(1, 2)$ and $g_s(1, 2)$.

**17–20** Use a tree diagram to write out the Chain Rule for the given case. Assume all functions are differentiable.

**17.** $u = f(x, y)$, where $x = x(r, s, t)$, $y = y(r, s, t)$

**18.** $R = f(x, y, z, t)$, where $x = x(u, v, w)$, $y = y(u, v, w)$, $z = z(u, v, w)$, $t = t(u, v, w)$

**19.** $w = f(r, s, t)$, where $r = r(x, y)$, $s = s(x, y)$, $t = t(x, y)$

**20.** $t = f(u, v, w)$, where $u = u(p, q, r, s)$, $v = v(p, q, r, s)$, $w = w(p, q, r, s)$

**21–26** Use the Chain Rule to find the indicated partial derivatives.

**21.** $z = x^2 + xy^3, \quad x = uv^2 + w^3, \quad y = u + ve^w;$

$\dfrac{\partial z}{\partial u}, \dfrac{\partial z}{\partial v}, \dfrac{\partial z}{\partial w}$ when $u = 2, v = 1, w = 0$

**22.** $u = \sqrt{r^2 + s^2}, \quad r = y + x\cos t, \quad s = x + y\sin t;$

$\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial u}{\partial t}$ when $x = 1, y = 2, t = 0$

**23.** $R = \ln(u^2 + v^2 + w^2),$

$u = x + 2y, \quad v = 2x - y, \quad w = 2xy;$

$\dfrac{\partial R}{\partial x}, \dfrac{\partial R}{\partial y}$ when $x = y = 1$

**24.** $M = xe^{y-z^2}, \quad x = 2uv, \quad y = u - v, \quad z = u + v;$

$\dfrac{\partial M}{\partial u}, \dfrac{\partial M}{\partial v}$ when $u = 3, v = -1$

**25.** $u = x^2 + yz, \quad x = pr\cos\theta, \quad y = pr\sin\theta, \quad z = p + r;$

$\dfrac{\partial u}{\partial p}, \dfrac{\partial u}{\partial r}, \dfrac{\partial u}{\partial \theta}$ when $p = 2, r = 3, \theta = 0$

**26.** $Y = w\tan^{-1}(uv), \quad u = r + s, \quad v = s + t, \quad w = t + r;$

$\dfrac{\partial Y}{\partial r}, \dfrac{\partial Y}{\partial s}, \dfrac{\partial Y}{\partial t}$ when $r = 1, s = 0, t = 1$

**27–30** Use Equation 6 to find $dy/dx$.

**27.** $\sqrt{xy} = 1 + x^2y$

**28.** $y^5 + x^2y^3 = 1 + ye^{x^2}$

**29.** $\cos(x - y) = xe^y$

**30.** $\sin x + \cos y = \sin x \cos y$

**31–34** Use Equations 7 to find $\partial z/\partial x$ and $\partial z/\partial y$.

**31.** $x^2 + y^2 + z^2 = 3xyz$

**32.** $xyz = \cos(x + y + z)$

**33.** $x - z = \arctan(yz)$

**34.** $yz = \ln(x + z)$

**35.** The temperature at a point $(x, y)$ is $T(x, y)$, measured in degrees Celsius. A bug crawls so that its position after $t$ seconds is given by $x = \sqrt{1 + t}$, $y = 2 + \frac{1}{3}t$, where $x$ and $y$ are measured in centimeters. The temperature function satisfies $T_x(2, 3) = 4$ and $T_y(2, 3) = 3$. How fast is the temperature rising on the bug's path after 3 seconds?

**36.** Wheat production $W$ in a given year depends on the average temperature $T$ and the annual rainfall $R$. Scientists estimate that the average temperature is rising at a rate of 0.15°C/year and rainfall is decreasing at a rate of 0.1 cm/year. They also estimate that, at current production levels, $\partial W/\partial T = -2$ and $\partial W/\partial R = 8$.

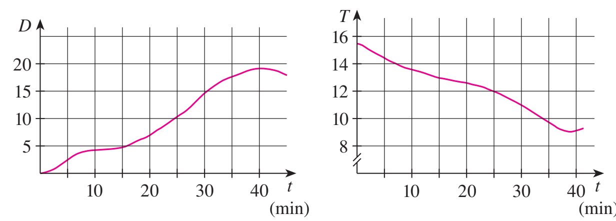
(a) What is the significance of the signs of these partial derivatives?

(b) Estimate the current rate of change of wheat production, $dW/dt$.

**37.** The speed of sound traveling through ocean water with salinity 35 parts per thousand has been modeled by the equation

$$C = 1449.2 + 4.6T - 0.055T^2 + 0.00029T^3 + 0.016D$$

where $C$ is the speed of sound (in meters per second), $T$ is the temperature (in degrees Celsius), and $D$ is the depth below the ocean surface (in meters). A scuba diver began a leisurely dive into the ocean water; the diver's depth and the surrounding water temperature over time are recorded in the following graphs. Estimate the rate of change (with respect to time) of the speed of sound through the ocean water experienced by the diver 20 minutes into the dive. What are the units?




**38.** The radius of a right circular cone is increasing at a rate of 1.8 in/s while its height is decreasing at a rate of 2.5 in/s. At what rate is the volume of the cone changing when the radius is 120 in. and the height is 140 in.?

**39.** The length $\ell$, width $w$, and height $h$ of a box change with time. At a certain instant the dimensions are $\ell = 1$ m and $w = h = 2$ m, and $\ell$ and $w$ are increasing at a rate of 2 m/s while $h$ is decreasing at a rate of 3 m/s. At that instant find the rates at which the following quantities are changing.

(a) The volume

(b) The surface area

(c) The length of a diagonal

**40.** The voltage $V$ in a simple electrical circuit is slowly decreasing as the battery wears out. The resistance $R$ is slowly increasing as the resistor heats up. Use Ohm's Law, $V = IR$, to find how the current $I$ is changing at the moment when $R = 400\ \Omega$, $I = 0.08$ A, $dV/dt = -0.01$ V/s, and $dR/dt = 0.03\ \Omega/\text{s}$.

**41.** The pressure of 1 mole of an ideal gas is increasing at a rate of 0.05 kPa/s and the temperature is increasing at a rate of 0.15 K/s. Use the equation in Example 2 to find the rate of change of the volume when the pressure is 20 kPa and the temperature is 320 K.

**42.** Car A is traveling north on Highway 16 and car B is traveling west on Highway 83. Each car is approaching the intersection of these highways. At a certain moment, car A is 0.3 km from the intersection and traveling at 90 km/h while car B is 0.4 km from the intersection and traveling at 80 km/h. How fast is the distance between the cars changing at that moment?

**43.** One side of a triangle is increasing at a rate of 3 cm/s and a second side is decreasing at a rate of 2 cm/s. If the area of the triangle remains constant, at what rate does the angle between the sides change when the first side is 20 cm long, the second side is 30 cm, and the angle is $\pi/6$?

**44.** If a sound with frequency $f_s$ is produced by a source traveling along a line with speed $v_s$ and an observer is traveling with speed $v_o$ along the same line from the opposite direction toward the source, then the frequency of the sound heard by the observer is

$$f_o = \left(\frac{c + v_o}{c - v_s}\right) f_s$$

where $c$ is the speed of sound, about 332 m/s. (This is the **Doppler effect**.) Suppose that, at a particular moment, you are in a train traveling at 34 m/s and accelerating at 1.2 m/s$^2$. A train is approaching you from the opposite direction on the other track at 40 m/s, accelerating at 1.4 m/s$^2$, and sounds its whistle, which has a frequency of 460 Hz. At that instant, what is the perceived frequency that you hear and how fast is it changing?

**45–48** Assume that all the given functions are differentiable.

**45.** If $z = f(x, y)$, where $x = r\cos\theta$ and $y = r\sin\theta$, (a) find $\partial z/\partial r$ and $\partial z/\partial\theta$ and (b) show that

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$$

**46.** If $u = f(x, y)$, where $x = e^s\cos t$ and $y = e^s\sin t$, show that

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = e^{-2s}\left[\left(\frac{\partial u}{\partial s}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2\right]$$

**47.** If $z = f(x - y)$, show that $\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = 0$.

**48.** If $z = f(x, y)$, where $x = s + t$ and $y = s - t$, show that

$$\left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 = \frac{\partial z}{\partial s}\frac{\partial z}{\partial t}$$

**49–54** Assume that all the given functions have continuous second-order partial derivatives.

**49.** Show that any function of the form

$$z = f(x + at) + g(x - at)$$

is a solution of the wave equation

$$\frac{\partial^2 z}{\partial t^2} = a^2\frac{\partial^2 z}{\partial x^2}$$

[*Hint:* Let $u = x + at$, $v = x - at$.]

**50.** If $u = f(x, y)$, where $x = e^s\cos t$ and $y = e^s\sin t$, show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^{-2s}\left[\frac{\partial^2 u}{\partial s^2} + \frac{\partial^2 u}{\partial t^2}\right]$$

**51.** If $z = f(x, y)$, where $x = r^2 + s^2$ and $y = 2rs$, find $\partial^2 z/\partial r\,\partial s$. (Compare with Example 7.)

**52.** If $z = f(x, y)$, where $x = r\cos\theta$ and $y = r\sin\theta$, find (a) $\partial z/\partial r$, (b) $\partial z/\partial\theta$, and (c) $\partial^2 z/\partial r\,\partial\theta$.

**53.** If $z = f(x, y)$, where $x = r\cos\theta$ and $y = r\sin\theta$, show that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 z}{\partial\theta^2} + \frac{1}{r}\frac{\partial z}{\partial r}$$

**54.** Suppose $z = f(x, y)$, where $x = g(s, t)$ and $y = h(s, t)$.
(a) Show that

$$\frac{\partial^2 z}{\partial t^2} = \frac{\partial^2 z}{\partial x^2}\left(\frac{\partial x}{\partial t}\right)^2 + 2\frac{\partial^2 z}{\partial x\,\partial y}\frac{\partial x}{\partial t}\frac{\partial y}{\partial t} + \frac{\partial^2 z}{\partial y^2}\left(\frac{\partial y}{\partial t}\right)^2 + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial t^2} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial t^2}$$

(b) Find a similar formula for $\partial^2 z/\partial s\,\partial t$.

**55.** A function $f$ is called ***homogeneous of degree n*** if it satisfies the equation $f(tx, ty) = t^n f(x, y)$ for all $t$, where $n$ is a positive integer and $f$ has continuous second-order partial derivatives.
(a) Verify that $f(x, y) = x^2y + 2xy^2 + 5y^3$ is homogeneous of degree 3.
(b) Show that if $f$ is homogeneous of degree $n$, then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x, y)$$

[*Hint:* Use the Chain Rule to differentiate $f(tx, ty)$ with respect to $t$.]

**56.** If $f$ is homogeneous of degree $n$, show that

$$x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x\,\partial y} + y^2\frac{\partial^2 f}{\partial y^2} = n(n-1)f(x, y)$$

**57.** If $f$ is homogeneous of degree $n$, show that

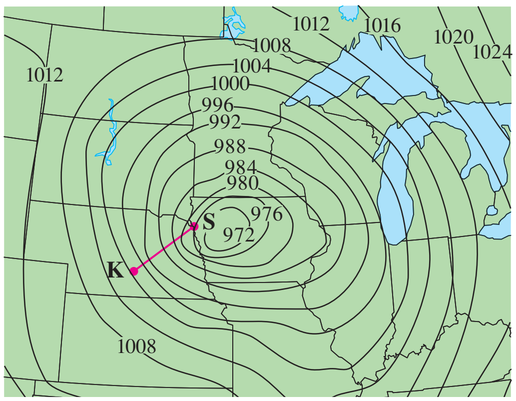
$$f_x(tx, ty) = t^{n-1}f_x(x, y)$$

**58.** Suppose that the equation $F(x, y, z) = 0$ implicitly defines each of the three variables $x$, $y$, and $z$ as functions of the other two: $z = f(x, y)$, $y = g(x, z)$, $x = h(y, z)$. If $F$ is differentiable and $F_x$, $F_y$, and $F_z$ are all nonzero, show that

$$\frac{\partial z}{\partial x}\frac{\partial x}{\partial y}\frac{\partial y}{\partial z} = -1$$

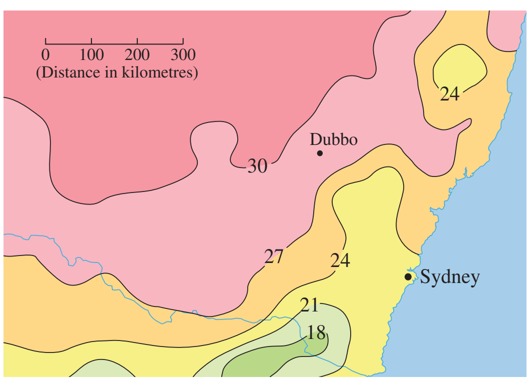
## 14.6 EXERCISES

1. Level curves for barometric pressure (in millibars) are shown for 6:00 AM on November 10, 1998. A deep low with pressure 972 mb is moving over northeast Iowa. The distance along the red line from K (Kearney, Nebraska) to S (Sioux City, Iowa) is 300 km. Estimate the value of the directional derivative of the pressure function at Kearney in the direction of Sioux City. What are the units of the directional derivative?




From *Meteorology Today*, 8E by C. Donald Ahrens (2007 Thomson Brooks/Cole).

2. The contour map shows the average maximum temperature for November 2004 (in °C). Estimate the value of the directional derivative of this temperature function at Dubbo, New South Wales, in the direction of Sydney. What are the units?

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3. A table of values for the wind-chill index $W = f(T, v)$ is given in Exercise 3 on page 888. Use the table to estimate the value of $D_{\mathbf{u}} f(-20, 30)$, where $\mathbf{u} = (\mathbf{i} + \mathbf{j})/\sqrt{2}$.

**4–6** Find the directional derivative of $f$ at the given point in the direction indicated by the angle $\theta$.

4. $f(x, y) = x^2y^3 - y^4$, $(2, 1)$, $\theta = \pi/4$

5. $f(x, y) = ye^{-x}$, $(0, 4)$, $\theta = 2\pi/3$

6. $f(x, y) = x\sin(xy)$, $(2, 0)$, $\theta = \pi/3$

**7–10**

(a) Find the gradient of $f$.

(b) Evaluate the gradient at the point $P$.

(c) Find the rate of change of $f$ at $P$ in the direction of the vector $\mathbf{u}$.

7. $f(x, y) = \sin(2x + 3y)$, $P(-6, 4)$, $\mathbf{u} = \frac{1}{2}\left(\sqrt{3}\,\mathbf{i} - \mathbf{j}\right)$

8. $f(x, y) = y^2/x$, $P(1, 2)$, $\mathbf{u} = \frac{1}{3}\left(2\mathbf{i} + \sqrt{5}\,\mathbf{j}\right)$

9. $f(x, y, z) = xe^{2yz}$, $P(3, 0, 2)$, $\mathbf{u} = \left\langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right\rangle$

10. $f(x, y, z) = \sqrt{x + yz}$, $P(1, 3, 1)$, $\mathbf{u} = \left\langle \frac{2}{7}, \frac{3}{7}, \frac{6}{7} \right\rangle$

**11–17** Find the directional derivative of the function at the given point in the direction of the vector $\mathbf{v}$.

11. $f(x, y) = 1 + 2x\sqrt{y}$, $(3, 4)$, $\mathbf{v} = \langle 4, -3 \rangle$

12. $f(x, y) = \ln(x^2 + y^2)$, $(2, 1)$, $\mathbf{v} = \langle -1, 2 \rangle$

13. $g(p, q) = p^4 - p^2q^3$, $(2, 1)$, $\mathbf{v} = \mathbf{i} + 3\mathbf{j}$

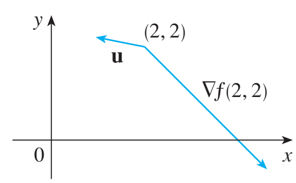
14. $g(r, s) = \tan^{-1}(rs)$, $(1, 2)$, $\mathbf{v} = 5\mathbf{i} + 10\mathbf{j}$

15. $f(x, y, z) = xe^y + ye^z + ze^x$, $(0, 0, 0)$, $\mathbf{v} = \langle 5, 1, -2 \rangle$

16. $f(x, y, z) = \sqrt{xyz}$, $(3, 2, 6)$, $\mathbf{v} = \langle -1, -2, 2 \rangle$

17. $g(x, y, z) = (x + 2y + 3z)^{3/2}$, $(1, 1, 2)$, $\mathbf{v} = 2\mathbf{j} - \mathbf{k}$

18. Use the figure to estimate $D_{\mathbf{u}} f(2, 2)$.




19. Find the directional derivative of $f(x, y) = \sqrt{xy}$ at $P(2, 8)$ in the direction of $Q(5, 4)$.

20. Find the directional derivative of $f(x, y, z) = xy + yz + zx$ at $P(1, -1, 3)$ in the direction of $Q(2, 4, 5)$.

**21–26** Find the maximum rate of change of $f$ at the given point and the direction in which it occurs.

21. $f(x, y) = y^2/x$, $(2, 4)$

22. $f(p, q) = qe^{-p} + pe^{-q}$, $(0, 0)$

23. $f(x, y) = \sin(xy)$, $(1, 0)$

24. $f(x, y, z) = (x + y)/z$, $(1, 1, -1)$

25. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$, $(3, 6, -2)$

26. $f(x, y, z) = \tan(x + 2y + 3z)$, $(-5, 1, 1)$

**27.** (a) Show that a differentiable function $f$ decreases most rapidly at $\mathbf{x}$ in the direction opposite to the gradient vector, that is, in the direction of $-\nabla f(\mathbf{x})$.
(b) Use the result of part (a) to find the direction in which the function $f(x, y) = x^4y - x^2y^3$ decreases fastest at the point $(2, -3)$.

**28.** Find the directions in which the directional derivative of $f(x, y) = ye^{-xy}$ at the point $(0, 2)$ has the value 1.

**29.** Find all points at which the direction of fastest change of the function $f(x, y) = x^2 + y^2 - 2x - 4y$ is $\mathbf{i} + \mathbf{j}$.

**30.** Near a buoy, the depth of a lake at the point with coordinates $(x, y)$ is $z = 200 + 0.02x^2 - 0.001y^3$, where $x$, $y$, and $z$ are measured in meters. A fisherman in a small boat starts at the point $(80, 60)$ and moves toward the buoy, which is located at $(0, 0)$. Is the water under the boat getting deeper or shallower when he departs? Explain.

**31.** The temperature $T$ in a metal ball is inversely proportional to the distance from the center of the ball, which we take to be the origin. The temperature at the point $(1, 2, 2)$ is $120°$.
(a) Find the rate of change of $T$ at $(1, 2, 2)$ in the direction toward the point $(2, 1, 3)$.
(b) Show that at any point in the ball the direction of greatest increase in temperature is given by a vector that points toward the origin.

**32.** The temperature at a point $(x, y, z)$ is given by

$$T(x, y, z) = 200e^{-x^2-3y^2-9z^2}$$

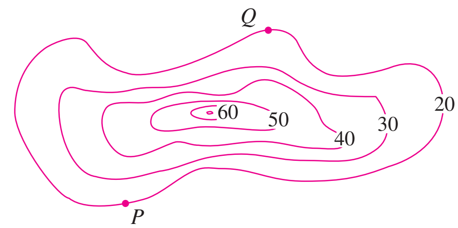
where $T$ is measured in °C and $x$, $y$, $z$ in meters.
(a) Find the rate of change of temperature at the point $P(2, -1, 2)$ in the direction toward the point $(3, -3, 3)$.
(b) In which direction does the temperature increase fastest at $P$?
(c) Find the maximum rate of increase at $P$.

**33.** Suppose that over a certain region of space the electrical potential $V$ is given by $V(x, y, z) = 5x^2 - 3xy + xyz$.
(a) Find the rate of change of the potential at $P(3, 4, 5)$ in the direction of the vector $\mathbf{v} = \mathbf{i} + \mathbf{j} - \mathbf{k}$.
(b) In which direction does $V$ change most rapidly at $P$?
(c) What is the maximum rate of change at $P$?

**34.** Suppose you are climbing a hill whose shape is given by the equation $z = 1000 - 0.005x^2 - 0.01y^2$, where $x$, $y$, and $z$ are measured in meters, and you are standing at a point with coordinates $(60, 40, 966)$. The positive $x$-axis points east and the positive $y$-axis points north.
(a) If you walk due south, will you start to ascend or descend? At what rate?
(b) If you walk northwest, will you start to ascend or descend? At what rate?
(c) In which direction is the slope largest? What is the rate of ascent in that direction? At what angle above the horizontal does the path in that direction begin?

**35.** Let $f$ be a function of two variables that has continuous partial derivatives and consider the points $A(1, 3)$, $B(3, 3)$, $C(1, 7)$, and $D(6, 15)$. The directional derivative of $f$ at $A$ in the direction of the vector $\overrightarrow{AB}$ is 3 and the directional derivative at $A$ in the direction of $\overrightarrow{AC}$ is 26. Find the directional derivative of $f$ at $A$ in the direction of the vector $\overrightarrow{AD}$.

**36.** For the given contour map draw the curves of steepest ascent starting at $P$ and at $Q$.

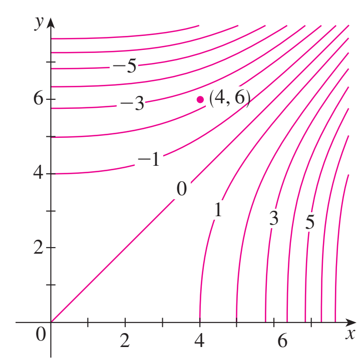



**37.** Show that the operation of taking the gradient of a function has the given property. Assume that $u$ and $v$ are differentiable functions of $x$ and $y$ and that $a$, $b$ are constants.

(a) $\nabla(au + bv) = a\,\nabla u + b\,\nabla v$ (b) $\nabla(uv) = u\,\nabla v + v\,\nabla u$

(c) $\nabla\left(\dfrac{u}{v}\right) = \dfrac{v\,\nabla u - u\,\nabla v}{v^2}$ (d) $\nabla u^n = nu^{n-1}\,\nabla u$

**38.** Sketch the gradient vector $\nabla f(4, 6)$ for the function $f$ whose level curves are shown. Explain how you chose the direction and length of this vector.




**39–44** Find equations of (a) the tangent plane and (b) the normal line to the given surface at the specified point.

**39.** $2(x - 2)^2 + (y - 1)^2 + (z - 3)^2 = 10, \quad (3, 3, 5)$

**40.** $y = x^2 - z^2, \quad (4, 7, 3)$

**41.** $x^2 - 2y^2 + z^2 + yz = 2, \quad (2, 1, -1)$

**42.** $x - z = 4\arctan(yz), \quad (1 + \pi, 1, 1)$

**43.** $z + 1 = xe^y\cos z, \quad (1, 0, 0)$

**44.** $yz = \ln(x + z), \quad (0, 0, 1)$

**45–46** Use a computer to graph the surface, the tangent plane, and the normal line on the same screen. Choose the domain carefully so that you avoid extraneous vertical planes. Choose the viewpoint so that you get a good view of all three objects.

**45.** $xy + yz + zx = 3$, $(1, 1, 1)$

**46.** $xyz = 6$, $(1, 2, 3)$

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**47.** If $f(x, y) = xy$, find the gradient vector $\nabla f(3, 2)$ and use it to find the tangent line to the level curve $f(x, y) = 6$ at the point $(3, 2)$. Sketch the level curve, the tangent line, and the gradient vector.

**48.** If $g(x, y) = x^2 + y^2 - 4x$, find the gradient vector $\nabla g(1, 2)$ and use it to find the tangent line to the level curve $g(x, y) = 1$ at the point $(1, 2)$. Sketch the level curve, the tangent line, and the gradient vector.

**49.** Show that the equation of the tangent plane to the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ at the point $(x_0, y_0, z_0)$ can be written as

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} + \frac{zz_0}{c^2} = 1$$

**50.** Find the equation of the tangent plane to the hyperboloid $x^2/a^2 + y^2/b^2 - z^2/c^2 = 1$ at $(x_0, y_0, z_0)$ and express it in a form similar to the one in Exercise 49.

**51.** Show that the equation of the tangent plane to the elliptic paraboloid $z/c = x^2/a^2 + y^2/b^2$ at the point $(x_0, y_0, z_0)$ can be written as

$$\frac{2xx_0}{a^2} + \frac{2yy_0}{b^2} = \frac{z + z_0}{c}$$

**52.** At what point on the paraboloid $y = x^2 + z^2$ is the tangent plane parallel to the plane $x + 2y + 3z = 1$?

**53.** Are there any points on the hyperboloid $x^2 - y^2 - z^2 = 1$ where the tangent plane is parallel to the plane $z = x + y$?

**54.** Show that the ellipsoid $3x^2 + 2y^2 + z^2 = 9$ and the sphere $x^2 + y^2 + z^2 - 8x - 6y - 8z + 24 = 0$ are tangent to each other at the point $(1, 1, 2)$. (This means that they have a common tangent plane at the point.)

**55.** Show that every plane that is tangent to the cone $x^2 + y^2 = z^2$ passes through the origin.

**56.** Show that every normal line to the sphere $x^2 + y^2 + z^2 = r^2$ passes through the center of the sphere.

**57.** Show that the sum of the $x$-, $y$-, and $z$-intercepts of any tangent plane to the surface $\sqrt{x} + \sqrt{y} + \sqrt{z} = \sqrt{c}$ is a constant.

**58.** Show that the pyramids cut off from the first octant by any tangent planes to the surface $xyz = 1$ at points in the first octant must all have the same volume.

**59.** Find parametric equations for the tangent line to the curve of intersection of the paraboloid $z = x^2 + y^2$ and the ellipsoid $4x^2 + y^2 + z^2 = 9$ at the point $(-1, 1, 2)$.

**60.** (a) The plane $y + z = 3$ intersects the cylinder $x^2 + y^2 = 5$ in an ellipse. Find parametric equations for the tangent line to this ellipse at the point $(1, 2, 1)$.
(b) Graph the cylinder, the plane, and the tangent line on the same screen.

**61.** (a) Two surfaces are called **orthogonal** at a point of intersection if their normal lines are perpendicular at that point. Show that surfaces with equations $F(x, y, z) = 0$ and $G(x, y, z) = 0$ are orthogonal at a point $P$ where $\nabla F \neq \mathbf{0}$ and $\nabla G \neq \mathbf{0}$ if and only if

$$F_x G_x + F_y G_y + F_z G_z = 0 \quad \text{at } P$$

(b) Use part (a) to show that the surfaces $z^2 = x^2 + y^2$ and $x^2 + y^2 + z^2 = r^2$ are orthogonal at every point of intersection. Can you see why this is true without using calculus?

**62.** (a) Show that the function $f(x, y) = \sqrt[3]{xy}$ is continuous and the partial derivatives $f_x$ and $f_y$ exist at the origin but the directional derivatives in all other directions do not exist.
(b) Graph $f$ near the origin and comment on how the graph confirms part (a).

**63.** Suppose that the directional derivatives of $f(x, y)$ are known at a given point in two nonparallel directions given by unit vectors $\mathbf{u}$ and $\mathbf{v}$. Is it possible to find $\nabla f$ at this point? If so, how would you do it?

**64.** Show that if $z = f(x, y)$ is differentiable at $\mathbf{x}_0 = \langle x_0, y_0 \rangle$, then

$$\lim_{\mathbf{x} \to \mathbf{x}_0} \frac{f(\mathbf{x}) - f(\mathbf{x}_0) - \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)}{|\mathbf{x} - \mathbf{x}_0|} = 0$$

[*Hint:* Use Definition 14.4.7 directly.]

## 14.7 MAXIMUM AND MINIMUM VALUES

As we saw in Chapter 4, one of the main uses of ordinary derivatives is in finding maximum and minimum values. In this section we see how to use partial derivatives to locate maxima and minima of functions of two variables. In particular, in Example 6 we will see how to maximize the volume of a box without a lid if we have a fixed amount of cardboard to work with.