15.7 EXERCISES

1–2 Plot the point whose cylindrical coordinates are given. Then find the rectangular coordinates of the point.

1. (a) $(2, \pi/4, 1)$ (b) $(4, -\pi/3, 5)$

2. (a) $(1, \pi, e)$ (b) $(1, 3\pi/2, 2)$

3–4 Change from rectangular to cylindrical coordinates.

3. (a) $(1, -1, 4)$ (b) $\left(-1, -\sqrt{3}, 2\right)$

4. (a) $\left(2\sqrt{3}, 2, -1\right)$ (b) $(4, -3, 2)$

5–6 Describe in words the surface whose equation is given.

5. $\theta = \pi/4$

6. $r = 5$

7–8 Identify the surface whose equation is given.

7. $z = 4 - r^2$

8. $2r^2 + z^2 = 1$

9–10 Write the equations in cylindrical coordinates.

9. (a) $z = x^2 + y^2$ (b) $x^2 + y^2 = 2y$

10. (a) $3x + 2y + z = 6$ (b) $-x^2 - y^2 + z^2 = 1$

11–12 Sketch the solid described by the given inequalities.

11. $0 \leqslant r \leqslant 2, \quad -\pi/2 \leqslant \theta \leqslant \pi/2, \quad 0 \leqslant z \leqslant 1$

12. $0 \leqslant \theta \leqslant \pi/2, \quad r \leqslant z \leqslant 2$

13. A cylindrical shell is 20 cm long, with inner radius 6 cm and outer radius 7 cm. Write inequalities that describe the shell in an appropriate coordinate system. Explain how you have positioned the coordinate system with respect to the shell.

14. Use a graphing device to draw the solid enclosed by the paraboloids $z = x^2 + y^2$ and $z = 5 - x^2 - y^2$.

15–16 Sketch the solid whose volume is given by the integral and evaluate the integral.

15. $\displaystyle\int_0^4 \int_0^{2\pi} \int_r^4 r\, dz\, d\theta\, dr$

16. $\displaystyle\int_0^{\pi/2} \int_0^2 \int_0^{9-r^2} r\, dz\, dr\, d\theta$

17–26 Use cylindrical coordinates.

17. Evaluate $\iiint_E \sqrt{x^2 + y^2}\, dV$, where E is the region that lies inside the cylinder $x^2 + y^2 = 16$ and between the planes $z = -5$ and $z = 4$.

18. Evaluate $\iiint_E (x^3 + xy^2)\, dV$, where E is the solid in the first octant that lies beneath the paraboloid $z = 1 - x^2 - y^2$.

19. Evaluate $\iiint_E e^z\, dV$, where E is enclosed by the paraboloid $z = 1 + x^2 + y^2$, the cylinder $x^2 + y^2 = 5$, and the xy-plane.

20. Evaluate $\iiint_E x\, dV$, where E is enclosed by the planes $z = 0$ and $z = x + y + 5$ and by the cylinders $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$.

21. Evaluate $\iiint_E x^2\, dV$, where E is the solid that lies within the cylinder $x^2 + y^2 = 1$, above the plane $z = 0$, and below the cone $z^2 = 4x^2 + 4y^2$.

22. Find the volume of the solid that lies within both the cylinder $x^2 + y^2 = 1$ and the sphere $x^2 + y^2 + z^2 = 4$.

23. (a) Find the volume of the region E bounded by the paraboloids $z = x^2 + y^2$ and $z = 36 - 3x^2 - 3y^2$.
(b) Find the centroid of E (the center of mass in the case where the density is constant).

24. (a) Find the volume of the solid that the cylinder $r = a \cos \theta$ cuts out of the sphere of radius a centered at the origin.
(b) Illustrate the solid of part (a) by graphing the sphere and the cylinder on the same screen.

25. Find the mass and center of mass of the solid S bounded by the paraboloid $z = 4x^2 + 4y^2$ and the plane $z = a$ $(a > 0)$ if S has constant density K.

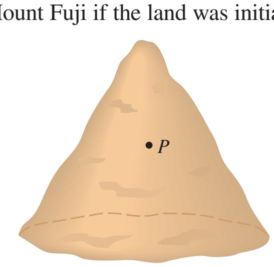
26. Find the mass of a ball B given by $x^2 + y^2 + z^2 \leqslant a^2$ if the density at any point is proportional to its distance from the z-axis.

27–28 Evaluate the integral by changing to cylindrical coordinates.

27. $\displaystyle\int_{-2}^{2} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_{\sqrt{x^2+y^2}}^{2} xz\, dz\, dx\, dy$

28. $\displaystyle\int_{-3}^{3} \int_0^{\sqrt{9-x^2}} \int_0^{9-x^2-y^2} \sqrt{x^2 + y^2}\, dz\, dy\, dx$

29. When studying the formation of mountain ranges, geologists estimate the amount of work required to lift a mountain from sea level. Consider a mountain that is essentially in the shape of a right circular cone. Suppose that the weight density of the material in the vicinity of a point P is $g(P)$ and the height is $h(P)$.
(a) Find a definite integral that represents the total work done in forming the mountain.
(b) Assume that Mount Fuji in Japan is in the shape of a right circular cone with radius 62,000 ft, height 12,400 ft, and density a constant 200 lb/ft^3. How much work was done in forming Mount Fuji if the land was initially at sea level?



15.8 EXERCISES

1–2 Plot the point whose spherical coordinates are given. Then find the rectangular coordinates of the point.

1. (a) $(1, 0, 0)$ (b) $(2, \pi/3, \pi/4)$

2. (a) $(5, \pi, \pi/2)$ (b) $(4, 3\pi/4, \pi/3)$

3–4 Change from rectangular to spherical coordinates.

3. (a) $\left(1, \sqrt{3}, 2\sqrt{3}\right)$ (b) $(0, -1, -1)$

4. (a) $\left(0, \sqrt{3}, 1\right)$ (b) $\left(-1, 1, \sqrt{6}\right)$

5–6 Describe in words the surface whose equation is given.

5. $\phi = \pi/3$

6. $\rho = 3$

7–8 Identify the surface whose equation is given.

7. $\rho = \sin\theta \sin\phi$

8. $\rho^2(\sin^2\phi \sin^2\theta + \cos^2\phi) = 9$

9–10 Write the equation in spherical coordinates.

9. (a) $z^2 = x^2 + y^2$ (b) $x^2 + z^2 = 9$

10. (a) $x^2 - 2x + y^2 + z^2 = 0$ (b) $x + 2y + 3z = 1$

11–14 Sketch the solid described by the given inequalities.

11. $\rho \leqslant 2, \quad 0 \leqslant \phi \leqslant \pi/2, \quad 0 \leqslant \theta \leqslant \pi/2$

12. $2 \leqslant \rho \leqslant 3, \quad \pi/2 \leqslant \phi \leqslant \pi$

13. $\rho \leqslant 1, \quad 3\pi/4 \leqslant \phi \leqslant \pi$

14. $\rho \leqslant 2, \quad \rho \leqslant \csc\phi$

15. A solid lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$. Write a description of the solid in terms of inequalities involving spherical coordinates.

16. (a) Find inequalities that describe a hollow ball with diameter 30 cm and thickness 0.5 cm. Explain how you have positioned the coordinate system that you have chosen.
(b) Suppose the ball is cut in half. Write inequalities that describe one of the halves.

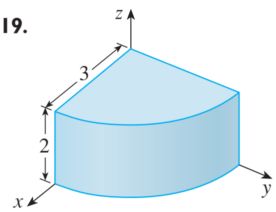
17–18 Sketch the solid whose volume is given by the integral and evaluate the integral.

17. $\displaystyle\int_0^{\pi/6} \int_0^{\pi/2} \int_0^3 \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$

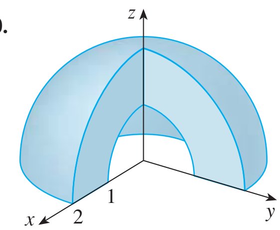
18. $\displaystyle\int_0^{2\pi} \int_{\pi/2}^{\pi} \int_1^2 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

19–20 Set up the triple integral of an arbitrary continuous function $f(x, y, z)$ in cylindrical or spherical coordinates over the solid shown.

19.



20.



21–34 Use spherical coordinates.

21. Evaluate $\iiint_B (x^2 + y^2 + z^2)^2 \, dV$, where B is the ball with center the origin and radius 5.

22. Evaluate $\iiint_H (9 - x^2 - y^2) \, dV$, where H is the solid hemisphere $x^2 + y^2 + z^2 \leqslant 9$, $z \geqslant 0$.

23. Evaluate $\iiint_E z \, dV$, where E lies between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$ in the first octant.

24. Evaluate $\iiint_E e^{\sqrt{x^2+y^2+z^2}} \, dV$, where E is enclosed by the sphere $x^2 + y^2 + z^2 = 9$ in the first octant.

25. Evaluate $\iiint_E x^2 \, dV$, where E is bounded by the xz-plane and the hemispheres $y = \sqrt{9 - x^2 - z^2}$ and $y = \sqrt{16 - x^2 - z^2}$.

26. Evaluate $\iiint_E xyz \, dV$, where E lies between the spheres $\rho = 2$ and $\rho = 4$ and above the cone $\phi = \pi/3$.

27. Find the volume of the part of the ball $\rho \leqslant a$ that lies between the cones $\phi = \pi/6$ and $\phi = \pi/3$.

28. Find the average distance from a point in a ball of radius a to its center.

29. (a) Find the volume of the solid that lies above the cone $\phi = \pi/3$ and below the sphere $\rho = 4\cos\phi$.
(b) Find the centroid of the solid in part (a).

30. Find the volume of the solid that lies within the sphere $x^2 + y^2 + z^2 = 4$, above the xy-plane, and below the cone $z = \sqrt{x^2 + y^2}$.

31. Find the centroid of the solid in Exercise 25.

32. Let H be a solid hemisphere of radius a whose density at any point is proportional to its distance from the center of the base.
(a) Find the mass of H.
(b) Find the center of mass of H.
(c) Find the moment of inertia of H about its axis.

33. (a) Find the centroid of a solid homogeneous hemisphere of radius a.
(b) Find the moment of inertia of the solid in part (a) about a diameter of its base.

12.5 EXERCISES

1. Determine whether each statement is true or false.
(a) Two lines parallel to a third line are parallel.
(b) Two lines perpendicular to a third line are parallel.
(c) Two planes parallel to a third plane are parallel.
(d) Two planes perpendicular to a third plane are parallel.
(e) Two lines parallel to a plane are parallel.
(f) Two lines perpendicular to a plane are parallel.
(g) Two planes parallel to a line are parallel.
(h) Two planes perpendicular to a line are parallel.
(i) Two planes either intersect or are parallel.
(j) Two lines either intersect or are parallel.
(k) A plane and a line either intersect or are parallel.

2–5 Find a vector equation and parametric equations for the line.

2. The line through the point $(6, -5, 2)$ and parallel to the vector $\langle 1, 3, -\frac{2}{3} \rangle$

3. The line through the point $(2, 2.4, 3.5)$ and parallel to the vector $3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

4. The line through the point $(0, 14, -10)$ and parallel to the line $x = -1 + 2t$, $y = 6 - 3t$, $z = 3 + 9t$

5. The line through the point $(1, 0, 6)$ and perpendicular to the plane $x + 3y + z = 5$

6–12 Find parametric equations and symmetric equations for the line.

6. The line through the origin and the point $(1, 2, 3)$

7. The line through the points $(1, 3, 2)$ and $(-4, 3, 0)$

8. The line through the points $(6, 1, -3)$ and $(2, 4, 5)$

9. The line through the points $(0, \frac{1}{2}, 1)$ and $(2, 1, -3)$

10. The line through $(2, 1, 0)$ and perpendicular to both $\mathbf{i} + \mathbf{j}$ and $\mathbf{j} + \mathbf{k}$

11. The line through $(1, -1, 1)$ and parallel to the line $x + 2 = \frac{1}{2}y = z - 3$

12. The line of intersection of the planes $x + y + z = 1$ and $x + z = 0$

13. Is the line through $(-4, -6, 1)$ and $(-2, 0 -3)$ parallel to the line through $(10, 18, 4)$ and $(5, 3, 14)$?

14. Is the line through $(4, 1, -1)$ and $(2, 5, 3)$ perpendicular to the line through $(-3, 2, 0)$ and $(5, 1, 4)$?

15. (a) Find symmetric equations for the line that passes through the point $(1, -5, 6)$ and is parallel to the vector $\langle -1, 2, -3 \rangle$.
(b) Find the points in which the required line in part (a) intersects the coordinate planes.

16. (a) Find parametric equations for the line through $(2, 4, 6)$ that is perpendicular to the plane $x - y + 3z = 7$.
(b) In what points does this line intersect the coordinate planes?

17. Find a vector equation for the line segment from $(2, -1, 4)$ to $(4, 6, 1)$.

18. Find parametric equations for the line segment from $(10, 3, 1)$ to $(5, 6, -3)$.

19–22 Determine whether the lines L_1 and L_2 are parallel, skew, or intersecting. If they intersect, find the point of intersection.

19. L_1: $x = -6t$, $y = 1 + 9t$, $z = -3t$
L_2: $x = 1 + 2s$, $y = 4 - 3s$, $z = s$

20. L_1: $x = 1 + 2t$, $y = 3t$, $z = 2 - t$
L_2: $x = -1 + s$, $y = 4 + s$, $z = 1 + 3s$

21. L_1: $\dfrac{x}{1} = \dfrac{y-1}{2} = \dfrac{z-2}{3}$

L_2: $\dfrac{x-3}{-4} = \dfrac{y-2}{-3} = \dfrac{z-1}{2}$

22. L_1: $\dfrac{x-1}{2} = \dfrac{y-3}{2} = \dfrac{z-2}{-1}$

L_2: $\dfrac{x-2}{1} = \dfrac{y-6}{-1} = \dfrac{z+2}{3}$

23–38 Find an equation of the plane.

23. The plane through the point $(6, 3, 2)$ and perpendicular to the vector $\langle -2, 1, 5 \rangle$

24. The plane through the point $(4, 0, -3)$ and with normal vector $\mathbf{j} + 2\mathbf{k}$

25. The plane through the point $(1, -1, 1)$ and with normal vector $\mathbf{i} + \mathbf{j} - \mathbf{k}$

26. The plane through the point $(-2, 8, 10)$ and perpendicular to the line $x = 1 + t$, $y = 2t$, $z = 4 - 3t$

27. The plane through the origin and parallel to the plane $2x - y + 3z = 1$

28. The plane through the point $(-1, 6, -5)$ and parallel to the plane $x + y + z + 2 = 0$

29. The plane through the point $(4, -2, 3)$ and parallel to the plane $3x - 7z = 12$

30. The plane that contains the line $x = 3 + 2t$, $y = t$, $z = 8 - t$ and is parallel to the plane $2x + 4y + 8z = 17$

31. The plane through the points $(0, 1, 1)$, $(1, 0, 1)$, and $(1, 1, 0)$

32. The plane through the origin and the points $(2, -4, 6)$ and $(5, 1, 3)$

33. The plane through the points $(3, -1, 2)$, $(8, 2, 4)$, and $(-1, -2, -3)$

34. The plane that passes through the point $(1, 2, 3)$ and contains the line $x = 3t$, $y = 1 + t$, $z = 2 - t$

35. The plane that passes through the point $(6, 0, -2)$ and contains the line $x = 4 - 2t$, $y = 3 + 5t$, $z = 7 + 4t$

36. The plane that passes through the point $(1, -1, 1)$ and contains the line with symmetric equations $x = 2y = 3z$

37. The plane that passes through the point $(-1, 2, 1)$ and contains the line of intersection of the planes $x + y - z = 2$ and $2x - y + 3z = 1$

38. The plane that passes through the line of intersection of the planes $x - z = 1$ and $y + 2z = 3$ and is perpendicular to the plane $x + y - 2z = 1$

39–42 Use intercepts to help sketch the plane.

39. $2x + 5y + z = 10$

40. $3x + y + 2z = 6$

41. $6x - 3y + 4z = 6$

42. $6x + 5y - 3z = 15$

43–45 Find the point at which the line intersects the given plane.

43. $x = 3 - t,\ y = 2 + t,\ z = 5t;\quad x - y + 2z = 9$

44. $x = 1 + 2t,\ y = 4t,\ z = 2 - 3t;\quad x + 2y - z + 1 = 0$

45. $x = y - 1 = 2z;\quad 4x - y + 3z = 8$

46. Where does the line through $(1, 0, 1)$ and $(4, -2, 2)$ intersect the plane $x + y + z = 6$?

47. Find direction numbers for the line of intersection of the planes $x + y + z = 1$ and $x + z = 0$.

48. Find the cosine of the angle between the planes $x + y + z = 0$ and $x + 2y + 3z = 1$.

49–54 Determine whether the planes are parallel, perpendicular, or neither. If neither, find the angle between them.

49. $x + 4y - 3z = 1,\quad -3x + 6y + 7z = 0$

50. $2z = 4y - x,\quad 3x - 12y + 6z = 1$

51. $x + y + z = 1,\quad x - y + z = 1$

52. $2x - 3y + 4z = 5,\quad x + 6y + 4z = 3$

53. $x = 4y - 2z,\quad 8y = 1 + 2x + 4z$

54. $x + 2y + 2z = 1,\quad 2x - y + 2z = 1$

55–56 (a) Find parametric equations for the line of intersection of the planes and (b) find the angle between the planes.

55. $x + y + z = 1,\quad x + 2y + 2z = 1$

56. $3x - 2y + z = 1,\quad 2x + y - 3z = 3$

57–58 Find symmetric equations for the line of intersection of the planes.

57. $5x - 2y - 2z = 1,\quad 4x + y + z = 6$

58. $z = 2x - y - 5,\quad z = 4x + 3y - 5$

59. Find an equation for the plane consisting of all points that are equidistant from the points $(1, 0, -2)$ and $(3, 4, 0)$.

60. Find an equation for the plane consisting of all points that are equidistant from the points $(2, 5, 5)$ and $(-6, 3, 1)$.

61. Find an equation of the plane with x-intercept a, y-intercept b, and z-intercept c.

62. (a) Find the point at which the given lines intersect:

$$\mathbf{r} = \langle 1, 1, 0 \rangle + t\langle 1, -1, 2 \rangle$$

$$\mathbf{r} = \langle 2, 0, 2 \rangle + s\langle -1, 1, 0 \rangle$$

(b) Find an equation of the plane that contains these lines.

63. Find parametric equations for the line through the point $(0, 1, 2)$ that is parallel to the plane $x + y + z = 2$ and perpendicular to the line $x = 1 + t$, $y = 1 - t$, $z = 2t$.

64. Find parametric equations for the line through the point $(0, 1, 2)$ that is perpendicular to the line $x = 1 + t$, $y = 1 - t$, $z = 2t$ and intersects this line.

65. Which of the following four planes are parallel? Are any of them identical?

P_1: $4x - 2y + 6z = 3$ $\qquad$ P_2: $4x - 2y - 2z = 6$

P_3: $-6x + 3y - 9z = 5$ $\qquad$ P_4: $z = 2x - y - 3$

66. Which of the following four lines are parallel? Are any of them identical?

$$L_1:\ x = 1 + t,\quad y = t,\quad z = 2 - 5t$$

$$L_2:\ x + 1 = y - 2 = 1 - z$$

$$L_3:\ x = 1 + t,\quad y = 4 + t,\quad z = 1 - t$$

$$L_4:\ \mathbf{r} = \langle 2, 1, -3 \rangle + t\langle 2, 2, -10 \rangle$$

67–68 Use the formula in Exercise 43 in Section 12.4 to find the distance from the point to the given line.

67. $(4, 1, -2);\quad x = 1 + t,\ y = 3 - 2t,\ z = 4 - 3t$

68. $(0, 1, 3);\quad x = 2t,\ y = 6 - 2t,\ z = 3 + t$

69–70 Find the distance from the point to the given plane.

69. $(1, -2, 4),\quad 3x + 2y + 6z = 5$

70. $(-6, 3, 5),\quad x - 2y - 4z = 8$

71–72 Find the distance between the given parallel planes.

71. $2x - 3y + z = 4,\quad 4x - 6y + 2z = 3$

72. $6z = 4y - 2x, \quad 9z = 1 - 3x + 6y$

73. Show that the distance between the parallel planes $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ is

$$D = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$$

74. Find equations of the planes that are parallel to the plane $x + 2y - 2z = 1$ and two units away from it.

75. Show that the lines with symmetric equations $x = y = z$ and $x + 1 = y/2 = z/3$ are skew, and find the distance between these lines.

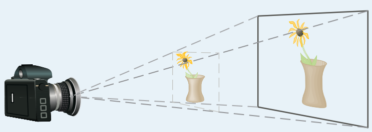
76. Find the distance between the skew lines with parametric equations $x = 1 + t$, $y = 1 + 6t$, $z = 2t$, and $x = 1 + 2s$, $y = 5 + 15s$, $z = -2 + 6s$.

77. If a, b, and c are not all 0, show that the equation $ax + by + cz + d = 0$ represents a plane and $\langle a, b, c \rangle$ is a normal vector to the plane.
Hint: Suppose $a \neq 0$ and rewrite the equation in the form

$$a\left(x + \frac{d}{a}\right) + b(y - 0) + c(z - 0) = 0$$

78. Give a geometric description of each family of planes.
(a) $x + y + z = c$ (b) $x + y + cz = 1$
(c) $y \cos\theta + z \sin\theta = 1$

LABORATORY PROJECT — PUTTING 3D IN PERSPECTIVE



Computer graphics programmers face the same challenge as the great painters of the past: how to represent a three-dimensional scene as a flat image on a two-dimensional plane (a screen or a canvas). To create the illusion of perspective, in which closer objects appear larger than those farther away, three-dimensional objects in the computer's memory are projected onto a rectangular screen window from a viewpoint where the eye, or camera, is located. The viewing volume—the portion of space that will be visible—is the region contained by the four planes that pass through the viewpoint and an edge of the screen window. If objects in the scene extend beyond these four planes, they must be truncated before pixel data are sent to the screen. These planes are therefore called *clipping planes*.

1. Suppose the screen is represented by a rectangle in the yz-plane with vertices $(0, \pm 400, 0)$ and $(0, \pm 400, 600)$, and the camera is placed at $(1000, 0, 0)$. A line L in the scene passes through the points $(230, -285, 102)$ and $(860, 105, 264)$. At what points should L be clipped by the clipping planes?

2. If the clipped line segment is projected on the screen window, identify the resulting line segment.

3. Use parametric equations to plot the edges of the screen window, the clipped line segment, and its projection on the screen window. Then add sight lines connecting the viewpoint to each end of the clipped segments to verify that the projection is correct.

4. A rectangle with vertices $(621, -147, 206)$, $(563, 31, 242)$, $(657, -111, 86)$, and $(599, 67, 122)$ is added to the scene. The line L intersects this rectangle. To make the rectangle appear opaque, a programmer can use *hidden line rendering,* which removes portions of objects that are behind other objects. Identify the portion of L that should be removed.

12.6 CYLINDERS AND QUADRIC SURFACES

We have already looked at two special types of surfaces: planes (in Section 12.5) and spheres (in Section 12.1). Here we investigate two other types of surfaces: cylinders and quadric surfaces.

In order to sketch the graph of a surface, it is useful to determine the curves of intersection of the surface with planes parallel to the coordinate planes. These curves are called **traces** (or cross-sections) of the surface.