4. By using the substitutions $x_i = \sqrt{r^2 - x_n^2 - x_{n-1}^2 - \cdots - x_{i+1}^2}\cos\theta_i$ and then applying Formulas 1 and 2 from Problem 3, we can write

$$V_n = \int_{-r}^{r}\int_{-\sqrt{r^2-x_n^2}}^{\sqrt{r^2-x_n^2}} \cdots \int_{-\sqrt{r^2-x_n^2-x_{n-1}^2-\cdots-x_3^2}}^{\sqrt{r^2-x_n^2-x_{n-1}^2-\cdots-x_3^2}} \int_{-\sqrt{r^2-x_n^2-x_{n-1}^2-\cdots-x_3^2-x_2^2}}^{\sqrt{r^2-x_n^2-x_{n-1}^2-\cdots-x_3^2-x_2^2}} dx_1\, dx_2 \cdots dx_{n-1}\, dx_n$$

$$= 2\left[\int_{-\pi/2}^{\pi/2} \cos^2\theta_2\, d\theta_2\right]\left[\int_{-\pi/2}^{\pi/2} \cos^3\theta_3\, d\theta_3\right]\cdots\left[\int_{-\pi/2}^{\pi/2} \cos^{n-1}\theta_{n-1}\, d\theta_{n-1}\right]\left[\int_{-\pi/2}^{\pi/2} \cos^n\theta_n\, d\theta_n\right] r^n$$

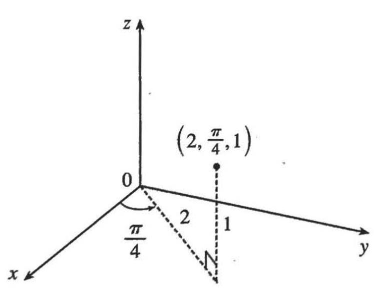
$$= \begin{cases} \left[2\cdot\frac{\pi}{2}\right]\left[\frac{2\cdot 2}{1\cdot 3}\cdot\frac{1\cdot 3\pi}{2\cdot 4}\right]\left[\frac{2\cdot 2\cdot 4}{1\cdot 3\cdot 5}\cdot\frac{1\cdot 3\cdot 5\pi}{2\cdot 4\cdot 6}\right]\cdots\left[\frac{2\cdot\cdots\cdot(n-2)}{1\cdot\cdots\cdot(n-1)}\cdot\frac{1\cdot\cdots\cdot(n-1)\,\pi}{2\cdot\cdots\cdot n}\right] r^n & n \text{ even} \\ 2\left[\frac{\pi}{2}\cdot\frac{2\cdot 2}{1\cdot 3}\right]\left[\frac{1\cdot 3\pi}{2\cdot 4}\cdot\frac{2\cdot 2\cdot 4}{1\cdot 3\cdot 5}\right]\cdots\left[\frac{1\cdot\cdots\cdot(n-2)\,\pi}{2\cdot\cdots\cdot(n-1)}\cdot\frac{2\cdot\cdots\cdot(n-1)}{1\cdot\cdots\cdot n}\right] r^n & n \text{ odd} \end{cases}$$

By canceling within each set of brackets, we find that

$$V_n = \begin{cases} \frac{2\pi}{2}\cdot\frac{2\pi}{4}\cdot\frac{2\pi}{6}\cdot\cdots\cdot\frac{2\pi}{n}r^n = \frac{(2\pi)^{n/2}}{2\cdot 4\cdot 6\cdot\cdots\cdot n}r^n = \frac{\pi^{n/2}}{\left(\frac{1}{2}n\right)!}r^n & n \text{ even} \\ 2\cdot\frac{2\pi}{3}\cdot\frac{2\pi}{5}\cdot\frac{2\pi}{7}\cdot\cdots\cdot\frac{2\pi}{n}r^n = \frac{2(2\pi)^{(n-1)/2}}{3\cdot 5\cdot 7\cdot\cdots\cdot n}r^n = \frac{2^n\left[\frac{1}{2}(n-1)\right]!\,\pi^{(n-1)/2}}{n!}r^n & n \text{ odd} \end{cases}$$

16.7 Triple Integrals in Cylindrical Coordinates ET 15.7

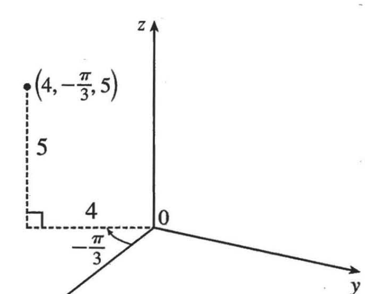
1. (a)



$x = 2\cos\frac{\pi}{4} = \sqrt{2}$, $y = 2\sin\frac{\pi}{4} = \sqrt{2}$, $z = 1$,

so the point is $(\sqrt{2}, \sqrt{2}, 1)$ in rectangular coordinates.

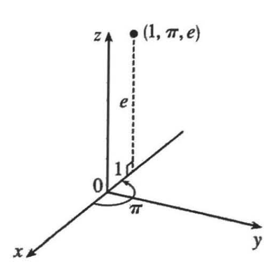
(b)



$x = 4\cos\left(-\frac{\pi}{3}\right) = 2$, $y = 4\sin\left(-\frac{\pi}{3}\right) = -2\sqrt{3}$,

and $z = 5$, so the point is $(2, -2\sqrt{3}, 5)$ in rectangular coordinates.

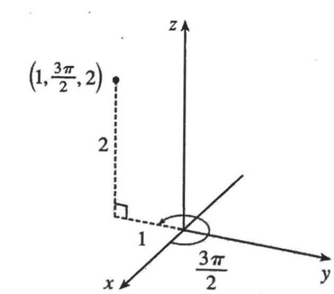
2. (a)



$x = 1\cos\pi = -1$, $y = 1\sin\pi = 0$, and $z = e$,

so the point is $(-1, 0, e)$ in rectangular coordinates.

(b)



$x = 1\cos\frac{3\pi}{2} = 0$, $y = 1\sin\frac{3\pi}{2} = -1$, $z = 2$,

so the point is $(0, -1, 2)$ in rectangular coordinates.

3. (a) $r^2 = x^2 + y^2 = 1^2 + (-1)^2 = 2$ so $r = \sqrt{2}$; $\tan\theta = \frac{y}{x} = \frac{-1}{1} = -1$ and the point $(1, -1)$ is in the fourth quadrant of the xy-plane, so $\theta = \frac{7\pi}{4} + 2n\pi$; $z = 4$. Thus, one set of cylindrical coordinates is $\left(\sqrt{2}, \frac{7\pi}{4}, 4\right)$.

(b) $r^2 = (-1)^2 + \left(-\sqrt{3}\right)^2 = 4$ so $r = 2$; $\tan\theta = \frac{-\sqrt{3}}{-1} = \sqrt{3}$ and the point $\left(-1, -\sqrt{3}\right)$ is in the third quadrant of the xy-plane, so $\theta = \frac{4\pi}{3} + 2n\pi$; $z = 2$. Thus, one set of cylindrical coordinates is $\left(2, \frac{4\pi}{3}, 2\right)$.

4. (a) $r^2 = \left(2\sqrt{3}\right)^2 + 2^2 = 16$ so $r = 4$; $\tan\theta = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}}$ and the point $\left(2\sqrt{3}, 2\right)$ is in the first quadrant of the xy-plane, so $\theta = \frac{\pi}{6} + 2n\pi$; $z = -1$. Thus, one set of cylindrical coordinates is $\left(4, \frac{\pi}{6}, -1\right)$.

(b) $r^2 = 4^2 + (-3)^2 = 25$ so $r = 5$; $\tan\theta = \frac{-3}{4}$ and the point $(4, -3)$ is in the fourth quadrant of the xy-plane, so $\theta = \tan^{-1}\left(-\frac{3}{4}\right) + 2n\pi \approx -0.64 + 2n\pi$; $z = 2$. Thus, one set of cylindrical coordinates is $\left(5, \tan^{-1}\left(-\frac{3}{4}\right) + 2\pi, 2\right) \approx (5, 5.64, 2)$.

5. Since $\theta = \frac{\pi}{4}$ but r and z may vary, the surface is a vertical half-plane including the z-axis and intersecting the xy-plane in the half-line $y = x$, $x \geq 0$.

6. Since $r = 5$, $x^2 + y^2 = 25$ and the surface is a circular cylinder with radius 5 and axis the z-axis.

7. $z = 4 - r^2 = 4 - (x^2 + y^2)$ or $4 - x^2 - y^2$, so the surface is a circular paraboloid with vertex $(0, 0, 4)$, axis the z-axis, and opening downward.

8. Since $2r^2 + z^2 = 1$ and $r^2 = x^2 + y^2$, we have $2(x^2 + y^2) + z^2 = 1$ or $2x^2 + 2y^2 + z^2 = 1$, an ellipsoid centered at the origin with intercepts $x = \pm\frac{1}{\sqrt{2}}$, $y = \pm\frac{1}{\sqrt{2}}$, $z = \pm 1$.

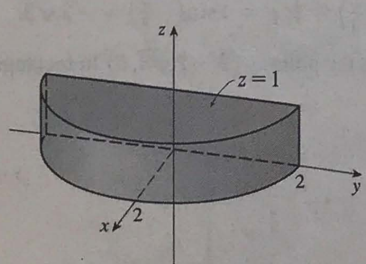
9. (a) $x^2 + y^2 = r^2$, so the equation becomes $z = r^2$.

(b) Substituting $x^2 + y^2 = r^2$ and $y = r\sin\theta$, the equation $x^2 + y^2 = 2y$ becomes $r^2 = 2r\sin\theta$ or $r = 2\sin\theta$.

10. (a) Substituting $x = r\cos\theta$ and $y = r\sin\theta$, the equation $3x + 2y + z = 6$ becomes $3r\cos\theta + 2r\sin\theta + z = 6$ or $z = 6 - r(3\cos\theta + 2\sin\theta)$.

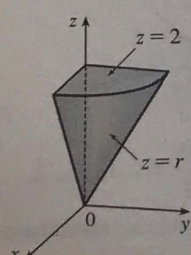
(b) The equation $-x^2 - y^2 + z^2 = 1$ can be written as $-(x^2 + y^2) + z^2 = 1$ which becomes $-r^2 + z^2 = 1$ or $z^2 = 1 + r^2$ in cylindrical coordinates.

11.



$0 \leq r \leq 2$ and $0 \leq z \leq 1$ describe a solid circular cylinder with radius 2, axis the z-axis, and height 1, but $-\pi/2 \leq \theta \leq \pi/2$ restricts the solid to the first and fourth quadrants of the xy-plane, so we have a half-cylinder.

12.



$z = r = \sqrt{x^2 + y^2}$ is a cone that opens upward. Thus $r \leq z \leq 2$ is the region above this cone and beneath the horizontal plane $z = 2$. $0 \leq \theta \leq \frac{\pi}{2}$ restricts the solid to that part of this region in the first octant.

we split the region of integration where the outside boundary changes from the vertical line $x = 1$ to the circle $x^2 + y^2 = a^2$ or $r = 1$. R_1 is a right triangle, so $\cos\theta = \frac{1}{a}$. Thus, the boundary between R_1 and R_2 is $\theta = \cos^{-1}\left(\frac{1}{a}\right)$ in polar coordinates, or $y = \sqrt{a^2 - 1}\,x$ in rectangular coordinates. Using rectangular coordinates for the region R_1 and polar coordinates for R_2, we find the total volume of the solid to be

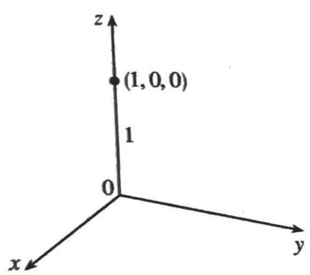
$$V = 16\left[\int_0^1 \int_0^{\sqrt{a^2-1}\,x} \sqrt{1 - x^2}\,dy\,dx + \int_{\cos^{-1}(1/a)}^{\pi/4} \int_0^a \sqrt{1 - r^2\cos^2\theta}\,r\,dr\,d\theta\right]$$

If $a \geq \sqrt{2}$, the cylinder $x^2 + y^2 = 1$ completely encloses the intersection of the other two cylinders, so the solid of intersection of the three cylinders coincides with the intersection of $x^2 + z^2 = 1$ and $y^2 + z^2 = 1$ as illustrated in Exercise 17.6.58 [ET 16.6.58]. Its volume is $V = 16\int_0^1 \int_0^x \sqrt{1 - x^2}\,dy\,dx$.

16.8 Triple Integrals in Spherical Coordinates

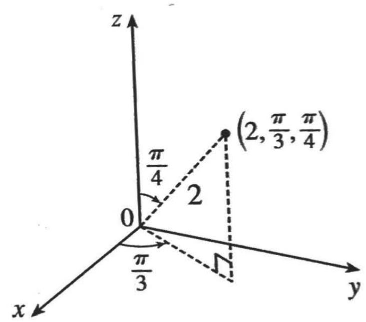
ET 15.8

1. (a)



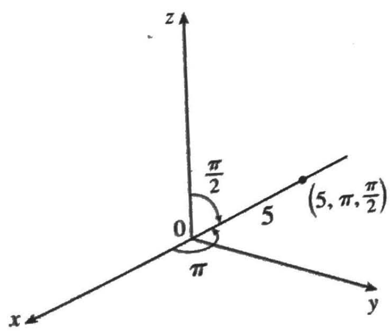
$x = \rho\sin\phi\cos\theta = (1)\sin 0\cos 0 = 0$,
$y = \rho\sin\phi\sin\theta = (1)\sin 0\sin 0 = 0$, and
$z = \rho\cos\phi = (1)\cos 0 = 1$ so the point is $(0, 0, 1)$ in rectangular coordinates.

(b)



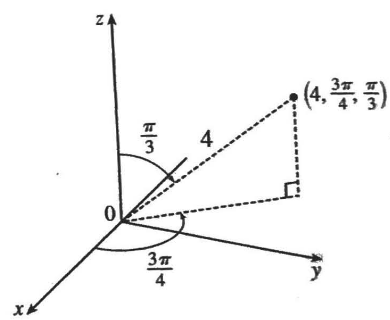
$x = 2\sin\frac{\pi}{4}\cos\frac{\pi}{3} = \frac{\sqrt{2}}{2}$, $y = 2\sin\frac{\pi}{4}\sin\frac{\pi}{3} = \frac{\sqrt{6}}{2}$,
$z = 2\cos\frac{\pi}{4} = \sqrt{2}$ so the point is $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{6}}{2}, \sqrt{2}\right)$ in rectangular coordinates.

2. (a)



$x = 5\sin\frac{\pi}{2}\cos\pi = -5$, $y = 5\sin\frac{\pi}{2}\sin\pi = 0$,
$z = 5\cos\frac{\pi}{2} = 0$ so the point is $(-5, 0, 0)$ in rectangular coordinates.

(b)



$x = 4\sin\frac{\pi}{3}\cos\frac{3\pi}{4} = 4\left(\frac{\sqrt{3}}{2}\right)\left(-\frac{\sqrt{2}}{2}\right) = -\sqrt{6}$,
$y = 4\sin\frac{\pi}{3}\sin\frac{3\pi}{4} = 4\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = \sqrt{6}$,
$z = 4\cos\frac{\pi}{3} = 4\left(\frac{1}{2}\right) = 2$ so the point is $(-\sqrt{6}, \sqrt{6}, 2)$ in rectangular coordinates.

3. (a) $\rho = \sqrt{x^2+y^2+z^2} = \sqrt{1+3+12} = 4$, $\cos\phi = \dfrac{z}{\rho} = \dfrac{2\sqrt{3}}{4} = \dfrac{\sqrt{3}}{2} \Rightarrow \phi = \dfrac{\pi}{6}$, and

$\cos\theta = \dfrac{x}{\rho\sin\phi} = \dfrac{1}{4\sin(\pi/6)} = \dfrac{1}{2} \Rightarrow \theta = \dfrac{\pi}{3}$ [since $y > 0$]. Thus spherical coordinates are $\left(4, \dfrac{\pi}{3}, \dfrac{\pi}{6}\right)$.

(b) $\rho = \sqrt{0+1+1} = \sqrt{2}$, $\cos\phi = \dfrac{-1}{\sqrt{2}} \Rightarrow \phi = \dfrac{3\pi}{4}$, and $\cos\theta = \dfrac{0}{\sqrt{2}\sin(3\pi/4)} = 0 \Rightarrow \theta = \dfrac{3\pi}{2}$ [since $y < 0$].

Thus spherical coordinates are $\left(\sqrt{2}, \dfrac{3\pi}{2}, \dfrac{3\pi}{4}\right)$.

4. (a) $\rho = \sqrt{x^2+y^2+z^2} = \sqrt{0+3+1} = 2$, $\cos\phi = \dfrac{z}{\rho} = \dfrac{1}{2} \Rightarrow \phi = \dfrac{\pi}{3}$, and $\cos\theta = \dfrac{x}{\rho\sin\phi} = \dfrac{0}{2\sin(\pi/3)} = 0 \Rightarrow$

$\theta = \dfrac{\pi}{2}$ [since $y > 0$]. Thus spherical coordinates are $\left(2, \dfrac{\pi}{2}, \dfrac{\pi}{3}\right)$.

(b) $\rho = \sqrt{1+1+6} = 2\sqrt{2}$, $\cos\phi = \dfrac{\sqrt{6}}{2\sqrt{2}} = \dfrac{\sqrt{3}}{2} \Rightarrow \phi = \dfrac{\pi}{6}$, and $\cos\theta = \dfrac{-1}{2\sqrt{2}\sin(\pi/6)} = -\dfrac{1}{\sqrt{2}} \Rightarrow$

$\theta = \dfrac{3\pi}{4}$ [since $y > 0$]. Thus spherical coordinates are $\left(2\sqrt{2}, \dfrac{3\pi}{4}, \dfrac{\pi}{6}\right)$.

5. Since $\phi = \frac{\pi}{3}$, the surface is the top half of the right circular cone with vertex at the origin and axis the positive z-axis.

6. Since $\rho = 3$, $x^2+y^2+z^2 = 9$ and the surface is a sphere with center the origin and radius 3.

7. $\rho = \sin\theta\sin\phi \Rightarrow \rho^2 = \rho\sin\theta\sin\phi \Leftrightarrow x^2+y^2+z^2 = y \Leftrightarrow x^2+y^2-y+\frac{1}{4}+z^2 = \frac{1}{4} \Leftrightarrow$

$x^2+(y-\frac{1}{2})^2+z^2 = \frac{1}{4}$. Therefore, the surface is a sphere of radius $\frac{1}{2}$ centered at $(0, \frac{1}{2}, 0)$.

8. $\rho^2(\sin^2\phi\sin^2\theta+\cos^2\phi) = 9 \Leftrightarrow (\rho\sin\phi\sin\theta)^2+(\rho\cos\phi)^2 = 9 \Leftrightarrow y^2+z^2 = 9$. Thus the surface is a circular cylinder of radius 3 with axis the x-axis.

9. (a) $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, and $z = \rho\cos\phi$, so the equation $z^2 = x^2+y^2$ becomes $(\rho\cos\phi)^2 = (\rho\sin\phi\cos\theta)^2+(\rho\sin\phi\sin\theta)^2$ or $\rho^2\cos^2\phi = \rho^2\sin^2\phi$. If $\rho \neq 0$, this becomes $\cos^2\phi = \sin^2\phi$. ($\rho = 0$ corresponds to the origin which is included in the surface.) There are many equivalent equations in spherical coordinates, such as $\tan^2\phi = 1$, $2\cos^2\phi = 1$, $\cos 2\phi = 0$, or even $\phi = \frac{\pi}{4}$, $\phi = \frac{3\pi}{4}$.

(b) $x^2+z^2 = 9 \Leftrightarrow (\rho\sin\phi\cos\theta)^2+(\rho\cos\phi)^2 = 9 \Leftrightarrow \rho^2\sin^2\phi\cos^2\theta+\rho^2\cos^2\phi = 9$ or $\rho^2(\sin^2\phi\cos^2\theta+\cos^2\phi) = 9$.

10. (a) $x^2-2x+y^2+z^2 = 0 \Leftrightarrow (x^2+y^2+z^2)-2x = 0 \Leftrightarrow \rho^2-2(\rho\sin\phi\cos\theta) = 0$ or $\rho = 2\sin\phi\cos\theta$.

(b) $x+2y+3z = 1 \Leftrightarrow \rho\sin\phi\cos\theta+2\rho\sin\phi\sin\theta+3\rho\cos\phi = 1$ or $\rho = 1/(\sin\phi\cos\theta+2\sin\phi\sin\theta+3\cos\phi)$.

11. $\rho = 2$ represents a sphere of radius 2, centered at the origin, so $\rho \le 2$ is this sphere and its interior. $0 \le \phi \le \frac{\pi}{2}$ restricts the solid to that portion of the region that lies on or above the xy-plane, and $0 \le \theta \le \frac{\pi}{2}$ further restricts the solid to the first octant. Thus the solid is the portion in the first octant of the solid ball centered at the origin with radius 2.