29. Here we need the volume of the solid lying under the surface $z = x\sec^2 y$ and above the rectangle $R = [0,2]\times[0,\pi/4]$ in the xy-plane.

$$V = \int_0^2\int_0^{\pi/4} x\sec^2 y\,dy\,dx = \int_0^2 x\,dx\int_0^{\pi/4}\sec^2 y\,dy = \left[\tfrac{1}{2}x^2\right]_0^2\,\left[\tan y\right]_0^{\pi/4}$$
$$= (2-0)(\tan\tfrac{\pi}{4} - \tan 0) = 2(1-0) = 2$$

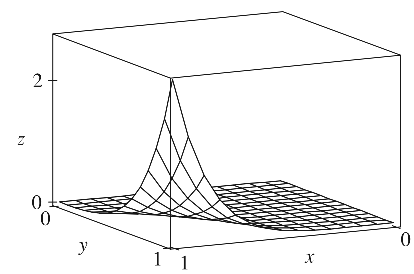
31. The solid lies below the surface $z = 2 + x^2 + (y-2)^2$ and above the plane $z = 1$ for $-1 \le x \le 1$, $0 \le y \le 4$. The volume of the solid is the difference in volumes between the solid that lies under $z = 2 + x^2 + (y-2)^2$ over the rectangle $R = [-1,1]\times[0,4]$ and the solid that lies under $z = 1$ over R.

$$V = \int_0^4\int_{-1}^1 [2 + x^2 + (y-2)^2]\,dx\,dy - \int_0^4\int_{-1}^1 (1)\,dx\,dy = \int_0^4 \left[2x + \tfrac{1}{3}x^3 + x(y-2)^2\right]_{x=-1}^{x=1}\,dy - \int_{-1}^1 dx\int_0^4 dy$$
$$= \int_0^4 \left[(2 + \tfrac{1}{3} + (y-2)^2) - (-2 - \tfrac{1}{3} - (y-2)^2)\right]dy - [x]_{-1}^1\,[y]_0^4$$
$$= \int_0^4 \left[\tfrac{14}{3} + 2(y-2)^2\right]dy - [1-(-1)][4-0] = \left[\tfrac{14}{3}y + \tfrac{2}{3}(y-2)^3\right]_0^4 - (2)(4)$$
$$= \left[\left(\tfrac{56}{3} + \tfrac{16}{3}\right) - \left(0 - \tfrac{16}{3}\right)\right] - 8 = \tfrac{88}{3} - 8 = \tfrac{64}{3}$$

33. In Maple, we can calculate the integral by defining the integrand as `f` and then using the command `int(int(f,x=0..1),y=0..1);`. In Mathematica, we can use the command

```
Integrate[f,{x,0,1},{y,0,1}]
```

We find that $\iint_R x^5y^3e^{xy}\,dA = 21e - 57 \approx 0.0839$. We can use `plot3d` (in Maple) or `Plot3D` (in Mathematica) to graph the function.



35. R is the rectangle $[-1,1]\times[0,5]$. Thus, $A(R) = 2\cdot 5 = 10$ and

$$f_{\text{ave}} = \frac{1}{A(R)}\iint_R f(x,y)\,dA = \tfrac{1}{10}\int_0^5\int_{-1}^1 x^2y\,dx\,dy = \tfrac{1}{10}\int_0^5\left[\tfrac{1}{3}x^3y\right]_{x=-1}^{x=1}dy = \tfrac{1}{10}\int_0^5 \tfrac{2}{3}y\,dy = \tfrac{1}{10}\left[\tfrac{1}{3}y^2\right]_0^5 = \tfrac{5}{6}.$$

37. Let $f(x,y) = \dfrac{x-y}{(x+y)^3}$. Then a CAS gives $\int_0^1\int_0^1 f(x,y)\,dy\,dx = \frac{1}{2}$ and $\int_0^1\int_0^1 f(x,y)\,dx\,dy = -\frac{1}{2}$.

To explain the seeming violation of Fubini's Theorem, note that f has an infinite discontinuity at $(0,0)$ and thus does not satisfy the conditions of Fubini's Theorem. In fact, both iterated integrals involve improper integrals which diverge at their lower limits of integration.

16.3 Double Integrals over General Regions — ET 15.3

1. $\int_0^4\int_0^{\sqrt{y}} xy^2\,dx\,dy = \int_0^4\left[\tfrac{1}{2}x^2y^2\right]_{x=0}^{x=\sqrt{y}}dy = \int_0^4 \tfrac{1}{2}y^2[(\sqrt{y}\,)^2 - 0^2]dy = \tfrac{1}{2}\int_0^4 y^3\,dy = \tfrac{1}{2}\left[\tfrac{1}{4}y^4\right]_0^4 = \tfrac{1}{2}(64-0) = 32$

3. $\int_0^1\int_{x^2}^x (1+2y)dy\,dx = \int_0^1\left[y + y^2\right]_{y=x^2}^{y=x}dx = \int_0^1\left[x + x^2 - x^2 - (x^2)^2\right]dx$

$$= \int_0^1 (x - x^4)dx = \left[\tfrac{1}{2}x^2 - \tfrac{1}{5}x^5\right]_0^1 = \tfrac{1}{2} - \tfrac{1}{5} - 0 + 0 = \tfrac{3}{10}$$

5. $\int_0^{\pi/2}\int_0^{\cos\theta} e^{\sin\theta}dr\,d\theta = \int_0^{\pi/2}\left[re^{\sin\theta}\right]_{r=0}^{r=\cos\theta}d\theta = \int_0^{\pi/2}(\cos\theta)\,e^{\sin\theta}d\theta = e^{\sin\theta}\Big]_0^{\pi/2} = e^{\sin(\pi/2)} - e^0 = e - 1$

7. $\iint_D y^2\,dA = \int_{-1}^{1}\int_{-y-2}^{y} y^2\,dx\,dy = \int_{-1}^{1}\left[xy^2\right]_{x=-y-2}^{x=y}dy = \int_{-1}^{1} y^2\,[y-(-y-2)]\,dy$

$= \int_{-1}^{1}(2y^3+2y^2)dy = \left[\frac{1}{2}y^4+\frac{2}{3}y^3\right]_{-1}^{1} = \frac{1}{2}+\frac{2}{3}-\frac{1}{2}+\frac{2}{3} = \frac{4}{3}$

9. $\iint_D x\,dA = \int_0^{\pi}\int_0^{\sin x} x\,dy\,dx = \int_0^{\pi}\,[xy]_{y=0}^{y=\sin x}\,dx = \int_0^{\pi} x\sin x\,dx$ [integrate by parts with $u=x,\ dv=\sin x\,dx$]

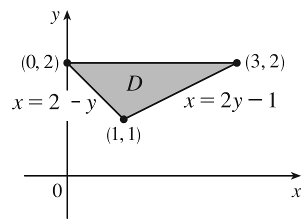
$= \left[-x\cos x+\sin x\right]_0^{\pi} = -\pi\cos\pi+\sin\pi+0-\sin 0 = \pi$

11. $\iint_D y^2 e^{xy}\,dA = \int_0^4\int_0^y y^2e^{xy}\,dx\,dy = \int_0^4\left[ye^{xy}\right]_{x=0}^{x=y}dy = \int_0^4\left(ye^{y^2}-y\right)dy$

$= \left[\frac{1}{2}e^{y^2}-\frac{1}{2}y^2\right]_0^4 = \frac{1}{2}e^{16}-8-\frac{1}{2}+0 = \frac{1}{2}e^{16}-\frac{17}{2}$

13. $\int_0^1\int_0^{x^2} x\cos y\,dy\,dx = \int_0^1\left[x\sin y\right]_{y=0}^{y=x^2}dx = \int_0^1 x\sin x^2\,dx = -\frac{1}{2}\cos x^2\Big]_0^1 = \frac{1}{2}(1-\cos 1)$

15.

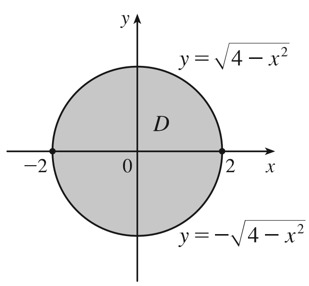


$$\int_1^2\int_{2-y}^{2y-1} y^3\,dx\,dy = \int_1^2\left[xy^3\right]_{x=2-y}^{x=2y-1}dy = \int_1^2\left[(2y-1)-(2-y)\right]y^3\,dy$$

$$= \int_1^2(3y^4-3y^3)\,dy = \left[\tfrac{3}{5}y^5-\tfrac{3}{4}y^4\right]_1^2$$

$$= \tfrac{96}{5}-12-\tfrac{3}{5}+\tfrac{3}{4} = \tfrac{147}{20}$$

17.



$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(2x-y)\,dy\,dx$$

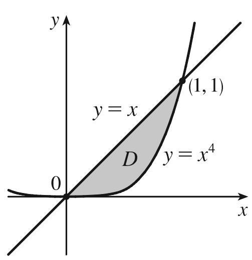
$$= \int_{-2}^{2}\left[2xy-\tfrac{1}{2}y^2\right]_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}}dx$$

$$= \int_{-2}^{2}\left[2x\sqrt{4-x^2}-\tfrac{1}{2}\left(4-x^2\right)+2x\sqrt{4-x^2}+\tfrac{1}{2}\left(4-x^2\right)\right]dx$$

$$= \int_{-2}^{2}4x\sqrt{4-x^2}\,dx = -\tfrac{4}{3}\left(4-x^2\right)^{3/2}\Big]_{-2}^{2} = 0$$

[Or, note that $4x\sqrt{4-x^2}$ is an odd function, so $\int_{-2}^{2}4x\sqrt{4-x^2}\,dx = 0$.]

19.

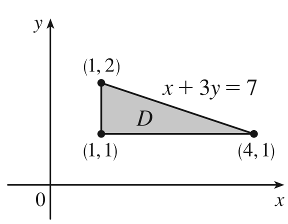


$$V = \int_0^1\int_{x^4}^{x}(x+2y)\,dy\,dx$$

$$= \int_0^1\left[xy+y^2\right]_{y=x^4}^{y=x}dx = \int_0^1(2x^2-x^5-x^8)\,dx$$

$$= \left[\tfrac{2}{3}x^3-\tfrac{1}{6}x^6-\tfrac{1}{9}x^9\right]_0^1 = \tfrac{2}{3}-\tfrac{1}{6}-\tfrac{1}{9} = \tfrac{7}{18}$$

21.

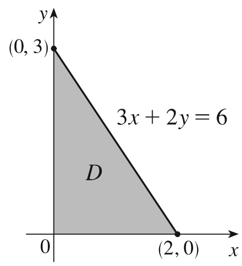


$$V = \int_1^2\int_1^{7-3y} xy\,dx\,dy = \int_1^2\left[\tfrac{1}{2}x^2y\right]_{x=1}^{x=7-3y}dy$$

$$= \tfrac{1}{2}\int_1^2(48y-42y^2+9y^3)\,dy$$

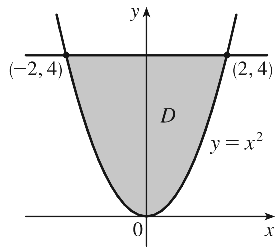
$$= \tfrac{1}{2}\left[24y^2-14y^3+\tfrac{9}{4}y^4\right]_1^2 = \tfrac{31}{8}$$

23.



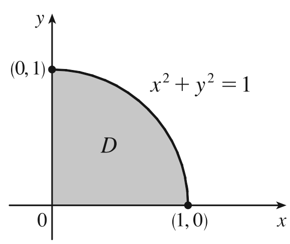
$$\begin{aligned} V &= \int_0^2 \int_0^{3-\frac{3}{2}x} (6-3x-2y)\,dy\,dx \\ &= \int_0^2 \left[6y-3xy-y^2\right]_{y=0}^{y=3-\frac{3}{2}x} dx \\ &= \int_0^2 \left[6(3-\tfrac{3}{2}x)-3x(3-\tfrac{3}{2}x)-(3-\tfrac{3}{2}x)^2\right] dx \\ &= \int_0^2 \left(\tfrac{9}{4}x^2-9x+9\right) dx = \left[\tfrac{3}{4}x^3-\tfrac{9}{2}x^2+9x\right]_0^2 = 6-0=6 \end{aligned}$$

25.



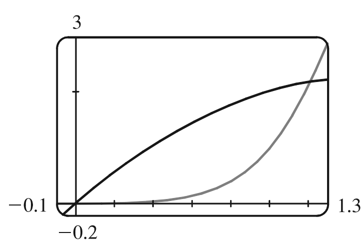
$$\begin{aligned} V &= \int_{-2}^2 \int_{x^2}^4 x^2\,dy\,dx \\ &= \int_{-2}^2 x^2\left[\,y\,\right]_{y=x^2}^{y=4} dx = \int_{-2}^2 (4x^2-x^4)\,dx \\ &= \left[\tfrac{4}{3}x^3-\tfrac{1}{5}x^5\right]_{-2}^2 = \tfrac{32}{3}-\tfrac{32}{5}+\tfrac{32}{3}-\tfrac{32}{5}=\tfrac{128}{15} \end{aligned}$$

27.



$$\begin{aligned} V &= \int_0^1 \int_0^{\sqrt{1-x^2}} y\,dy\,dx = \int_0^1 \left[\frac{y^2}{2}\right]_{y=0}^{y=\sqrt{1-x^2}} dx \\ &= \int_0^1 \frac{1-x^2}{2}\,dx = \tfrac{1}{2}\left[x-\tfrac{1}{3}x^3\right]_0^1 = \tfrac{1}{3} \end{aligned}$$

29.



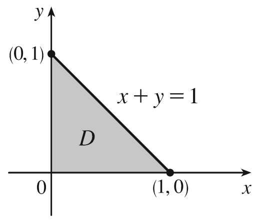
From the graph, it appears that the two curves intersect at $x=0$ and at $x\approx 1.213$. Thus the desired integral is

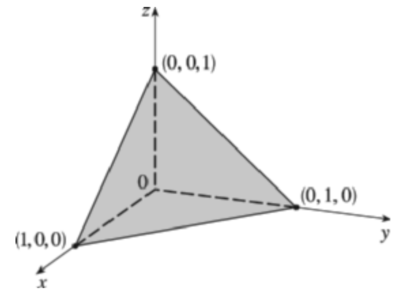
$$\begin{aligned} \iint_D x\,dA &\approx \int_0^{1.213}\int_{x^4}^{3x-x^2} x\,dy\,dx = \int_0^{1.213}\left[\,xy\,\right]_{y=x^4}^{y=3x-x^2} dx \\ &= \int_0^{1.213}(3x^2-x^3-x^5)\,dx = \left[x^3-\tfrac{1}{4}x^4-\tfrac{1}{6}x^6\right]_0^{1.213} \\ &\approx 0.713 \end{aligned}$$

31. The two bounding curves $y=1-x^2$ and $y=x^2-1$ intersect at $(\pm1,0)$ with $1-x^2\ge x^2-1$ on $[-1,1]$. Within this region, the plane $z=2x+2y+10$ is above the plane $z=2-x-y$, so

$$\begin{aligned} V &= \int_{-1}^1\int_{x^2-1}^{1-x^2}(2x+2y+10)\,dy\,dx - \int_{-1}^1\int_{x^2-1}^{1-x^2}(2-x-y)\,dy\,dx \\ &= \int_{-1}^1\int_{x^2-1}^{1-x^2}(2x+2y+10-(2-x-y))\,dy\,dx \\ &= \int_{-1}^1\int_{x^2-1}^{1-x^2}(3x+3y+8)\,dy\,dx = \int_{-1}^1\left[3xy+\tfrac{3}{2}y^2+8y\right]_{y=x^2-1}^{y=1-x^2} dx \\ &= \int_{-1}^1\left[3x(1-x^2)+\tfrac{3}{2}(1-x^2)^2+8(1-x^2)-3x(x^2-1)-\tfrac{3}{2}(x^2-1)^2-8(x^2-1)\right] dx \\ &= \int_{-1}^1(-6x^3-16x^2+6x+16)\,dx = \left[-\tfrac{3}{2}x^4-\tfrac{16}{3}x^3+3x^2+16x\right]_{-1}^1 \\ &= -\tfrac{3}{2}-\tfrac{16}{3}+3+16+\tfrac{3}{2}-\tfrac{16}{3}-3+16=\tfrac{64}{3} \end{aligned}$$

33. The solid lies below the plane $z = 1 - x - y$ or $x + y + z = 1$ and above the region $D = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le 1 - x\}$ in the xy-plane. The solid is a tetrahedron.





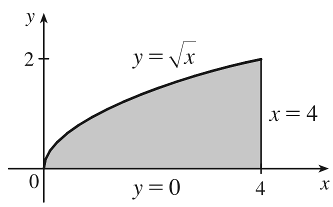
35. The two bounding curves $y = x^3 - x$ and $y = x^2 + x$ intersect at the origin and at $x = 2$, with $x^2 + x > x^3 - x$ on $(0, 2)$. Using a CAS, we find that the volume is

$$V = \int_0^2 \int_{x^3 - x}^{x^2 + x} z \, dy \, dx = \int_0^2 \int_{x^3 - x}^{x^2 + x} (x^3 y^4 + xy^2) \, dy \, dx = \frac{13{,}984{,}735{,}616}{14{,}549{,}535}$$

37. The two surfaces intersect in the circle $x^2 + y^2 = 1$, $z = 0$ and the region of integration is the disk D: $x^2 + y^2 \le 1$.

Using a CAS, the volume is $\iint_D (1 - x^2 - y^2) \, dA = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1 - x^2 - y^2) \, dy \, dx = \dfrac{\pi}{2}$.

39.

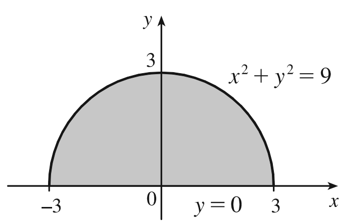


Because the region of integration is

$D = \{(x, y) \mid 0 \le y \le \sqrt{x}, 0 \le x \le 4\} = \{(x, y) \mid y^2 \le x \le 4, 0 \le y \le 2\}$

we have $\int_0^4 \int_0^{\sqrt{x}} f(x, y) \, dy \, dx = \iint_D f(x, y) \, dA = \int_0^2 \int_{y^2}^4 f(x, y) \, dx \, dy$.

41.



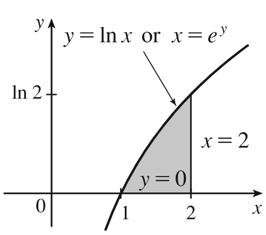
Because the region of integration is

$$\begin{aligned} D &= \left\{(x, y) \mid -\sqrt{9 - y^2} \le x \le \sqrt{9 - y^2}, 0 \le y \le 3\right\} \\ &= \left\{(x, y) \mid 0 \le y \le \sqrt{9 - x^2}, -3 \le x \le 3\right\} \end{aligned}$$

we have

$$\begin{aligned} \int_0^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} f(x, y) \, dx \, dy &= \iint_D f(x, y) \, dA \\ &= \int_{-3}^{3} \int_0^{\sqrt{9-x^2}} f(x, y) \, dy \, dx \end{aligned}$$

43.



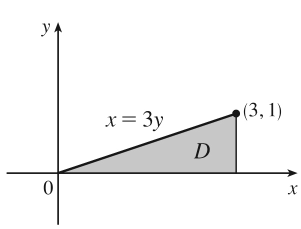
Because the region of integration is

$D = \{(x, y) \mid 0 \le y \le \ln x, 1 \le x \le 2\} = \{(x, y) \mid e^y \le x \le 2, 0 \le y \le \ln 2\}$

we have

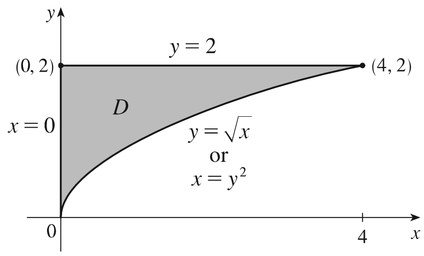
$$\int_1^2 \int_0^{\ln x} f(x, y) \, dy \, dx = \iint_D f(x, y) \, dA = \int_0^{\ln 2} \int_{e^y}^2 f(x, y) \, dx \, dy$$

45.



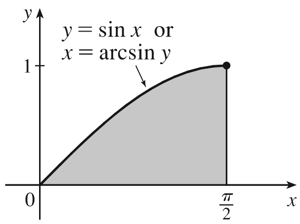
$$\begin{aligned} \int_0^1 \int_{3y}^3 e^{x^2} \, dx \, dy &= \int_0^3 \int_0^{x/3} e^{x^2} \, dy \, dx = \int_0^3 \left[e^{x^2} y\right]_{y=0}^{y=x/3} dx \\ &= \int_0^3 \left(\frac{x}{3}\right) e^{x^2} \, dx = \tfrac{1}{6} \, e^{x^2}\Big]_0^3 = \frac{e^9 - 1}{6} \end{aligned}$$

47.



$$\int_0^4 \int_{\sqrt{x}}^2 \frac{1}{y^3+1}\,dy\,dx = \int_0^2 \int_0^{y^2} \frac{1}{y^3+1}\,dx\,dy$$
$$= \int_0^2 \frac{1}{y^3+1}\,\big[x\big]_{x=0}^{x=y^2}\,dy = \int_0^2 \frac{y^2}{y^3+1}\,dy$$
$$= \tfrac{1}{3}\ln\left|y^3+1\right|\Big]_0^2 = \tfrac{1}{3}(\ln 9 - \ln 1) = \tfrac{1}{3}\ln 9$$

49.



$$\int_0^1 \int_{\arcsin y}^{\pi/2} \cos x\,\sqrt{1+\cos^2 x}\,dx\,dy$$
$$= \int_0^{\pi/2} \int_0^{\sin x} \cos x\,\sqrt{1+\cos^2 x}\,dy\,dx$$
$$= \int_0^{\pi/2} \cos x\,\sqrt{1+\cos^2 x}\,\big[y\big]_{y=0}^{y=\sin x}\,dx$$
$$= \int_0^{\pi/2} \cos x\,\sqrt{1+\cos^2 x}\,\sin x\,dx \qquad \begin{bmatrix}\text{Let } u=\cos x,\, du=-\sin x\,dx,\\ dx = du/(-\sin x)\end{bmatrix}$$
$$= \int_1^0 -u\,\sqrt{1+u^2}\,du = -\tfrac{1}{3}\left(1+u^2\right)^{3/2}\Big]_1^0$$
$$= \tfrac{1}{3}\left(\sqrt{8}-1\right) = \tfrac{1}{3}\left(2\sqrt{2}-1\right)$$

51. $D = \{(x,y) \mid 0 \le x \le 1,\ -x+1 \le y \le 1\} \cup \{(x,y) \mid -1 \le x \le 0, x+1 \le y \le 1\}$

$\cup\,\{(x,y) \mid 0 \le x \le 1,\ -1 \le y \le x-1\} \cup \{(x,y) \mid -1 \le x \le 0,\ -1 \le y \le -x-1\}$, all type I.

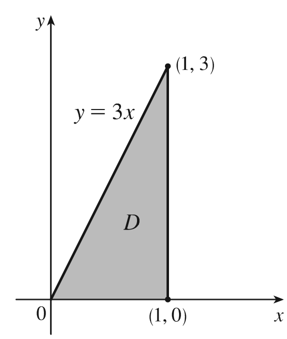
$$\iint_D x^2\,dA = \int_0^1\int_{1-x}^1 x^2\,dy\,dx + \int_{-1}^0\int_{x+1}^1 x^2\,dy\,dx + \int_0^1\int_{-1}^{x-1} x^2\,dy\,dx + \int_{-1}^0\int_{-1}^{-x-1} x^2\,dy\,dx$$
$$= 4\int_0^1\int_{1-x}^1 x^2\,dy\,dx \qquad \text{[by symmetry of the regions and because } f(x,y)=x^2\ge 0\text{]}$$
$$= 4\int_0^1 x^3\,dx = 4\left[\tfrac{1}{4}x^4\right]_0^1 = 1$$

53. Here $Q = \left\{(x,y) \mid x^2+y^2 \le \frac{1}{4}, x \ge 0, y \ge 0\right\}$, and $0 \le (x^2+y^2)^2 \le \left(\frac{1}{4}\right)^2 \quad\Rightarrow\quad -\frac{1}{16} \le -(x^2+y^2)^2 \le 0$ so $e^{-1/16} \le e^{-(x^2+y^2)^2} \le e^0 = 1$ since e^t is an increasing function. We have $A(Q) = \frac{1}{4}\pi\left(\frac{1}{2}\right)^2 = \frac{\pi}{16}$, so by Property 11, $e^{-1/16}\,A(Q) \le \iint_Q e^{-(x^2+y^2)^2}\,dA \le 1\cdot A(Q) \quad\Rightarrow\quad \frac{\pi}{16}e^{-1/16} \le \iint_Q e^{-(x^2+y^2)^2}\,dA \le \frac{\pi}{16}$ or we can say $0.1844 < \iint_Q e^{-(x^2+y^2)^2}\,dA < 0.1964$. (We have rounded the lower bound down and the upper bound up to preserve the inequalities.)

55. The average value of a function f of two variables defined on a rectangle R was defined in Section 16.1 [ET 15.1] as $f_{\text{ave}} = \frac{1}{A(R)}\iint_R f(x,y)dA$. Extending this definition to general regions D, we have $f_{\text{ave}} = \frac{1}{A(D)}\iint_D f(x,y)dA$.

Here $D = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le 3x\}$, so $A(D) = \frac{1}{2}(1)(3) = \frac{3}{2}$ and

$$f_{\text{ave}} = \tfrac{1}{A(D)}\iint_D f(x,y)dA = \tfrac{1}{3/2}\int_0^1\int_0^{3x} xy\,dy\,dx$$
$$= \tfrac{2}{3}\int_0^1 \left[\tfrac{1}{2}xy^2\right]_{y=0}^{y=3x} dx = \tfrac{1}{3}\int_0^1 9x^3\,dx = \tfrac{3}{4}x^4\Big]_0^1 = \tfrac{3}{4}$$



57. Since $m \le f(x,y) \le M$, $\iint_D m\,dA \le \iint_D f(x,y)\,dA \le \iint_D M\,dA$ by (8) $\Rightarrow$ $m\iint_D 1\,dA \le \iint_D f(x,y)\,dA \le M\iint_D 1\,dA$ by (7) $\Rightarrow$ $mA(D) \le \iint_D f(x,y)\,dA \le MA(D)$ by (10).

59. $\iint_D (x^2 \tan x + y^3 + 4)\, dA = \iint_D x^2 \tan x\, dA + \iint_D y^3\, dA + \iint_D 4\, dA$. But $x^2 \tan x$ is an odd function of x and D is symmetric with respect to the y-axis, so $\iint_D x^2 \tan x\, dA = 0$. Similarly, y^3 is an odd function of y and D is symmetric with respect to the x-axis, so $\iint_D y^3\, dA = 0$. Thus

$$\iint_D (x^2 \tan x + y^3 + 4)\, dA = 4 \iint_D dA = 4(\text{area of } D) = 4 \cdot \pi\left(\sqrt{2}\right)^2 = 8\pi$$

61. Since $\sqrt{1 - x^2 - y^2} \geq 0$, we can interpret $\iint_D \sqrt{1 - x^2 - y^2}\, dA$ as the volume of the solid that lies below the graph of $z = \sqrt{1 - x^2 - y^2}$ and above the region D in the xy-plane. $z = \sqrt{1 - x^2 - y^2}$ is equivalent to $x^2 + y^2 + z^2 = 1$, $z \geq 0$ which meets the xy-plane in the circle $x^2 + y^2 = 1$, the boundary of D. Thus, the solid is an upper hemisphere of radius 1 which has volume $\frac{1}{2}\left[\frac{4}{3}\pi(1)^3\right] = \frac{2}{3}\pi$.

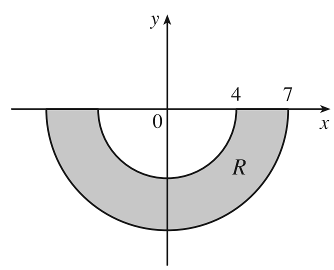
16.4 Double Integrals in Polar Coordinates ET 15.4

1. The region R is more easily described by polar coordinates: $R = \left\{(r, \theta) \mid 0 \leq r \leq 4, 0 \leq \theta \leq \frac{3\pi}{2}\right\}$.
Thus $\iint_R f(x, y)\, dA = \int_0^{3\pi/2} \int_0^4 f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$.

3. The region R is more easily described by rectangular coordinates: $R = \left\{(x, y) \mid -1 \leq x \leq 1, 0 \leq y \leq \frac{1}{2}x + \frac{1}{2}\right\}$.
Thus $\iint_R f(x, y)\, dA = \int_{-1}^{1} \int_0^{(x+1)/2} f(x, y)\, dy\, dx$.

5. The integral $\int_\pi^{2\pi} \int_4^7 r\, dr\, d\theta$ represents the area of the region $R = \{(r, \theta) \mid 4 \leq r \leq 7, \pi \leq \theta \leq 2\pi\}$, the lower half of a ring.

$$\int_\pi^{2\pi} \int_4^7 r\, dr\, d\theta = \left(\int_\pi^{2\pi} d\theta\right)\left(\int_4^7 r\, dr\right)$$
$$= \left[\,\theta\,\right]_\pi^{2\pi} \left[\tfrac{1}{2}r^2\right]_4^7 = \pi \cdot \tfrac{1}{2}(49 - 16) = \tfrac{33\pi}{2}$$



7. The disk D can be described in polar coordinates as $D = \{(r, \theta) \mid 0 \leq r \leq 3, 0 \leq \theta \leq 2\pi\}$. Then

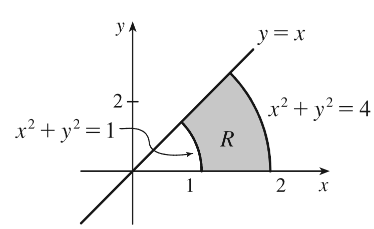
$$\iint_D xy\, dA = \int_0^{2\pi} \int_0^3 (r\cos\theta)(r\sin\theta)\, r\, dr\, d\theta = \left(\int_0^{2\pi} \sin\theta\cos\theta\, d\theta\right)\left(\int_0^3 r^3\, dr\right) = \left[\tfrac{1}{2}\sin^2\theta\right]_0^{2\pi} \left[\tfrac{1}{4}r^4\right]_0^3 = 0.$$

9. $$\iint_R \cos(x^2 + y^2)\, dA = \int_0^\pi \int_0^3 \cos(r^2)\, r\, dr\, d\theta = \left(\int_0^\pi d\theta\right)\left(\int_0^3 r\cos(r^2)\, dr\right)$$
$$= \left[\,\theta\,\right]_0^\pi \left[\tfrac{1}{2}\sin(r^2)\right]_0^3 = \pi \cdot \tfrac{1}{2}(\sin 9 - \sin 0) = \tfrac{\pi}{2}\sin 9$$

11. $$\iint_D e^{-x^2 - y^2}\, dA = \int_{-\pi/2}^{\pi/2} \int_0^2 e^{-r^2}\, r\, dr\, d\theta = \left(\int_{-\pi/2}^{\pi/2} d\theta\right)\left(\int_0^2 re^{-r^2}\, dr\right)$$
$$= \left[\,\theta\,\right]_{-\pi/2}^{\pi/2} \left[-\tfrac{1}{2}e^{-r^2}\right]_0^2 = \pi\left(-\tfrac{1}{2}\right)(e^{-4} - e^0) = \tfrac{\pi}{2}(1 - e^{-4})$$

13. R is the region shown in the figure, and can be described by $R = \{(r, \theta) \mid 0 \leq \theta \leq \pi/4, 1 \leq r \leq 2\}$. Thus $\iint_R \arctan(y/x)\, dA = \int_0^{\pi/4} \int_1^2 \arctan(\tan\theta)\, r\, dr\, d\theta$ since $y/x = \tan\theta$. Also, $\arctan(\tan\theta) = \theta$ for $0 \leq \theta \leq \pi/4$, so the integral becomes

$$\int_0^{\pi/4} \int_1^2 \theta\, r\, dr\, d\theta = \int_0^{\pi/4} \theta\, d\theta \int_1^2 r\, dr = \left[\tfrac{1}{2}\theta^2\right]_0^{\pi/4} \left[\tfrac{1}{2}r^2\right]_1^2 = \tfrac{\pi^2}{32} \cdot \tfrac{3}{2} = \tfrac{3}{64}\pi^2.$$



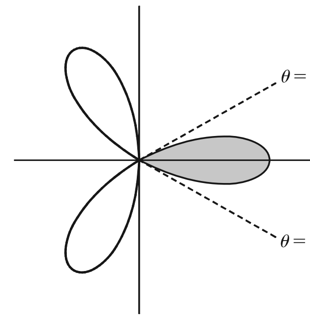
15. One loop is given by the region

$D = \{(r,\theta)\,|\,-\pi/6 \le \theta \le \pi/6, 0 \le r \le \cos 3\theta\}$, so the area is

$$\iint_D dA = \int_{-\pi/6}^{\pi/6}\int_0^{\cos 3\theta} r\,dr\,d\theta = \int_{-\pi/6}^{\pi/6}\left[\frac{1}{2}r^2\right]_{r=0}^{r=\cos 3\theta} d\theta$$

$$= \int_{-\pi/6}^{\pi/6} \frac{1}{2}\cos^2 3\theta\,d\theta = 2\int_0^{\pi/6}\frac{1}{2}\left(\frac{1+\cos 6\theta}{2}\right)d\theta$$

$$= \frac{1}{2}\left[\theta + \frac{1}{6}\sin 6\theta\right]_0^{\pi/6} = \frac{\pi}{12}$$



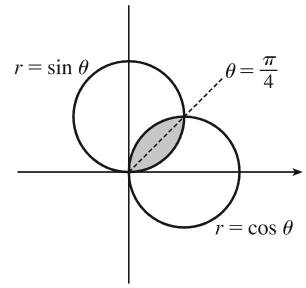
17. By symmetry,

$$A = 2\int_0^{\pi/4}\int_0^{\sin\theta} r\,dr\,d\theta = 2\int_0^{\pi/4}\left[\tfrac{1}{2}r^2\right]_{r=0}^{r=\sin\theta} d\theta$$

$$= \int_0^{\pi/4}\sin^2\theta\,d\theta = \int_0^{\pi/4}\tfrac{1}{2}(1-\cos 2\theta)\,d\theta$$

$$= \tfrac{1}{2}\left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/4}$$

$$= \tfrac{1}{2}\left[\tfrac{\pi}{4} - \tfrac{1}{2}\sin\tfrac{\pi}{2} - 0 + \tfrac{1}{2}\sin 0\right] = \tfrac{1}{8}(\pi - 2)$$



19. $V = \iint_{x^2+y^2\le 4}\sqrt{x^2+y^2}\,dA = \int_0^{2\pi}\int_0^2\sqrt{r^2}\,r\,dr\,d\theta = \int_0^{2\pi}d\theta\int_0^2 r^2\,dr = \left[\,\theta\,\right]_0^{2\pi}\left[\tfrac{1}{3}r^3\right]_0^2 = 2\pi\left(\tfrac{8}{3}\right) = \tfrac{16}{3}\pi$

21. The hyperboloid of two sheets $-x^2-y^2+z^2=1$ intersects the plane $z=2$ when $-x^2-y^2+4=1$ or $x^2+y^2=3$. So the solid region lies above the surface $z=\sqrt{1+x^2+y^2}$ and below the plane $z=2$ for $x^2+y^2\le 3$, and its volume is

$$V = \iint_{x^2+y^2\le 3}\left(2-\sqrt{1+x^2+y^2}\right)dA = \int_0^{2\pi}\int_0^{\sqrt{3}}\left(2-\sqrt{1+r^2}\right)r\,dr\,d\theta$$

$$= \int_0^{2\pi}d\theta\int_0^{\sqrt{3}}\left(2r - r\sqrt{1+r^2}\right)dr = \left[\,\theta\,\right]_0^{2\pi}\left[r^2 - \tfrac{1}{3}(1+r^2)^{3/2}\right]_0^{\sqrt{3}}$$

$$= 2\pi\left(3 - \tfrac{8}{3} - 0 + \tfrac{1}{3}\right) = \tfrac{4}{3}\pi$$

23. By symmetry,

$$V = 2\iint_{x^2+y^2\le a^2}\sqrt{a^2-x^2-y^2}\,dA = 2\int_0^{2\pi}\int_0^a\sqrt{a^2-r^2}\,r\,dr\,d\theta = 2\int_0^{2\pi}d\theta\int_0^a r\sqrt{a^2-r^2}\,dr$$

$$= 2\left[\,\theta\,\right]_0^{2\pi}\left[-\tfrac{1}{3}(a^2-r^2)^{3/2}\right]_0^a = 2(2\pi)\left(0+\tfrac{1}{3}a^3\right) = \tfrac{4\pi}{3}a^3$$

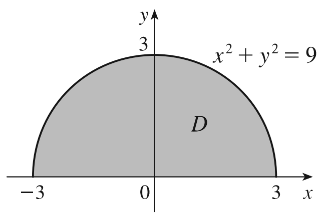
25. The cone $z=\sqrt{x^2+y^2}$ intersects the sphere $x^2+y^2+z^2=1$ when $x^2+y^2+\left(\sqrt{x^2+y^2}\right)^2=1$ or $x^2+y^2=\tfrac{1}{2}$. So

$$V = \iint_{x^2+y^2\le 1/2}\left(\sqrt{1-x^2-y^2}-\sqrt{x^2+y^2}\right)dA = \int_0^{2\pi}\int_0^{1/\sqrt{2}}\left(\sqrt{1-r^2}-r\right)r\,dr\,d\theta$$

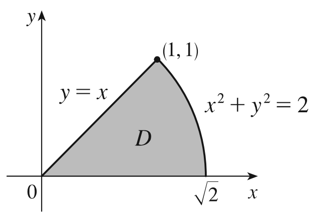
$$= \int_0^{2\pi}d\theta\int_0^{1/\sqrt{2}}\left(r\sqrt{1-r^2}-r^2\right)dr = \left[\,\theta\,\right]_0^{2\pi}\left[-\tfrac{1}{3}(1-r^2)^{3/2}-\tfrac{1}{3}r^3\right]_0^{1/\sqrt{2}} = 2\pi\left(-\tfrac{1}{3}\right)\left(\tfrac{1}{\sqrt{2}}-1\right) = \tfrac{\pi}{3}\left(2-\sqrt{2}\right)$$

27. The given solid is the region inside the cylinder $x^2 + y^2 = 4$ between the surfaces $z = \sqrt{64 - 4x^2 - 4y^2}$ and $z = -\sqrt{64 - 4x^2 - 4y^2}$. So

$$V = \iint_{x^2+y^2\le 4} \left[\sqrt{64 - 4x^2 - 4y^2} - \left(-\sqrt{64 - 4x^2 - 4y^2}\right)\right] dA = \iint_{x^2+y^2\le 4} 2\sqrt{64 - 4x^2 - 4y^2}\, dA$$
$$= 4\int_0^{2\pi}\int_0^2 \sqrt{16 - r^2}\, r\, dr\, d\theta = 4\int_0^{2\pi} d\theta \int_0^2 r\sqrt{16 - r^2}\, dr = 4\left[\,\theta\,\right]_0^{2\pi}\left[-\tfrac{1}{3}(16 - r^2)^{3/2}\right]_0^2$$
$$= 8\pi\left(-\tfrac{1}{3}\right)(12^{3/2} - 16^{2/3}) = \tfrac{8\pi}{3}\left(64 - 24\sqrt{3}\right)$$

29. 

$$\int_{-3}^{3}\int_0^{\sqrt{9-x^2}} \sin(x^2 + y^2)dy\, dx = \int_0^{\pi}\int_0^3 \sin\left(r^2\right) r\, dr\, d\theta$$
$$= \int_0^{\pi} d\theta \int_0^3 r\sin\left(r^2\right)\, dr = [\theta]_0^{\pi}\left[-\tfrac{1}{2}\cos\left(r^2\right)\right]_0^3$$
$$= \pi\left(-\tfrac{1}{2}\right)(\cos 9 - 1) = \tfrac{\pi}{2}(1 - \cos 9)$$

31. 

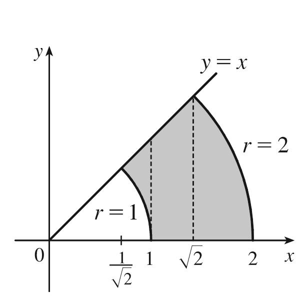
$$\int_0^{\pi/4}\int_0^{\sqrt{2}} (r\cos\theta + r\sin\theta)\, r\, dr\, d\theta = \int_0^{\pi/4}(\cos\theta + \sin\theta)\, d\theta \int_0^{\sqrt{2}} r^2\, dr$$
$$= [\sin\theta - \cos\theta]_0^{\pi/4}\left[\tfrac{1}{3}r^3\right]_0^{\sqrt{2}}$$
$$= \left[\tfrac{\sqrt{2}}{2} - \tfrac{\sqrt{2}}{2} - 0 + 1\right]\cdot\tfrac{1}{3}\left(2\sqrt{2} - 0\right) = \tfrac{2\sqrt{2}}{3}$$

33. The surface of the water in the pool is a circular disk D with radius 20 ft. If we place D on coordinate axes with the origin at the center of D and define $f(x, y)$ to be the depth of the water at (x, y), then the volume of water in the pool is the volume of the solid that lies above $D = \left\{(x, y) \mid x^2 + y^2 \le 400\right\}$ and below the graph of $f(x, y)$. We can associate north with the positive y-direction, so we are given that the depth is constant in the x-direction and the depth increases linearly in the y-direction from $f(0, -20) = 2$ to $f(0, 20) = 7$. The trace in the yz-plane is a line segment from $(0, -20, 2)$ to $(0, 20, 7)$. The slope of this line is $\frac{7 - 2}{20 - (-20)} = \frac{1}{8}$, so an equation of the line is $z - 7 = \frac{1}{8}(y - 20) \;\Rightarrow\; z = \frac{1}{8}y + \frac{9}{2}$. Since $f(x, y)$ is independent of x, $f(x, y) = \frac{1}{8}y + \frac{9}{2}$. Thus the volume is given by $\iint_D f(x, y)\, dA$, which is most conveniently evaluated using polar coordinates. Then $D = \{(r, \theta) \mid 0 \le r \le 20, 0 \le \theta \le 2\pi\}$ and substituting $x = r\cos\theta$, $y = r\sin\theta$ the integral becomes

$$\int_0^{2\pi}\int_0^{20}\left(\tfrac{1}{8}r\sin\theta + \tfrac{9}{2}\right) r\, dr\, d\theta = \int_0^{2\pi}\left[\tfrac{1}{24}r^3\sin\theta + \tfrac{9}{4}r^2\right]_{r=0}^{r=20} d\theta = \int_0^{2\pi}\left(\tfrac{1000}{3}\sin\theta + 900\right) d\theta$$
$$= \left[-\tfrac{1000}{3}\cos\theta + 900\theta\right]_0^{2\pi} = 1800\pi$$

Thus the pool contains $1800\pi \approx 5655$ ft^3 of water.

35.
$$\int_{1/\sqrt{2}}^{1}\int_{\sqrt{1-x^2}}^{x} xy\, dy\, dx + \int_1^{\sqrt{2}}\int_0^x xy\, dy\, dx + \int_{\sqrt{2}}^{2}\int_0^{\sqrt{4-x^2}} xy\, dy\, dx$$
$$= \int_0^{\pi/4}\int_1^2 r^3\cos\theta\sin\theta\, dr\, d\theta = \int_0^{\pi/4}\left[\frac{r^4}{4}\cos\theta\sin\theta\right]_{r=1}^{r=2} d\theta$$
$$= \frac{15}{4}\int_0^{\pi/4}\sin\theta\cos\theta\, d\theta = \frac{15}{4}\left[\frac{\sin^2\theta}{2}\right]_0^{\pi/4} = \frac{15}{16}$$



37. (a) We integrate by parts with $u = x$ and $dv = xe^{-x^2}\,dx$. Then $du = dx$ and $v = -\frac{1}{2}e^{-x^2}$, so

$$\begin{aligned}\int_0^\infty x^2e^{-x^2}\,dx &= \lim_{t\to\infty}\int_0^t x^2e^{-x^2}\,dx = \lim_{t\to\infty}\left(-\tfrac{1}{2}xe^{-x^2}\Big]_0^t + \int_0^t \tfrac{1}{2}e^{-x^2}\,dx\right)\\ &= \lim_{t\to\infty}\left(-\tfrac{1}{2}te^{-t^2}\right) + \tfrac{1}{2}\int_0^\infty e^{-x^2}\,dx = 0 + \tfrac{1}{2}\int_0^\infty e^{-x^2}\,dx \qquad \text{[by l'Hospital's Rule]}\\ &= \tfrac{1}{4}\int_{-\infty}^\infty e^{-x^2}\,dx \qquad \text{[since } e^{-x^2} \text{ is an even function]}\\ &= \tfrac{1}{4}\sqrt{\pi} \qquad \text{[by Exercise 36(c)]}\end{aligned}$$

(b) Let $u = \sqrt{x}$. Then $u^2 = x \quad\Rightarrow\quad dx = 2u\,du \quad\Rightarrow$

$$\int_0^\infty \sqrt{x}e^{-x}\,dx = \lim_{t\to\infty}\int_0^t \sqrt{x}\,e^{-x}\,dx = \lim_{t\to\infty}\int_0^{\sqrt{t}} ue^{-u^2}2u\,du = 2\int_0^\infty u^2e^{-u^2}\,du = 2\left(\tfrac{1}{4}\sqrt{\pi}\right) \quad \text{[by part(a)]} \;= \tfrac{1}{2}\sqrt{\pi}.$$

16.5 Applications of Double Integrals ET 15.5

1. $Q = \iint_D \sigma(x,y)\,dA = \int_1^3\int_0^2 (2xy+y^2)\,dy\,dx = \int_1^3\left[xy^2 + \frac{1}{3}y^3\right]_{y=0}^{y=2}dx$

$= \int_1^3\left(4x+\frac{8}{3}\right)dx = \left[2x^2+\frac{8}{3}x\right]_1^3 = 16 + \frac{16}{3} = \frac{64}{3}$ C

3. $m = \iint_D \rho(x,y)\,dA = \int_0^2\int_{-1}^1 xy^2\,dy\,dx = \int_0^2 x\,dx\int_{-1}^1 y^2\,dy = \left[\frac{1}{2}x^2\right]_0^2\left[\frac{1}{3}y^3\right]_{-1}^1 = 2\cdot\frac{2}{3} = \frac{4}{3}$,

$\overline{x} = \frac{1}{m}\iint_D x\rho(x,y)\,dA = \frac{3}{4}\int_0^2\int_{-1}^1 x^2y^2\,dy\,dx = \frac{3}{4}\int_0^2 x^2\,dx\int_{-1}^1 y^2\,dy = \frac{3}{4}\left[\frac{1}{3}x^3\right]_0^2\left[\frac{1}{3}y^3\right]_{-1}^1 = \frac{3}{4}\cdot\frac{8}{3}\cdot\frac{2}{3} = \frac{4}{3}$,

$\overline{y} = \frac{1}{m}\iint_D y\rho(x,y)\,dA = \frac{3}{4}\int_0^2\int_{-1}^1 xy^3\,dy\,dx = \frac{3}{4}\int_0^2 x\,dx\int_{-1}^1 y^3\,dy = \frac{3}{4}\left[\frac{1}{2}x^2\right]_0^2\left[\frac{1}{4}y^4\right]_{-1}^1 = \frac{3}{4}\cdot 2\cdot 0 = 0.$

Hence, $(\overline{x},\overline{y}) = \left(\frac{4}{3},0\right)$.

5. $m = \int_0^2\int_{x/2}^{3-x}(x+y)\,dy\,dx = \int_0^2\left[xy+\frac{1}{2}y^2\right]_{y=x/2}^{y=3-x}dx = \int_0^2\left[x\left(3-\frac{3}{2}x\right)+\frac{1}{2}(3-x)^2-\frac{1}{8}x^2\right]dx$

$= \int_0^2\left(-\frac{9}{8}x^2+\frac{9}{2}\right)dx = \left[-\frac{9}{8}\left(\frac{1}{3}x^3\right)+\frac{9}{2}x\right]_0^2 = 6,$

$M_y = \int_0^2\int_{x/2}^{3-x}(x^2+xy)\,dy\,dx = \int_0^2\left[x^2y+\frac{1}{2}xy^2\right]_{y=x/2}^{y=3-x}dx = \int_0^2\left(\frac{9}{2}x-\frac{9}{8}x^3\right)dx = \frac{9}{2}$,

$M_x = \int_0^2\int_{x/2}^{3-y}(xy+y^2)\,dy\,dx = \int_0^2\left[\frac{1}{2}xy^2+\frac{1}{3}y^3\right]_{y=x/2}^{y=3-x}dx = \int_0^2\left(9-\frac{9}{2}x\right)dx = 9.$

Hence $m = 6$, $(\overline{x},\overline{y}) = \left(\dfrac{M_y}{m},\dfrac{M_x}{m}\right) = \left(\dfrac{3}{4},\dfrac{3}{2}\right)$.

7. $m = \int_0^1\int_0^{e^x} y\,dy\,dx = \int_0^1\left[\frac{1}{2}y^2\right]_{y=0}^{y=e^x}dx = \frac{1}{2}\int_0^1 e^{2x}\,dx = \frac{1}{4}e^{2x}\Big]_0^1 = \frac{1}{4}(e^2-1)$,

$M_y = \int_0^1\int_0^{e^x} xy\,dy\,dx = \frac{1}{2}\int_0^1 xe^{2x}\,dx = \frac{1}{2}\left[\frac{1}{2}xe^{2x}-\frac{1}{4}e^{2x}\right]_0^1 = \frac{1}{8}(e^2+1)$,

$M_x = \int_0^1\int_0^{e^x} y^2\,dy\,dx = \int_0^1\left[\frac{1}{3}y^3\right]_{y=0}^{y=e^x}dx = \frac{1}{3}\int_0^1 e^{3x}\,dx = \frac{1}{3}\left[\frac{1}{3}e^{3x}\right]_0^1 = \frac{1}{9}(e^3-1).$

Hence $m = \frac{1}{4}(e^2-1)$, $(\overline{x},\overline{y}) = \left(\dfrac{\frac{1}{8}(e^2+1)}{\frac{1}{4}(e^2-1)},\dfrac{\frac{1}{9}(e^3-1)}{\frac{1}{4}(e^2-1)}\right) = \left(\dfrac{e^2+1}{2(e^2-1)},\dfrac{4(e^3-1)}{9(e^2-1)}\right)$.