59. $\iint_D (x^2 \tan x + y^3 + 4)\ dA = \iint_D x^2 \tan x\, dA + \iint_D y^3\, dA + \iint_D 4\, dA$. But $x^2 \tan x$ is an odd function of x and D is symmetric with respect to the y-axis, so $\iint_D x^2 \tan x\, dA = 0$. Similarly, y^3 is an odd function of y and D is symmetric with respect to the x-axis, so $\iint_D y^3\, dA = 0$. Thus

$$\iint_D (x^2 \tan x + y^3 + 4)\, dA = 4 \iint_D\ dA = 4(\text{area of } D) = 4 \cdot \pi\left(\sqrt{2}\right)^2 = 8\pi$$

61. Since $\sqrt{1 - x^2 - y^2} \geq 0$, we can interpret $\iint_D \sqrt{1 - x^2 - y^2}\, dA$ as the volume of the solid that lies below the graph of $z = \sqrt{1 - x^2 - y^2}$ and above the region D in the xy-plane. $z = \sqrt{1 - x^2 - y^2}$ is equivalent to $x^2 + y^2 + z^2 = 1$, $z \geq 0$ which meets the xy-plane in the circle $x^2 + y^2 = 1$, the boundary of D. Thus, the solid is an upper hemisphere of radius 1 which has volume $\frac{1}{2}\left[\frac{4}{3}\pi(1)^3\right] = \frac{2}{3}\pi$.

16.4 Double Integrals in Polar Coordinates **ET 15.4**

1. The region R is more easily described by polar coordinates: $R = \left\{(r, \theta) \mid 0 \leq r \leq 4, 0 \leq \theta \leq \frac{3\pi}{2}\right\}$.

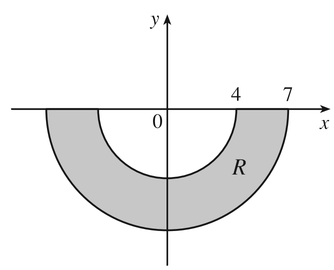
Thus $\iint_R f(x, y)\, dA = \int_0^{3\pi/2} \int_0^4 f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$.

3. The region R is more easily described by rectangular coordinates: $R = \left\{(x, y) \mid -1 \leq x \leq 1, 0 \leq y \leq \frac{1}{2}x + \frac{1}{2}\right\}$.

Thus $\iint_R f(x, y)\, dA = \int_{-1}^1 \int_0^{(x+1)/2} f(x, y)\, dy\, dx$.

5. The integral $\int_\pi^{2\pi} \int_4^7 r\, dr\, d\theta$ represents the area of the region $R = \{(r, \theta) \mid 4 \leq r \leq 7, \pi \leq \theta \leq 2\pi\}$, the lower half of a ring.

$$\int_\pi^{2\pi} \int_4^7 r\, dr\, d\theta = \left(\int_\pi^{2\pi} d\theta\right)\left(\int_4^7 r\, dr\right) = \left[\,\theta\,\right]_\pi^{2\pi} \left[\tfrac{1}{2}r^2\right]_4^7 = \pi \cdot \tfrac{1}{2}(49 - 16) = \tfrac{33\pi}{2}$$



7. The disk D can be described in polar coordinates as $D = \{(r, \theta) \mid 0 \leq r \leq 3, 0 \leq \theta \leq 2\pi\}$. Then

$$\iint_D xy\, dA = \int_0^{2\pi} \int_0^3 (r\cos\theta)(r\sin\theta)\, r\, dr\, d\theta = \left(\int_0^{2\pi} \sin\theta\cos\theta\, d\theta\right)\left(\int_0^3 r^3\, dr\right) = \left[\tfrac{1}{2}\sin^2\theta\right]_0^{2\pi} \left[\tfrac{1}{4}r^4\right]_0^3 = 0.$$

9. $\iint_R \cos(x^2 + y^2)\, dA = \int_0^\pi \int_0^3 \cos(r^2)\, r\, dr\, d\theta = \left(\int_0^\pi d\theta\right)\left(\int_0^3 r\cos(r^2)\, dr\right)$

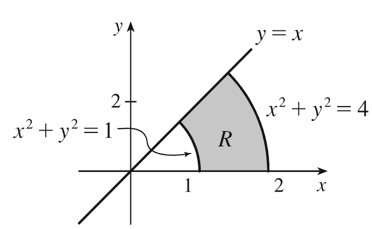
$$= \left[\,\theta\,\right]_0^\pi \left[\tfrac{1}{2}\sin(r^2)\right]_0^3 = \pi \cdot \tfrac{1}{2}(\sin 9 - \sin 0) = \tfrac{\pi}{2}\sin 9$$

11. $\iint_D e^{-x^2-y^2}\, dA = \int_{-\pi/2}^{\pi/2} \int_0^2 e^{-r^2}\, r\, dr\, d\theta = \left(\int_{-\pi/2}^{\pi/2} d\theta\right)\left(\int_0^2 re^{-r^2}\, dr\right)$

$$= \left[\,\theta\,\right]_{-\pi/2}^{\pi/2} \left[-\tfrac{1}{2}e^{-r^2}\right]_0^2 = \pi\left(-\tfrac{1}{2}\right)(e^{-4} - e^0) = \tfrac{\pi}{2}(1 - e^{-4})$$

13. R is the region shown in the figure, and can be described by $R = \{(r, \theta) \mid 0 \leq \theta \leq \pi/4, 1 \leq r \leq 2\}$. Thus $\iint_R \arctan(y/x)\, dA = \int_0^{\pi/4} \int_1^2 \arctan(\tan\theta)\, r\, dr\, d\theta$ since $y/x = \tan\theta$. Also, $\arctan(\tan\theta) = \theta$ for $0 \leq \theta \leq \pi/4$, so the integral becomes

$\int_0^{\pi/4} \int_1^2 \theta\, r\, dr\, d\theta = \int_0^{\pi/4} \theta\, d\theta\, \int_1^2 r\, dr = \left[\frac{1}{2}\theta^2\right]_0^{\pi/4} \left[\frac{1}{2}r^2\right]_1^2 = \frac{\pi^2}{32} \cdot \frac{3}{2} = \frac{3}{64}\pi^2$.



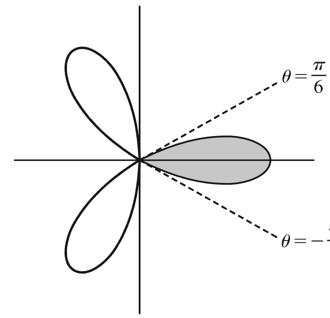
15. One loop is given by the region

$D = \{(r,\theta) \mid -\pi/6 \le \theta \le \pi/6, 0 \le r \le \cos 3\theta\}$, so the area is

$$\iint_D dA = \int_{-\pi/6}^{\pi/6} \int_0^{\cos 3\theta} r\,dr\,d\theta = \int_{-\pi/6}^{\pi/6} \left[\frac{1}{2}r^2\right]_{r=0}^{r=\cos 3\theta} d\theta$$

$$= \int_{-\pi/6}^{\pi/6} \frac{1}{2}\cos^2 3\theta\,d\theta = 2\int_0^{\pi/6} \frac{1}{2}\left(\frac{1+\cos 6\theta}{2}\right) d\theta$$

$$= \frac{1}{2}\left[\theta + \frac{1}{6}\sin 6\theta\right]_0^{\pi/6} = \frac{\pi}{12}$$



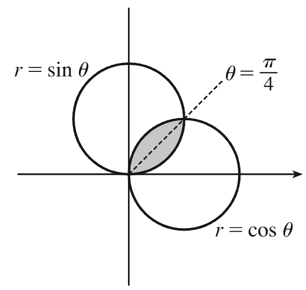
17. By symmetry,

$$A = 2\int_0^{\pi/4}\int_0^{\sin\theta} r\,dr\,d\theta = 2\int_0^{\pi/4}\left[\tfrac{1}{2}r^2\right]_{r=0}^{r=\sin\theta} d\theta$$

$$= \int_0^{\pi/4} \sin^2\theta\,d\theta = \int_0^{\pi/4} \tfrac{1}{2}(1-\cos 2\theta)\,d\theta$$

$$= \tfrac{1}{2}\left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/4}$$

$$= \tfrac{1}{2}\left[\tfrac{\pi}{4} - \tfrac{1}{2}\sin\tfrac{\pi}{2} - 0 + \tfrac{1}{2}\sin 0\right] = \tfrac{1}{8}(\pi - 2)$$



19. $V = \iint_{x^2+y^2\le 4} \sqrt{x^2+y^2}\,dA = \int_0^{2\pi}\int_0^2 \sqrt{r^2}\,r\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^2 r^2\,dr = \left[\theta\right]_0^{2\pi}\left[\tfrac{1}{3}r^3\right]_0^2 = 2\pi\left(\tfrac{8}{3}\right) = \tfrac{16}{3}\pi$

21. The hyperboloid of two sheets $-x^2 - y^2 + z^2 = 1$ intersects the plane $z = 2$ when $-x^2 - y^2 + 4 = 1$ or $x^2 + y^2 = 3$. So the solid region lies above the surface $z = \sqrt{1+x^2+y^2}$ and below the plane $z = 2$ for $x^2 + y^2 \le 3$, and its volume is

$$V = \iint_{x^2+y^2\le 3} \left(2 - \sqrt{1+x^2+y^2}\right) dA = \int_0^{2\pi}\int_0^{\sqrt{3}} (2 - \sqrt{1+r^2})\,r\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\sqrt{3}} (2r - r\sqrt{1+r^2})dr = \left[\theta\right]_0^{2\pi}\left[r^2 - \tfrac{1}{3}(1+r^2)^{3/2}\right]_0^{\sqrt{3}}$$

$$= 2\pi\left(3 - \tfrac{8}{3} - 0 + \tfrac{1}{3}\right) = \tfrac{4}{3}\pi$$

23. By symmetry,

$$V = 2\iint_{x^2+y^2\le a^2} \sqrt{a^2-x^2-y^2}\,dA = 2\int_0^{2\pi}\int_0^a \sqrt{a^2-r^2}\,r\,dr\,d\theta = 2\int_0^{2\pi} d\theta \int_0^a r\sqrt{a^2-r^2}\,dr$$

$$= 2\left[\theta\right]_0^{2\pi}\left[-\tfrac{1}{3}(a^2-r^2)^{3/2}\right]_0^a = 2(2\pi)\left(0 + \tfrac{1}{3}a^3\right) = \tfrac{4\pi}{3}a^3$$

25. The cone $z = \sqrt{x^2+y^2}$ intersects the sphere $x^2+y^2+z^2 = 1$ when $x^2 + y^2 + \left(\sqrt{x^2+y^2}\right)^2 = 1$ or $x^2+y^2 = \frac{1}{2}$. So

$$V = \iint_{x^2+y^2\le 1/2} \left(\sqrt{1-x^2-y^2} - \sqrt{x^2+y^2}\right) dA = \int_0^{2\pi}\int_0^{1/\sqrt{2}} \left(\sqrt{1-r^2} - r\right) r\,dr\,d\theta$$

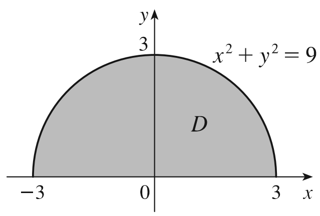
$$= \int_0^{2\pi} d\theta \int_0^{1/\sqrt{2}} \left(r\sqrt{1-r^2} - r^2\right) dr = \left[\theta\right]_0^{2\pi}\left[-\tfrac{1}{3}(1-r^2)^{3/2} - \tfrac{1}{3}r^3\right]_0^{1/\sqrt{2}} = 2\pi\left(-\tfrac{1}{3}\right)\left(\tfrac{1}{\sqrt{2}} - 1\right) = \tfrac{\pi}{3}\left(2 - \sqrt{2}\right)$$

27. The given solid is the region inside the cylinder $x^2 + y^2 = 4$ between the surfaces $z = \sqrt{64 - 4x^2 - 4y^2}$ and $z = -\sqrt{64 - 4x^2 - 4y^2}$. So

$$V = \iint\limits_{x^2+y^2\le 4} \left[\sqrt{64-4x^2-4y^2} - \left(-\sqrt{64-4x^2-4y^2}\right)\right] dA = \iint\limits_{x^2+y^2\le 4} 2\sqrt{64-4x^2-4y^2}\, dA$$

$$= 4\int_0^{2\pi}\int_0^2 \sqrt{16-r^2}\, r\, dr\, d\theta = 4\int_0^{2\pi} d\theta \int_0^2 r\sqrt{16-r^2}\, dr = 4\left[\,\theta\,\right]_0^{2\pi}\left[-\tfrac{1}{3}(16-r^2)^{3/2}\right]_0^2$$

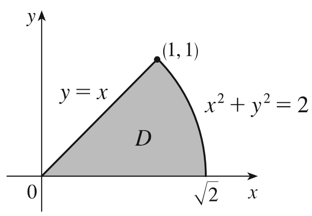
$$= 8\pi\left(-\tfrac{1}{3}\right)(12^{3/2} - 16^{2/3}) = \tfrac{8\pi}{3}\left(64 - 24\sqrt{3}\right)$$

29. 

$$\int_{-3}^{3}\int_0^{\sqrt{9-x^2}} \sin(x^2+y^2)\,dy\,dx = \int_0^{\pi}\int_0^3 \sin\left(r^2\right) r\,dr\,d\theta$$

$$= \int_0^{\pi} d\theta \int_0^3 r\sin\left(r^2\right)\,dr = [\theta]_0^{\pi}\left[-\tfrac{1}{2}\cos\left(r^2\right)\right]_0^3$$

$$= \pi\left(-\tfrac{1}{2}\right)(\cos 9 - 1) = \tfrac{\pi}{2}(1-\cos 9)$$

31. 

$$\int_0^{\pi/4}\int_0^{\sqrt{2}} (r\cos\theta + r\sin\theta)\, r\,dr\,d\theta = \int_0^{\pi/4} (\cos\theta + \sin\theta)\,d\theta \int_0^{\sqrt{2}} r^2\,dr$$

$$= [\sin\theta - \cos\theta]_0^{\pi/4}\left[\tfrac{1}{3}r^3\right]_0^{\sqrt{2}}$$

$$= \left[\tfrac{\sqrt{2}}{2} - \tfrac{\sqrt{2}}{2} - 0 + 1\right]\cdot \tfrac{1}{3}\left(2\sqrt{2} - 0\right) = \tfrac{2\sqrt{2}}{3}$$

33. The surface of the water in the pool is a circular disk D with radius 20 ft. If we place D on coordinate axes with the origin at the center of D and define $f(x,y)$ to be the depth of the water at (x,y), then the volume of water in the pool is the volume of the solid that lies above $D = \left\{(x,y) \mid x^2+y^2 \le 400\right\}$ and below the graph of $f(x,y)$. We can associate north with the positive y-direction, so we are given that the depth is constant in the x-direction and the depth increases linearly in the y-direction from $f(0,-20) = 2$ to $f(0,20) = 7$. The trace in the yz-plane is a line segment from $(0,-20,2)$ to $(0,20,7)$. The slope of this line is $\frac{7-2}{20-(-20)} = \frac{1}{8}$, so an equation of the line is $z - 7 = \frac{1}{8}(y-20) \;\Rightarrow\; z = \frac{1}{8}y + \frac{9}{2}$. Since $f(x,y)$ is independent of x, $f(x,y) = \frac{1}{8}y + \frac{9}{2}$. Thus the volume is given by $\iint_D f(x,y)\,dA$, which is most conveniently evaluated using polar coordinates. Then $D = \{(r,\theta) \mid 0 \le r \le 20, 0 \le \theta \le 2\pi\}$ and substituting $x = r\cos\theta$, $y = r\sin\theta$ the integral becomes

$$\int_0^{2\pi}\int_0^{20}\left(\tfrac{1}{8}r\sin\theta + \tfrac{9}{2}\right) r\,dr\,d\theta = \int_0^{2\pi}\left[\tfrac{1}{24}r^3\sin\theta + \tfrac{9}{4}r^2\right]_{r=0}^{r=20} d\theta = \int_0^{2\pi}\left(\tfrac{1000}{3}\sin\theta + 900\right)d\theta$$

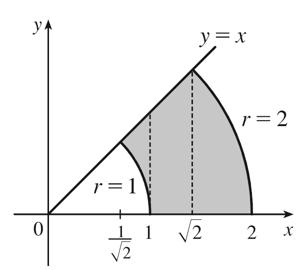
$$= \left[-\tfrac{1000}{3}\cos\theta + 900\theta\right]_0^{2\pi} = 1800\pi$$

Thus the pool contains $1800\pi \approx 5655$ ft^3 of water.

35. $$\int_{1/\sqrt{2}}^{1}\int_{\sqrt{1-x^2}}^{x} xy\,dy\,dx + \int_1^{\sqrt{2}}\int_0^x xy\,dy\,dx + \int_{\sqrt{2}}^{2}\int_0^{\sqrt{4-x^2}} xy\,dy\,dx$$

$$= \int_0^{\pi/4}\int_1^2 r^3\cos\theta\sin\theta\,dr\,d\theta = \int_0^{\pi/4}\left[\frac{r^4}{4}\cos\theta\sin\theta\right]_{r=1}^{r=2} d\theta$$

$$= \frac{15}{4}\int_0^{\pi/4}\sin\theta\cos\theta\,d\theta = \frac{15}{4}\left[\frac{\sin^2\theta}{2}\right]_0^{\pi/4} = \frac{15}{16}$$



37. (a) We integrate by parts with $u = x$ and $dv = xe^{-x^2}\,dx$. Then $du = dx$ and $v = -\frac{1}{2}e^{-x^2}$, so

$$\begin{aligned}\int_0^\infty x^2e^{-x^2}\,dx &= \lim_{t\to\infty}\int_0^t x^2e^{-x^2}\,dx = \lim_{t\to\infty}\left(\left[-\tfrac{1}{2}xe^{-x^2}\right]_0^t + \int_0^t \tfrac{1}{2}e^{-x^2}\,dx\right)\\ &= \lim_{t\to\infty}\left(-\tfrac{1}{2}te^{-t^2}\right) + \tfrac{1}{2}\int_0^\infty e^{-x^2}\,dx = 0 + \tfrac{1}{2}\int_0^\infty e^{-x^2}\,dx \qquad \text{[by l'Hospital's Rule]}\\ &= \tfrac{1}{4}\int_{-\infty}^\infty e^{-x^2}\,dx \qquad [\text{since } e^{-x^2} \text{ is an even function}]\\ &= \tfrac{1}{4}\sqrt{\pi} \qquad \text{[by Exercise 36(c)]}\end{aligned}$$

(b) Let $u = \sqrt{x}$. Then $u^2 = x \;\Rightarrow\; dx = 2u\,du \;\Rightarrow$

$$\int_0^\infty \sqrt{x}e^{-x}\,dx = \lim_{t\to\infty}\int_0^t \sqrt{x}\,e^{-x}\,dx = \lim_{t\to\infty}\int_0^{\sqrt{t}} ue^{-u^2}2u\,du = 2\int_0^\infty u^2e^{-u^2}\,du = 2\left(\tfrac{1}{4}\sqrt{\pi}\right) \quad \text{[by part(a)]} \;= \tfrac{1}{2}\sqrt{\pi}.$$

16.5 Applications of Double Integrals — ET 15.5

1. $Q = \iint_D \sigma(x,y)\,dA = \int_1^3\int_0^2 (2xy+y^2)\,dy\,dx = \int_1^3 \left[xy^2 + \frac{1}{3}y^3\right]_{y=0}^{y=2} dx$

$= \int_1^3 \left(4x + \frac{8}{3}\right) dx = \left[2x^2 + \frac{8}{3}x\right]_1^3 = 16 + \frac{16}{3} = \frac{64}{3}$ C

3. $m = \iint_D \rho(x,y)\,dA = \int_0^2\int_{-1}^1 xy^2\,dy\,dx = \int_0^2 x\,dx\,\int_{-1}^1 y^2\,dy = \left[\frac{1}{2}x^2\right]_0^2\left[\frac{1}{3}y^3\right]_{-1}^1 = 2\cdot\frac{2}{3} = \frac{4}{3}$,

$\overline{x} = \frac{1}{m}\iint_D x\rho(x,y)\,dA = \frac{3}{4}\int_0^2\int_{-1}^1 x^2y^2\,dy\,dx = \frac{3}{4}\int_0^2 x^2\,dx\,\int_{-1}^1 y^2\,dy = \frac{3}{4}\left[\frac{1}{3}x^3\right]_0^2\left[\frac{1}{3}y^3\right]_{-1}^1 = \frac{3}{4}\cdot\frac{8}{3}\cdot\frac{2}{3} = \frac{4}{3}$,

$\overline{y} = \frac{1}{m}\iint_D y\rho(x,y)\,dA = \frac{3}{4}\int_0^2\int_{-1}^1 xy^3\,dy\,dx = \frac{3}{4}\int_0^2 x\,dx\,\int_{-1}^1 y^3\,dy = \frac{3}{4}\left[\frac{1}{2}x^2\right]_0^2\left[\frac{1}{4}y^4\right]_{-1}^1 = \frac{3}{4}\cdot 2\cdot 0 = 0$.

Hence, $(\overline{x},\overline{y}) = \left(\frac{4}{3},0\right)$.

5. $m = \int_0^2\int_{x/2}^{3-x}(x+y)\,dy\,dx = \int_0^2\left[xy + \frac{1}{2}y^2\right]_{y=x/2}^{y=3-x} dx = \int_0^2\left[x\left(3-\frac{3}{2}x\right) + \frac{1}{2}(3-x)^2 - \frac{1}{8}x^2\right] dx$

$= \int_0^2\left(-\frac{9}{8}x^2 + \frac{9}{2}\right) dx = \left[-\frac{9}{8}\left(\frac{1}{3}x^3\right) + \frac{9}{2}x\right]_0^2 = 6$,

$M_y = \int_0^2\int_{x/2}^{3-x}(x^2+xy)\,dy\,dx = \int_0^2\left[x^2y + \frac{1}{2}xy^2\right]_{y=x/2}^{y=3-x} dx = \int_0^2\left(\frac{9}{2}x - \frac{9}{8}x^3\right) dx = \frac{9}{2}$,

$M_x = \int_0^2\int_{x/2}^{3-y}(xy+y^2)\,dy\,dx = \int_0^2\left[\frac{1}{2}xy^2 + \frac{1}{3}y^3\right]_{y=x/2}^{y=3-x} dx = \int_0^2\left(9 - \frac{9}{2}x\right) dx = 9$.

Hence $m = 6$, $(\overline{x},\overline{y}) = \left(\dfrac{M_y}{m},\dfrac{M_x}{m}\right) = \left(\dfrac{3}{4},\dfrac{3}{2}\right)$.

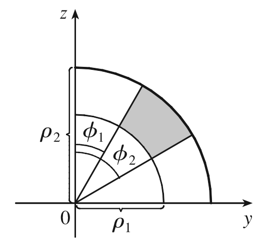
7. $m = \int_0^1\int_0^{e^x} y\,dy\,dx = \int_0^1\left[\frac{1}{2}y^2\right]_{y=0}^{y=e^x} dx = \frac{1}{2}\int_0^1 e^{2x}\,dx = \frac{1}{4}e^{2x}\Big]_0^1 = \frac{1}{4}(e^2-1)$,

$M_y = \int_0^1\int_0^{e^x} xy\,dy\,dx = \frac{1}{2}\int_0^1 xe^{2x}\,dx = \frac{1}{2}\left[\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}\right]_0^1 = \frac{1}{8}(e^2+1)$,

$M_x = \int_0^1\int_0^{e^x} y^2\,dy\,dx = \int_0^1\left[\frac{1}{3}y^3\right]_{y=0}^{y=e^x} dx = \frac{1}{3}\int_0^1 e^{3x}\,dx = \frac{1}{3}\left[\frac{1}{3}e^{3x}\right]_0^1 = \frac{1}{9}(e^3-1)$.

Hence $m = \frac{1}{4}(e^2-1)$, $(\overline{x},\overline{y}) = \left(\dfrac{\frac{1}{8}(e^2+1)}{\frac{1}{4}(e^2-1)},\dfrac{\frac{1}{9}(e^3-1)}{\frac{1}{4}(e^2-1)}\right) = \left(\dfrac{e^2+1}{2(e^2-1)},\dfrac{4(e^3-1)}{9(e^2-1)}\right)$.

(b) The wedge in question is the shaded area rotated from $\theta = \theta_1$ to $\theta = \theta_2$.
Letting



V_{ij} = volume of the region bounded by the sphere of radius ρ_i and the cone with angle ϕ_j ($\theta = \theta_1$ to θ_2)

and letting V be the volume of the wedge, we have

$$\begin{aligned} V &= (V_{22} - V_{21}) - (V_{12} - V_{11}) \\ &= \tfrac{1}{3}(\theta_2 - \theta_1)\left[\rho_2^3(1 - \cos\phi_2) - \rho_2^3(1 - \cos\phi_1) - \rho_1^3(1 - \cos\phi_2) + \rho_1^3(1 - \cos\phi_1)\right] \\ &= \tfrac{1}{3}(\theta_2 - \theta_1)\left[\left(\rho_2^3 - \rho_1^3\right)(1 - \cos\phi_2) - \left(\rho_2^3 - \rho_1^3\right)(1 - \cos\phi_1)\right] = \tfrac{1}{3}(\theta_2 - \theta_1)\left[\left(\rho_2^3 - \rho_1^3\right)(\cos\phi_1 - \cos\phi_2)\right] \end{aligned}$$

Or: Show that $V = \displaystyle\int_{\theta_1}^{\theta_2}\int_{\rho_1\sin\phi_1}^{\rho_2\sin\phi_2}\int_{r\cot\phi_2}^{r\cot\phi_1} r\,dz\,dr\,d\theta$.

(c) By the Mean Value Theorem with $f(\rho) = \rho^3$ there exists some $\tilde{\rho}$ with $\rho_1 \le \tilde{\rho} \le \rho_2$ such that $f(\rho_2) - f(\rho_1) = f'(\tilde{\rho})(\rho_2 - \rho_1)$ or $\rho_1^3 - \rho_2^3 = 3\tilde{\rho}^2\Delta\rho$. Similarly there exists ϕ with $\phi_1 \le \tilde{\phi} \le \phi_2$ such that $\cos\phi_2 - \cos\phi_1 = \left(-\sin\tilde{\phi}\right)\Delta\phi$. Substituting into the result from (b) gives

$\Delta V = (\tilde{\rho}^2\,\Delta\rho)(\theta_2 - \theta_1)(\sin\tilde{\phi})\,\Delta\phi = \tilde{\rho}^2\sin\tilde{\phi}\,\Delta\rho\,\Delta\phi\,\Delta\theta$.

16.9 Change of Variables in Multiple Integrals ET 15.9

1. $x = 5u - v,\ y = u + 3v$.

The Jacobian is $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} 5 & -1 \\ 1 & 3 \end{vmatrix} = 5(3) - (-1)(1) = 16$.

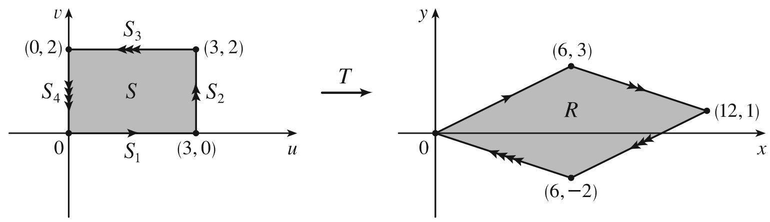
3. $x = e^{-r}\sin\theta,\ y = e^{r}\cos\theta$.

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \partial x/\partial r & \partial x/\partial\theta \\ \partial y/\partial r & \partial y/\partial\theta \end{vmatrix} = \begin{vmatrix} -e^{-r}\sin\theta & e^{-r}\cos\theta \\ e^{r}\cos\theta & -e^{r}\sin\theta \end{vmatrix} = e^{-r}e^{r}\sin^2\theta - e^{-r}e^{r}\cos^2\theta = \sin^2\theta - \cos^2\theta \text{ or } -\cos 2\theta$$

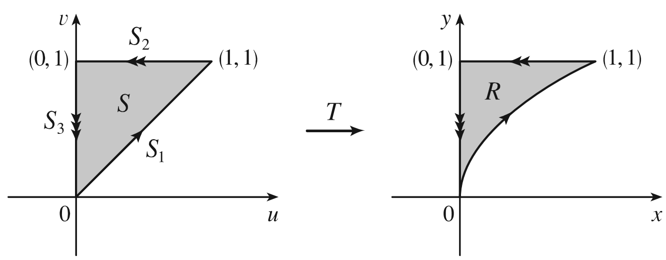
5. $x = u/v,\ y = v/w,\ z = w/u$.

$$\begin{aligned} \frac{\partial(x,y,z)}{\partial(u,v,w)} &= \begin{vmatrix} \partial x/\partial u & \partial x/\partial v & \partial x/\partial w \\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{vmatrix} = \begin{vmatrix} 1/v & -u/v^2 & 0 \\ 0 & 1/w & -v/w^2 \\ -w/u^2 & 0 & 1/u \end{vmatrix} \\ &= \frac{1}{v}\begin{vmatrix} 1/w & -v/w^2 \\ 0 & 1/u \end{vmatrix} - \left(-\frac{u}{v^2}\right)\begin{vmatrix} 0 & -v/w^2 \\ -w/u^2 & 1/u \end{vmatrix} + 0\begin{vmatrix} 0 & 1/w \\ -w/u^2 & 0 \end{vmatrix} \\ &= \frac{1}{v}\left(\frac{1}{uw} - 0\right) + \frac{u}{v^2}\left(0 - \frac{v}{u^2w}\right) + 0 = \frac{1}{uvw} - \frac{1}{uvw} = 0 \end{aligned}$$

7. The transformation maps the boundary of S to the boundary of the image R, so we first look at side S_1 in the uv-plane. S_1 is described by $v = 0$ $[0 \le u \le 3]$, so $x = 2u + 3v = 2u$ and $y = u - v = u$. Eliminating u, we have $x = 2y$, $0 \le x \le 6$. S_2 is the line segment $u = 3$, $0 \le v \le 2$, so $x = 6 + 3v$ and $y = 3 - v$. Then $v = 3 - y \Rightarrow x = 6 + 3(3 - y) = 15 - 3y$, $6 \le x \le 12$. S_3 is the line segment $v = 2$, $0 \le u \le 3$, so $x = 2u + 6$ and $y = u - 2$, giving $u = y + 2 \Rightarrow x = 2y + 10$, $6 \le x \le 12$. Finally, S_4 is the segment $u = 0$, $0 \le v \le 2$, so $x = 3v$ and $y = -v \Rightarrow x = -3y$, $0 \le x \le 6$. The image of set S is the region R shown in the xy-plane, a parallelogram bounded by these four segments.



9. S_1 is the line segment $u = v$, $0 \le u \le 1$, so $y = v = u$ and $x = u^2 = y^2$. Since $0 \le u \le 1$, the image is the portion of the parabola $x = y^2$, $0 \le y \le 1$. S_2 is the segment $v = 1$, $0 \le u \le 1$, thus $y = v = 1$ and $x = u^2$, so $0 \le x \le 1$. The image is the line segment $y = 1$, $0 \le x \le 1$. S_3 is the segment $u = 0$, $0 \le v \le 1$, so $x = u^2 = 0$ and $y = v \Rightarrow 0 \le y \le 1$. The image is the segment $x = 0$, $0 \le y \le 1$. Thus, the image of S is the region R in the first quadrant bounded by the parabola $x = y^2$, the y-axis, and the line $y = 1$.



11. $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$ and $x - 3y = (2u + v) - 3(u + 2v) = -u - 5v$. To find the region S in the uv-plane that corresponds to R we first find the corresponding boundary under the given transformation. The line through $(0, 0)$ and $(2, 1)$ is $y = \frac{1}{2}x$ which is the image of $u + 2v = \frac{1}{2}(2u + v) \Rightarrow v = 0$; the line through $(2, 1)$ and $(1, 2)$ is $x + y = 3$ which is the image of $(2u + v) + (u + 2v) = 3 \Rightarrow u + v = 1$; the line through $(0, 0)$ and $(1, 2)$ is $y = 2x$ which is the image of $u + 2v = 2(2u + v) \Rightarrow u = 0$. Thus S is the triangle $0 \le v \le 1 - u$, $0 \le u \le 1$ in the uv-plane and

$$\iint_R (x - 3y)\,dA = \int_0^1 \int_0^{1-u} (-u - 5v)\,|3|\,dv\,du = -3\int_0^1 \left[uv + \tfrac{5}{2}v^2\right]_{v=0}^{v=1-u} du$$

$$= -3\int_0^1 \left(u - u^2 + \tfrac{5}{2}(1-u)^2\right) du = -3\left[\tfrac{1}{2}u^2 - \tfrac{1}{3}u^3 - \tfrac{5}{6}(1-u)^3\right]_0^1 = -3\left(\tfrac{1}{2} - \tfrac{1}{3} + \tfrac{5}{6}\right) = -3$$

13. $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6$, $x^2 = 4u^2$ and the planar ellipse $9x^2 + 4y^2 \le 36$ is the image of the disk $u^2 + v^2 \le 1$. Thus

$$\iint_R x^2\,dA = \iint_{u^2+v^2 \le 1} (4u^2)(6)\,du\,dv = \int_0^{2\pi}\int_0^1 (24r^2\cos^2\theta)\,r\,dr\,d\theta = 24\int_0^{2\pi}\cos^2\theta\,d\theta \int_0^1 r^3\,dr$$

$$= 24\left[\tfrac{1}{2}x + \tfrac{1}{4}\sin 2x\right]_0^{2\pi} \left[\tfrac{1}{4}r^4\right]_0^1 = 24(\pi)\left(\tfrac{1}{4}\right) = 6\pi$$

15. $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/v & -u/v^2 \\ 0 & 1 \end{vmatrix} = \dfrac{1}{v}$, $xy = u$, $y = x$ is the image of the parabola $v^2 = u$, $y = 3x$ is the image of the parabola $v^2 = 3u$, and the hyperbolas $xy = 1$, $xy = 3$ are the images of the lines $u = 1$ and $u = 3$ respectively. Thus

$$\iint_R xy\,dA = \int_1^3 \int_{\sqrt{u}}^{\sqrt{3u}} u\left(\frac{1}{v}\right) dv\,du = \int_1^3 u\left(\ln\sqrt{3u} - \ln\sqrt{u}\right) du = \int_1^3 u \ln\sqrt{3}\,du = 4\ln\sqrt{3} = 2\ln 3.$$

17. (a) $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$ and since $u = \dfrac{x}{a}$, $v = \dfrac{y}{b}$, $w = \dfrac{z}{c}$ the solid enclosed by the ellipsoid is the image of the ball $u^2 + v^2 + w^2 \le 1$. So

$$\iiint_E dV = \iiint_{u^2+v^2+w^2\le 1} abc\,du\,dv\,dw = (abc)(\text{volume of the ball}) = \tfrac{4}{3}\pi abc$$

(b) If we approximate the surface of the earth by the ellipsoid $\dfrac{x^2}{6378^2} + \dfrac{y^2}{6378^2} + \dfrac{z^2}{6356^2} = 1$, then we can estimate the volume of the earth by finding the volume of the solid E enclosed by the ellipsoid. From part (a), this is $\iiint_E dV = \frac{4}{3}\pi(6378)(6378)(6356) \approx 1.083 \times 10^{12}\text{ km}^3$.

19. Letting $u = x - 2y$ and $v = 3x - y$, we have $x = \frac{1}{5}(2v - u)$ and $y = \frac{1}{5}(v - 3u)$. Then $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -1/5 & 2/5 \\ -3/5 & 1/5 \end{vmatrix} = \dfrac{1}{5}$ and R is the image of the rectangle enclosed by the lines $u = 0$, $u = 4$, $v = 1$, and $v = 8$. Thus

$$\iint_R \frac{x-2y}{3x-y}\,dA = \int_0^4 \int_1^8 \frac{u}{v}\left|\frac{1}{5}\right| dv\;du = \frac{1}{5}\int_0^4 u\,du \int_1^8 \frac{1}{v}\,dv = \tfrac{1}{5}\left[\tfrac{1}{2}u^2\right]_0^4 \left[\ln|v|\right]_1^8 = \tfrac{8}{5}\ln 8.$$

21. Letting $u = y - x$, $v = y + x$, we have $y = \frac{1}{2}(u + v)$, $x = \frac{1}{2}(v - u)$. Then $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{vmatrix} = -\dfrac{1}{2}$ and R is the image of the trapezoidal region with vertices $(-1,1)$, $(-2,2)$, $(2,2)$, and $(1,1)$. Thus

$$\iint_R \cos\frac{y-x}{y+x}\,dA = \int_1^2 \int_{-v}^{v} \cos\frac{u}{v}\left|-\frac{1}{2}\right| du\,dv = \frac{1}{2}\int_1^2 \left[v\sin\frac{u}{v}\right]_{u=-v}^{u=v} dv = \frac{1}{2}\int_1^2 2v\sin(1)\,dv = \tfrac{3}{2}\sin 1$$

23. Let $u = x + y$ and $v = -x + y$. Then $u + v = 2y \;\Rightarrow\; y = \frac{1}{2}(u + v)$ and $u - v = 2x \;\Rightarrow\; x = \frac{1}{2}(u - v)$.

$\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{vmatrix} = \dfrac{1}{2}$. Now $|u| = |x + y| \le |x| + |y| \le 1 \;\Rightarrow\; -1 \le u \le 1$, and

$|v| = |-x + y| \le |x| + |y| \le 1 \;\Rightarrow\; -1 \le v \le 1$. R is the image of the square region with vertices $(1,1)$, $(1,-1)$, $(-1,-1)$, and $(-1,1)$.

So $\iint_R e^{x+y}\,dA = \frac{1}{2}\int_{-1}^{1}\int_{-1}^{1} e^u\,du\,dv = \frac{1}{2}\left[e^u\right]_{-1}^{1}\left[v\right]_{-1}^{1} = e - e^{-1}$.