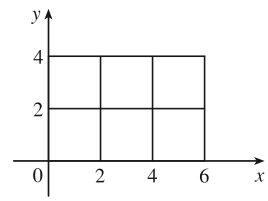
16 □ MULTIPLE INTEGRALS □ ET 15

16.1 Double Integrals over Rectangles ET 15.1

1. (a) The subrectangles are shown in the figure.



The surface is the graph of $f(x, y) = xy$ and $\Delta A = 4$, so we estimate

$$V \approx \sum_{i=1}^{3} \sum_{j=1}^{2} f(x_i, y_j)\, \Delta A$$

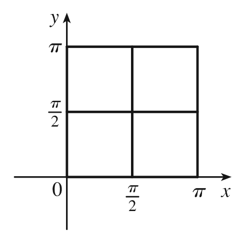
$$= f(2,2)\,\Delta A + f(2,4)\,\Delta A + f(4,2)\,\Delta A + f(4,4)\,\Delta A + f(6,2)\,\Delta A + f(6,4)\,\Delta A$$

$$= 4(4) + 8(4) + 8(4) + 16(4) + 12(4) + 24(4) = 288$$

(b) $$V \approx \sum_{i=1}^{3} \sum_{j=1}^{2} f\left(\overline{x}_i, \overline{y}_j\right) \Delta A = f(1,1)\,\Delta A + f(1,3)\,\Delta A + f(3,1)\,\Delta A + f(3,3)\,\Delta A + f(5,1)\,\Delta A + f(5,3)\,\Delta A$$

$$= 1(4) + 3(4) + 3(4) + 9(4) + 5(4) + 15(4) = 144$$

3. (a) The subrectangles are shown in the figure. Since $\Delta A = \pi^2/4$, we estimate

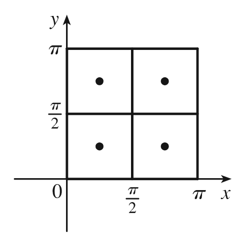


$$\iint_R \sin(x+y)\,dA \approx \sum_{i=1}^{2} \sum_{j=1}^{2} f\left(x_{ij}^*, y_{ij}^*\right) \Delta A$$

$$= f(0,0)\,\Delta A + f\left(0, \tfrac{\pi}{2}\right) \Delta A + f\left(\tfrac{\pi}{2}, 0\right) \Delta A + f\left(\tfrac{\pi}{2}, \tfrac{\pi}{2}\right) \Delta A$$

$$= 0\left(\tfrac{\pi^2}{4}\right) + 1\left(\tfrac{\pi^2}{4}\right) + 1\left(\tfrac{\pi^2}{4}\right) + 0\left(\tfrac{\pi^2}{4}\right) = \tfrac{\pi^2}{2} \approx 4.935$$

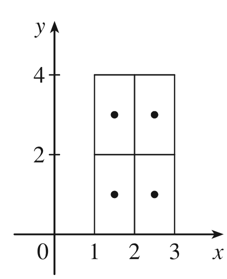
(b) $$\iint_R \sin(x+y)\,dA \approx \sum_{i=1}^{2} \sum_{j=1}^{2} f\left(\overline{x}_i, \overline{y}_j\right) \Delta A$$



$$= f\left(\tfrac{\pi}{4}, \tfrac{\pi}{4}\right) \Delta A + f\left(\tfrac{\pi}{4}, \tfrac{3\pi}{4}\right) \Delta A + f\left(\tfrac{3\pi}{4}, \tfrac{\pi}{4}\right) \Delta A + f\left(\tfrac{3\pi}{4}, \tfrac{3\pi}{4}\right) \Delta A$$

$$= 1\left(\tfrac{\pi^2}{4}\right) + 0\left(\tfrac{\pi^2}{4}\right) + 0\left(\tfrac{\pi^2}{4}\right) + (-1)\left(\tfrac{\pi^2}{4}\right) = 0$$

5. (a) Each subrectangle and its midpoint are shown in the figure. The area of each subrectangle is $\Delta A = 2$, so we evaluate f at each midpoint and estimate



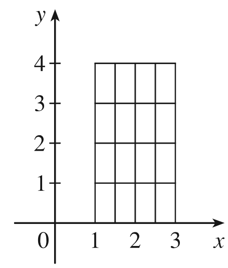
$$\iint_R f(x,y)\,dA \approx \sum_{i=1}^{2} \sum_{j=1}^{2} f\left(\overline{x}_i, \overline{y}_j\right) \Delta A$$

$$= f(1.5, 1)\,\Delta A + f(1.5, 3)\,\Delta A$$

$$+\ f(2.5, 1)\,\Delta A + f(2.5, 3)\,\Delta A$$

$$= 1(2) + (-8)(2) + 5(2) + (-1)(2) = -6$$

(b) The subrectangles are shown in the figure. In each subrectangle, the sample point farthest from the origin is the upper right corner, and the area of each subrectangle is $\Delta A = \frac{1}{2}$. Thus we estimate



$$\begin{aligned}
\iint_R f(x,y)\,dA &\approx \sum_{i=1}^{4}\sum_{j=1}^{4} f(x_i,y_j)\,\Delta A \\
&= f(1.5,1)\,\Delta A + f(1.5,2)\,\Delta A + f(1.5,3)\,\Delta A + f(1.5,4)\,\Delta A \\
&\quad + f(2,1)\,\Delta A + f(2,2)\,\Delta A + f(2,3)\,\Delta A + f(2,4)\,\Delta A \\
&\quad + f(2.5,1)\,\Delta A + f(2.5,2)\,\Delta A + f(2.5,3)\,\Delta A + f(2.5,4)\,\Delta A \\
&\quad + f(3,1)\,\Delta A + f(3,2)\,\Delta A + f(3,3)\,\Delta A + f(3,4)\,\Delta A \\
&= 1\left(\tfrac{1}{2}\right) + (-4)\left(\tfrac{1}{2}\right) + (-8)\left(\tfrac{1}{2}\right) + (-6)\left(\tfrac{1}{2}\right) + 3\left(\tfrac{1}{2}\right) + 0\left(\tfrac{1}{2}\right) + (-5)\left(\tfrac{1}{2}\right) + (-8)\left(\tfrac{1}{2}\right) \\
&\quad + 5\left(\tfrac{1}{2}\right) + 3\left(\tfrac{1}{2}\right) + (-1)\left(\tfrac{1}{2}\right) + (-4)\left(\tfrac{1}{2}\right) + 8\left(\tfrac{1}{2}\right) + 6\left(\tfrac{1}{2}\right) + 3\left(\tfrac{1}{2}\right) + 0\left(\tfrac{1}{2}\right) \\
&= -3.5
\end{aligned}$$

7. The values of $f(x,y) = \sqrt{52 - x^2 - y^2}$ get smaller as we move farther from the origin, so on any of the subrectangles in the problem, the function will have its largest value at the lower left corner of the subrectangle and its smallest value at the upper right corner, and any other value will lie between these two. So using these subrectangles we have $U < V < L$. (Note that this is true no matter how R is divided into subrectangles.)

9. (a) With $m = n = 2$, we have $\Delta A = 4$. Using the contour map to estimate the value of f at the center of each subrectangle, we have

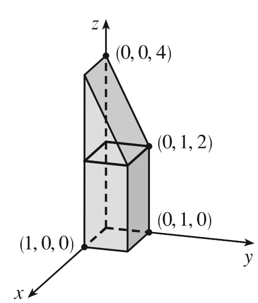
$$\iint_R f(x,y)\,dA \approx \sum_{i=1}^{2}\sum_{j=1}^{2} f\left(\overline{x}_i, \overline{y}_j\right)\Delta A = \Delta A[f(1,1) + f(1,3) + f(3,1) + f(3,3)] \approx 4(27 + 4 + 14 + 17) = 248$$

(b) $f_{\text{ave}} = \frac{1}{A(R)} \iint_R f(x,y)\,dA \approx \frac{1}{16}(248) = 15.5$

11. $z = 3 > 0$, so we can interpret the integral as the volume of the solid S that lies below the plane $z = 3$ and above the rectangle $[-2,2] \times [1,6]$. S is a rectangular solid, thus $\iint_R 3\,dA = 4 \cdot 5 \cdot 3 = 60$.

13. $z = f(x,y) = 4 - 2y \geq 0$ for $0 \leq y \leq 1$. Thus the integral represents the volume of that part of the rectangular solid $[0,1] \times [0,1] \times [0,4]$ which lies below the plane $z = 4 - 2y$. So

$$\iint_R (4 - 2y)\,dA = (1)(1)(2) + \tfrac{1}{2}(1)(1)(2) = 3$$



15. To calculate the estimates using a programmable calculator, we can use an algorithm similar to that of Exercise 5.1.7 [ET 5.1.7]. In Maple, we can define the function $f(x,y) = \sqrt{1 + xe^{-y}}$ (calling it `f`), load the `student` package, and then use the command

```
middlesum(middlesum(f,x=0..1,m),
                         y=0..1,m);
```

to get the estimate with $n = m^2$ squares of equal size. Mathematica has no special Riemann sum command, but we can define `f` and then use nested `Sum` commands to calculate the estimates.

n	estimate
1	1.141606
4	1.143191
16	1.143535
64	1.143617
256	1.143637
1024	1.143642

17. If we divide R into mn subrectangles, $\iint_R k\,dA \approx \sum\limits_{i=1}^{m} \sum\limits_{j=1}^{n} f\left(x_{ij}^*, y_{ij}^*\right) \Delta A$ for any choice of sample points $\left(x_{ij}^*, y_{ij}^*\right)$.

But $f\left(x_{ij}^*, y_{ij}^*\right) = k$ always and $\sum\limits_{i=1}^{m} \sum\limits_{j=1}^{n} \Delta A =$ area of $R = (b-a)(d-c)$. Thus, no matter how we choose the sample

points, $\sum\limits_{i=1}^{m} \sum\limits_{j=1}^{n} f\left(x_{ij}^*, y_{ij}^*\right) \Delta A = k \sum\limits_{i=1}^{m} \sum\limits_{j=1}^{n} \Delta A = k(b-a)(d-c)$ and so

$$\iint_R k\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f\left(x_{ij}^*, y_{ij}^*\right) \Delta A = \lim_{m,n\to\infty} k \sum_{i=1}^{m} \sum_{j=1}^{n} \Delta A = \lim_{m,n\to\infty} k(b-a)(d-c) = k(b-a)(d-c).$$

16.2 Iterated Integrals ET 15.2

1. $\int_0^5 12x^2y^3\,dx = \left[12\,\dfrac{x^3}{3}\,y^3\right]_{x=0}^{x=5} = \left[4x^3y^3\right]_{x=0}^{x=5} = 4(5)^3\,y^3 - 4(0)^3\,y^3 = 500y^3$,

$\int_0^1 12x^2y^3\,dy = \left[12x^2\,\dfrac{y^4}{4}\right]_{y=0}^{y=1} = \left[3x^2y^4\right]_{y=0}^{y=1} = 3x^2(1)^4 - 3x^2(0)^4 = 3x^2$

3. $\int_1^3 \int_0^1 (1+4xy)\,dx\,dy = \int_1^3 \left[x + 2x^2y\right]_{x=0}^{x=1} dy = \int_1^3 (1+2y)\,dy = \left[y + y^2\right]_1^3 = (3+9) - (1+1) = 10$

5. $\int_0^2 \int_0^{\pi/2} x \sin y\,dy\,dx = \int_0^2 x\,dx \int_0^{\pi/2} \sin y\,dy$ [as in Example 5] $= \left[\dfrac{x^2}{2}\right]_0^2 \left[-\cos y\right]_0^{\pi/2} = (2-0)(0+1) = 2$

7. $\int_0^2 \int_0^1 (2x+y)^8 dx\,dy = \displaystyle\int_0^2 \left[\frac{1}{2}\,\frac{(2x+y)^9}{9}\right]_{x=0}^{x=1} dy$ [substitute $u = 2x+y \;\Rightarrow\; dx = \frac{1}{2}\,du$]

$$= \frac{1}{18}\int_0^2 \left[(2+y)^9 - (0+y)^9\right] dy = \frac{1}{18}\left[\frac{(2+y)^{10}}{10} - \frac{y^{10}}{10}\right]_0^2$$

$$= \tfrac{1}{180}\left[(4^{10} - 2^{10}) - (2^{10} - 0^{10})\right] = \tfrac{1{,}046{,}528}{180} = \tfrac{261{,}632}{45}$$

9. $\displaystyle\int_1^4 \int_1^2 \left(\frac{x}{y} + \frac{y}{x}\right) dy\,dx = \int_1^4 \left[x \ln|y| + \frac{1}{x}\cdot\frac{1}{2}\,y^2\right]_{y=1}^{y=2} dx = \int_1^4 \left(x \ln 2 + \frac{3}{2x}\right) dx = \left[\tfrac{1}{2}x^2 \ln 2 + \tfrac{3}{2}\ln|x|\right]_1^4$

$$= 8\ln 2 + \tfrac{3}{2}\ln 4 - \tfrac{1}{2}\ln 2 = \tfrac{15}{2}\ln 2 + 3\ln 4^{1/2} = \tfrac{21}{2}\ln 2$$

11. $\int_0^1 \int_0^1 (u-v)^5 du\, dv = \int_0^1 \left[\frac{1}{6}(u-v)^6\right]_{u=0}^{u=1} dv = \frac{1}{6}\int_0^1 \left[(1-v)^6 - (0-v)^6\right] dv$

$= \frac{1}{6}\int_0^1 \left[(1-v)^6 - v^6\right] dv = \frac{1}{6}\left[-\frac{1}{7}(1-v)^7 - \frac{1}{7}v^7\right]_0^1$

$= -\frac{1}{42}\left[(0+1) - (1+0)\right] = 0$

13. $\int_0^2 \int_0^\pi r\sin^2\theta\, d\theta\, dr = \int_0^2 r\, dr \int_0^\pi \sin^2\theta\, d\theta$ [as in Example 5] $= \int_0^2 r\, dr \int_0^\pi \frac{1}{2}(1-\cos 2\theta)\, d\theta$

$= \left[\frac{1}{2}r^2\right]_0^2 \cdot \frac{1}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^\pi = (2-0)\cdot\frac{1}{2}\left[\left(\pi - \frac{1}{2}\sin 2\pi\right) - \left(0 - \frac{1}{2}\sin 0\right)\right]$

$= 2\cdot\frac{1}{2}[(\pi - 0) - (0-0)] = \pi$

15. $\iint_R (6x^2y^3 - 5y^4)\, dA = \int_0^3 \int_0^1 (6x^2y^3 - 5y^4)\, dy\, dx = \int_0^3 \left[\frac{3}{2}x^2y^4 - y^5\right]_{y=0}^{y=1} dx = \int_0^3 \left(\frac{3}{2}x^2 - 1\right) dx$

$= \left[\frac{1}{2}x^3 - x\right]_0^3 = \frac{27}{2} - 3 = \frac{21}{2}$

17. $\displaystyle\iint_R \frac{xy^2}{x^2+1}\, dA = \int_0^1 \int_{-3}^3 \frac{xy^2}{x^2+1}\, dy\, dx = \int_0^1 \frac{x}{x^2+1}\, dx \int_{-3}^3 y^2\, dy = \left[\frac{1}{2}\ln(x^2+1)\right]_0^1 \left[\frac{1}{3}y^3\right]_{-3}^3$

$= \frac{1}{2}(\ln 2 - \ln 1)\cdot\frac{1}{3}(27+27) = 9\ln 2$

19. $\int_0^{\pi/6} \int_0^{\pi/3} x\sin(x+y)\, dy\, dx$

$= \int_0^{\pi/6} \left[-x\cos(x+y)\right]_{y=0}^{y=\pi/3} dx = \int_0^{\pi/6} \left[x\cos x - x\cos\left(x+\frac{\pi}{3}\right)\right] dx$

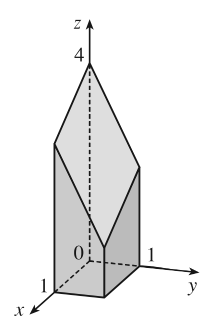
$= x\left[\sin x - \sin\left(x+\frac{\pi}{3}\right)\right]_0^{\pi/6} - \int_0^{\pi/6} \left[\sin x - \sin\left(x+\frac{\pi}{3}\right)\right] dx$ [by integrating by parts separately for each term]

$= \frac{\pi}{6}\left[\frac{1}{2} - 1\right] - \left[-\cos x + \cos\left(x+\frac{\pi}{3}\right)\right]_0^{\pi/6} = -\frac{\pi}{12} - \left[-\frac{\sqrt{3}}{2} + 0 - \left(-1 + \frac{1}{2}\right)\right] = \frac{\sqrt{3}-1}{2} - \frac{\pi}{12}$

21. $\iint_R xye^{x^2y}\, dA = \int_0^2 \int_0^1 xye^{x^2y}\, dx\, dy = \int_0^2 \left[\frac{1}{2}e^{x^2y}\right]_{x=0}^{x=1} dy = \frac{1}{2}\int_0^2 (e^y - 1)\, dy = \frac{1}{2}\left[e^y - y\right]_0^2$

$= \frac{1}{2}[(e^2-2) - (1-0)] = \frac{1}{2}(e^2 - 3)$

23. $z = f(x,y) = 4 - x - 2y \geq 0$ for $0 \leq x \leq 1$ and $0 \leq y \leq 1$. So the solid is the region in the first octant which lies below the plane $z = 4 - x - 2y$ and above $[0,1]\times[0,1]$.



25. $V = \iint_R (12 - 3x - 2y)\, dA = \int_{-2}^3 \int_0^1 (12 - 3x - 2y)\, dx\, dy = \int_{-2}^3 \left[12x - \frac{3}{2}x^2 - 2xy\right]_{x=0}^{x=1} dy$

$= \int_{-2}^3 \left(\frac{21}{2} - 2y\right) dy = \left[\frac{21}{2}y - y^2\right]_{-2}^3 = \frac{95}{2}$

27. $V = \int_{-2}^2 \int_{-1}^1 \left(1 - \frac{1}{4}x^2 - \frac{1}{9}y^2\right) dx\, dy = 4\int_0^2 \int_0^1 \left(1 - \frac{1}{4}x^2 - \frac{1}{9}y^2\right) dx\, dy$

$= 4\int_0^2 \left[x - \frac{1}{12}x^3 - \frac{1}{9}y^2x\right]_{x=0}^{x=1} dy = 4\int_0^2 \left(\frac{11}{12} - \frac{1}{9}y^2\right) dy = 4\left[\frac{11}{12}y - \frac{1}{27}y^3\right]_0^2 = 4\cdot\frac{83}{54} = \frac{166}{27}$

29. Here we need the volume of the solid lying under the surface $z = x\sec^2 y$ and above the rectangle $R = [0,2] \times [0,\pi/4]$ in the xy-plane.

$$V = \int_0^2 \int_0^{\pi/4} x\sec^2 y\,dy\,dx = \int_0^2 x\,dx \int_0^{\pi/4} \sec^2 y\,dy = \left[\tfrac{1}{2}x^2\right]_0^2 \left[\tan y\right]_0^{\pi/4}$$
$$= (2-0)(\tan\tfrac{\pi}{4} - \tan 0) = 2(1-0) = 2$$

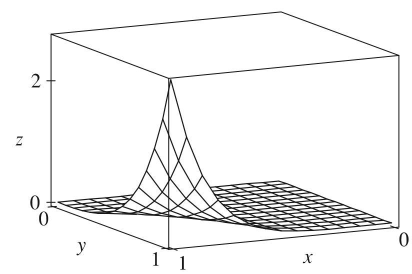
31. The solid lies below the surface $z = 2 + x^2 + (y-2)^2$ and above the plane $z = 1$ for $-1 \le x \le 1$, $0 \le y \le 4$. The volume of the solid is the difference in volumes between the solid that lies under $z = 2 + x^2 + (y-2)^2$ over the rectangle $R = [-1,1] \times [0,4]$ and the solid that lies under $z = 1$ over R.

$$V = \int_0^4 \int_{-1}^1 [2 + x^2 + (y-2)^2]\,dx\,dy - \int_0^4 \int_{-1}^1 (1)\,dx\,dy = \int_0^4 \left[2x + \tfrac{1}{3}x^3 + x(y-2)^2\right]_{x=-1}^{x=1} dy - \int_{-1}^1 dx \int_0^4 dy$$
$$= \int_0^4 \left[(2 + \tfrac{1}{3} + (y-2)^2) - (-2 - \tfrac{1}{3} - (y-2)^2)\right] dy - [x]_{-1}^1\,[y]_0^4$$
$$= \int_0^4 \left[\tfrac{14}{3} + 2(y-2)^2\right] dy - [1-(-1)][4-0] = \left[\tfrac{14}{3}y + \tfrac{2}{3}(y-2)^3\right]_0^4 - (2)(4)$$
$$= \left[\left(\tfrac{56}{3} + \tfrac{16}{3}\right) - \left(0 - \tfrac{16}{3}\right)\right] - 8 = \tfrac{88}{3} - 8 = \tfrac{64}{3}$$

33. In Maple, we can calculate the integral by defining the integrand as `f` and then using the command `int(int(f,x=0..1),y=0..1);`.
In Mathematica, we can use the command

```
Integrate[f,{x,0,1},{y,0,1}]
```

We find that $\iint_R x^5 y^3 e^{xy}\,dA = 21e - 57 \approx 0.0839$. We can use `plot3d` (in Maple) or `Plot3D` (in Mathematica) to graph the function.



35. R is the rectangle $[-1,1] \times [0,5]$. Thus, $A(R) = 2 \cdot 5 = 10$ and

$$f_{\text{ave}} = \frac{1}{A(R)} \iint_R f(x,y)\,dA = \tfrac{1}{10}\int_0^5 \int_{-1}^1 x^2 y\,dx\,dy = \tfrac{1}{10}\int_0^5 \left[\tfrac{1}{3}x^3 y\right]_{x=-1}^{x=1} dy = \tfrac{1}{10}\int_0^5 \tfrac{2}{3}y\,dy = \tfrac{1}{10}\left[\tfrac{1}{3}y^2\right]_0^5 = \tfrac{5}{6}.$$

37. Let $f(x,y) = \dfrac{x-y}{(x+y)^3}$. Then a CAS gives $\int_0^1 \int_0^1 f(x,y)\,dy\,dx = \frac{1}{2}$ and $\int_0^1 \int_0^1 f(x,y)\,dx\,dy = -\frac{1}{2}$.

To explain the seeming violation of Fubini's Theorem, note that f has an infinite discontinuity at $(0,0)$ and thus does not satisfy the conditions of Fubini's Theorem. In fact, both iterated integrals involve improper integrals which diverge at their lower limits of integration.

16.3 Double Integrals over General Regions ET 15.3

1. $\int_0^4 \int_0^{\sqrt{y}} xy^2\,dx\,dy = \int_0^4 \left[\tfrac{1}{2}x^2y^2\right]_{x=0}^{x=\sqrt{y}} dy = \int_0^4 \tfrac{1}{2}y^2[(\sqrt{y})^2 - 0^2]dy = \tfrac{1}{2}\int_0^4 y^3\,dy = \tfrac{1}{2}\left[\tfrac{1}{4}y^4\right]_0^4 = \tfrac{1}{2}(64-0) = 32$

3. $\int_0^1 \int_{x^2}^x (1+2y)dy\,dx = \int_0^1 \left[y + y^2\right]_{y=x^2}^{y=x} dx = \int_0^1 \left[x + x^2 - x^2 - (x^2)^2\right] dx$

$$= \int_0^1 (x - x^4)dx = \left[\tfrac{1}{2}x^2 - \tfrac{1}{5}x^5\right]_0^1 = \tfrac{1}{2} - \tfrac{1}{5} - 0 + 0 = \tfrac{3}{10}$$

5. $\int_0^{\pi/2} \int_0^{\cos\theta} e^{\sin\theta}dr\,d\theta = \int_0^{\pi/2} \left[re^{\sin\theta}\right]_{r=0}^{r=\cos\theta} d\theta = \int_0^{\pi/2} (\cos\theta)\,e^{\sin\theta}d\theta = e^{\sin\theta}\Big]_0^{\pi/2} = e^{\sin(\pi/2)} - e^0 = e - 1$

7. $\iint_D y^2\,dA = \int_{-1}^{1}\int_{-y-2}^{y} y^2\,dx\,dy = \int_{-1}^{1}\left[xy^2\right]_{x=-y-2}^{x=y} dy = \int_{-1}^{1} y^2\left[y-(-y-2)\right]dy$

$= \int_{-1}^{1}(2y^3+2y^2)dy = \left[\frac{1}{2}y^4+\frac{2}{3}y^3\right]_{-1}^{1} = \frac{1}{2}+\frac{2}{3}-\frac{1}{2}+\frac{2}{3} = \frac{4}{3}$

9. $\iint_D x\,dA = \int_0^{\pi}\int_0^{\sin x} x\,dy\,dx = \int_0^{\pi}\left[xy\right]_{y=0}^{y=\sin x} dx = \int_0^{\pi} x\sin x\,dx$ [integrate by parts with $u = x$, $dv = \sin x\,dx$]

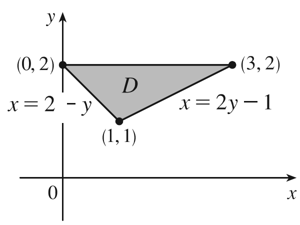
$= \left[-x\cos x+\sin x\right]_0^{\pi} = -\pi\cos\pi+\sin\pi+0-\sin 0 = \pi$

11. $\iint_D y^2 e^{xy}\,dA = \int_0^4\int_0^y y^2 e^{xy}\,dx\,dy = \int_0^4\left[ye^{xy}\right]_{x=0}^{x=y} dy = \int_0^4\left(ye^{y^2}-y\right)dy$

$= \left[\frac{1}{2}e^{y^2}-\frac{1}{2}y^2\right]_0^4 = \frac{1}{2}e^{16}-8-\frac{1}{2}+0 = \frac{1}{2}e^{16}-\frac{17}{2}$

13. $\int_0^1\int_0^{x^2} x\cos y\,dy\,dx = \int_0^1\left[x\sin y\right]_{y=0}^{y=x^2} dx = \int_0^1 x\sin x^2\,dx = -\frac{1}{2}\cos x^2\Big]_0^1 = \frac{1}{2}(1-\cos 1)$

15.

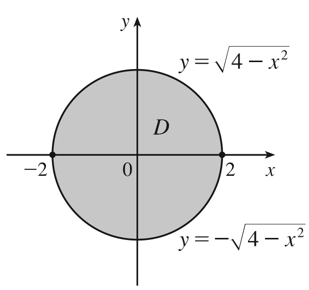


$$\int_1^2\int_{2-y}^{2y-1} y^3\,dx\,dy = \int_1^2\left[xy^3\right]_{x=2-y}^{x=2y-1} dy = \int_1^2\left[(2y-1)-(2-y)\right]y^3\,dy$$

$$= \int_1^2(3y^4-3y^3)\,dy = \left[\frac{3}{5}y^5-\frac{3}{4}y^4\right]_1^2$$

$$= \frac{96}{5}-12-\frac{3}{5}+\frac{3}{4} = \frac{147}{20}$$

17.



$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(2x-y)\,dy\,dx$$

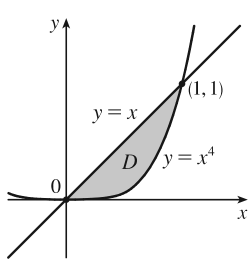
$$= \int_{-2}^{2}\left[2xy-\frac{1}{2}y^2\right]_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}} dx$$

$$= \int_{-2}^{2}\left[2x\sqrt{4-x^2}-\frac{1}{2}\left(4-x^2\right)+2x\sqrt{4-x^2}+\frac{1}{2}\left(4-x^2\right)\right]dx$$

$$= \int_{-2}^{2}4x\sqrt{4-x^2}\,dx = -\frac{4}{3}\left(4-x^2\right)^{3/2}\Big]_{-2}^{2} = 0$$

[Or, note that $4x\sqrt{4-x^2}$ is an odd function, so $\int_{-2}^{2}4x\sqrt{4-x^2}\,dx = 0$.]

19.

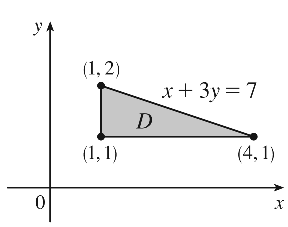


$$V = \int_0^1\int_{x^4}^{x}(x+2y)\,dy\,dx$$

$$= \int_0^1\left[xy+y^2\right]_{y=x^4}^{y=x} dx = \int_0^1(2x^2-x^5-x^8)\,dx$$

$$= \left[\frac{2}{3}x^3-\frac{1}{6}x^6-\frac{1}{9}x^9\right]_0^1 = \frac{2}{3}-\frac{1}{6}-\frac{1}{9} = \frac{7}{18}$$

21.

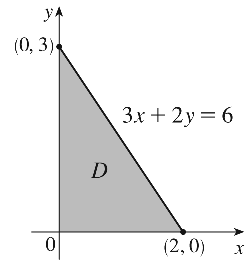


$$V = \int_1^2\int_1^{7-3y} xy\,dx\,dy = \int_1^2\left[\frac{1}{2}x^2y\right]_{x=1}^{x=7-3y} dy$$

$$= \frac{1}{2}\int_1^2(48y-42y^2+9y^3)\,dy$$

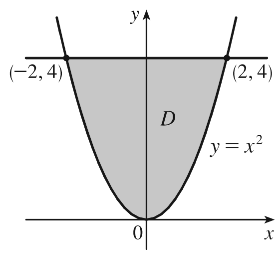
$$= \frac{1}{2}\left[24y^2-14y^3+\frac{9}{4}y^4\right]_1^2 = \frac{31}{8}$$

23.



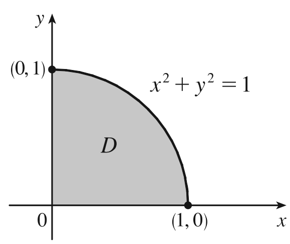
$$\begin{aligned} V &= \int_0^2 \int_0^{3-\frac{3}{2}x} (6-3x-2y)\,dy\,dx \\ &= \int_0^2 \left[6y-3xy-y^2\right]_{y=0}^{y=3-\frac{3}{2}x}\,dx \\ &= \int_0^2 \left[6(3-\tfrac{3}{2}x)-3x(3-\tfrac{3}{2}x)-(3-\tfrac{3}{2}x)^2\right]dx \\ &= \int_0^2 \left(\tfrac{9}{4}x^2-9x+9\right)dx = \left[\tfrac{3}{4}x^3-\tfrac{9}{2}x^2+9x\right]_0^2 = 6-0=6 \end{aligned}$$

25.



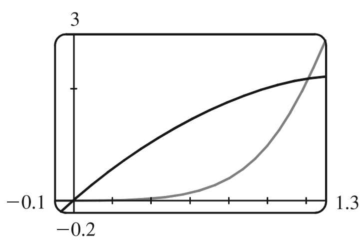
$$\begin{aligned} V &= \int_{-2}^2 \int_{x^2}^4 x^2\,dy\,dx \\ &= \int_{-2}^2 x^2\left[\,y\,\right]_{y=x^2}^{y=4}\,dx = \int_{-2}^2 (4x^2-x^4)\,dx \\ &= \left[\tfrac{4}{3}x^3-\tfrac{1}{5}x^5\right]_{-2}^2 = \tfrac{32}{3}-\tfrac{32}{5}+\tfrac{32}{3}-\tfrac{32}{5}=\tfrac{128}{15} \end{aligned}$$

27.



$$\begin{aligned} V &= \int_0^1 \int_0^{\sqrt{1-x^2}} y\,dy\,dx = \int_0^1 \left[\frac{y^2}{2}\right]_{y=0}^{y=\sqrt{1-x^2}}\,dx \\ &= \int_0^1 \frac{1-x^2}{2}\,dx = \tfrac{1}{2}\left[x-\tfrac{1}{3}x^3\right]_0^1 = \tfrac{1}{3} \end{aligned}$$

29.



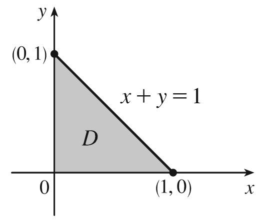
From the graph, it appears that the two curves intersect at $x=0$ and at $x\approx 1.213$. Thus the desired integral is

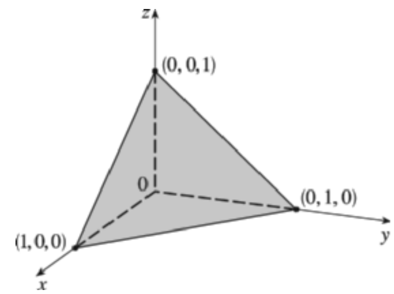
$$\begin{aligned} \iint_D x\,dA &\approx \int_0^{1.213}\int_{x^4}^{3x-x^2} x\,dy\,dx = \int_0^{1.213}\left[\,xy\,\right]_{y=x^4}^{y=3x-x^2}\,dx \\ &= \int_0^{1.213}(3x^2-x^3-x^5)\,dx = \left[x^3-\tfrac{1}{4}x^4-\tfrac{1}{6}x^6\right]_0^{1.213} \\ &\approx 0.713 \end{aligned}$$

31. The two bounding curves $y=1-x^2$ and $y=x^2-1$ intersect at $(\pm 1,0)$ with $1-x^2\geq x^2-1$ on $[-1,1]$. Within this region, the plane $z=2x+2y+10$ is above the plane $z=2-x-y$, so

$$\begin{aligned} V &= \int_{-1}^1\int_{x^2-1}^{1-x^2}(2x+2y+10)\,dy\,dx - \int_{-1}^1\int_{x^2-1}^{1-x^2}(2-x-y)\,dy\,dx \\ &= \int_{-1}^1\int_{x^2-1}^{1-x^2}(2x+2y+10-(2-x-y))\,dy\,dx \\ &= \int_{-1}^1\int_{x^2-1}^{1-x^2}(3x+3y+8)\,dy\,dx = \int_{-1}^1\left[3xy+\tfrac{3}{2}y^2+8y\right]_{y=x^2-1}^{y=1-x^2}\,dx \\ &= \int_{-1}^1\left[3x(1-x^2)+\tfrac{3}{2}(1-x^2)^2+8(1-x^2)-3x(x^2-1)-\tfrac{3}{2}(x^2-1)^2-8(x^2-1)\right]dx \\ &= \int_{-1}^1(-6x^3-16x^2+6x+16)\,dx = \left[-\tfrac{3}{2}x^4-\tfrac{16}{3}x^3+3x^2+16x\right]_{-1}^1 \\ &= -\tfrac{3}{2}-\tfrac{16}{3}+3+16+\tfrac{3}{2}-\tfrac{16}{3}-3+16=\tfrac{64}{3} \end{aligned}$$

33. The solid lies below the plane $z = 1 - x - y$ or $x + y + z = 1$ and above the region $D = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le 1 - x\}$ in the xy-plane. The solid is a tetrahedron.





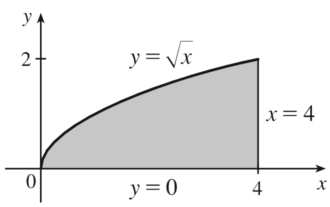
35. The two bounding curves $y = x^3 - x$ and $y = x^2 + x$ intersect at the origin and at $x = 2$, with $x^2 + x > x^3 - x$ on $(0, 2)$. Using a CAS, we find that the volume is

$$V = \int_0^2 \int_{x^3 - x}^{x^2 + x} z\, dy\, dx = \int_0^2 \int_{x^3 - x}^{x^2 + x} (x^3 y^4 + x y^2)\, dy\, dx = \frac{13{,}984{,}735{,}616}{14{,}549{,}535}$$

37. The two surfaces intersect in the circle $x^2 + y^2 = 1$, $z = 0$ and the region of integration is the disk D: $x^2 + y^2 \le 1$.

Using a CAS, the volume is $\iint_D (1 - x^2 - y^2)\, dA = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1 - x^2 - y^2)\, dy\, dx = \frac{\pi}{2}$.

39.

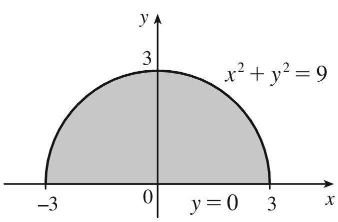


Because the region of integration is

$D = \{(x, y) \mid 0 \le y \le \sqrt{x}, 0 \le x \le 4\} = \left\{(x, y) \mid y^2 \le x \le 4, 0 \le y \le 2\right\}$

we have $\int_0^4 \int_0^{\sqrt{x}} f(x, y)\, dy\, dx = \iint_D f(x, y)\, dA = \int_0^2 \int_{y^2}^4 f(x, y)\, dx\, dy$.

41.



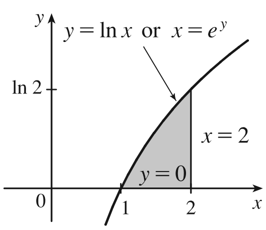
Because the region of integration is

$$\begin{aligned} D &= \left\{(x, y) \mid -\sqrt{9 - y^2} \le x \le \sqrt{9 - y^2}, 0 \le y \le 3\right\} \\ &= \left\{(x, y) \mid 0 \le y \le \sqrt{9 - x^2}, -3 \le x \le 3\right\} \end{aligned}$$

we have

$$\begin{aligned} \int_0^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} f(x, y)\, dx\, dy &= \iint_D f(x, y)\, dA \\ &= \int_{-3}^{3} \int_0^{\sqrt{9-x^2}} f(x, y)\, dy\, dx \end{aligned}$$

43.



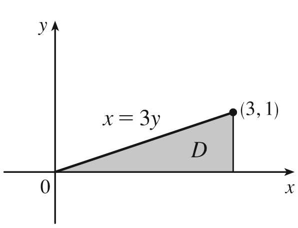
Because the region of integration is

$D = \{(x, y) \mid 0 \le y \le \ln x, 1 \le x \le 2\} = \{(x, y) \mid e^y \le x \le 2, 0 \le y \le \ln 2\}$

we have

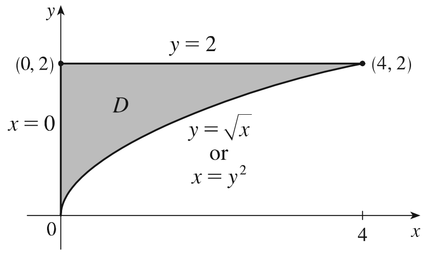
$$\int_1^2 \int_0^{\ln x} f(x, y)\, dy\, dx = \iint_D f(x, y)\, dA = \int_0^{\ln 2} \int_{e^y}^2 f(x, y)\, dx\, dy$$

45.



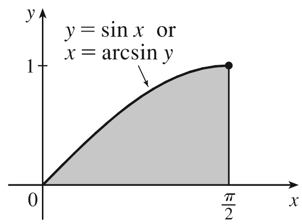
$$\begin{aligned} \int_0^1 \int_{3y}^3 e^{x^2}\, dx\, dy &= \int_0^3 \int_0^{x/3} e^{x^2}\, dy\, dx = \int_0^3 \left[e^{x^2} y\right]_{y=0}^{y=x/3} dx \\ &= \int_0^3 \left(\frac{x}{3}\right) e^{x^2}\, dx = \tfrac{1}{6}\, e^{x^2}\Big]_0^3 = \frac{e^9 - 1}{6} \end{aligned}$$

47.



$$\int_0^4 \int_{\sqrt{x}}^2 \frac{1}{y^3+1}\,dy\,dx = \int_0^2 \int_0^{y^2} \frac{1}{y^3+1}\,dx\,dy = \int_0^2 \frac{1}{y^3+1}\,\Big[x\Big]_{x=0}^{x=y^2} dy = \int_0^2 \frac{y^2}{y^3+1}\,dy = \tfrac{1}{3}\ln\left|y^3+1\right|\Big]_0^2 = \tfrac{1}{3}(\ln 9 - \ln 1) = \tfrac{1}{3}\ln 9$$

49.



$$\int_0^1 \int_{\arcsin y}^{\pi/2} \cos x\,\sqrt{1+\cos^2 x}\,dx\,dy$$
$$= \int_0^{\pi/2} \int_0^{\sin x} \cos x\,\sqrt{1+\cos^2 x}\,dy\,dx$$
$$= \int_0^{\pi/2} \cos x\,\sqrt{1+\cos^2 x}\,\Big[y\Big]_{y=0}^{y=\sin x}\,dx$$
$$= \int_0^{\pi/2} \cos x\,\sqrt{1+\cos^2 x}\,\sin x\,dx \qquad \left[\begin{array}{c}\text{Let } u=\cos x,\ du=-\sin x\,dx,\\ dx = du/(-\sin x)\end{array}\right]$$
$$= \int_1^0 -u\,\sqrt{1+u^2}\,du = -\tfrac{1}{3}\left(1+u^2\right)^{3/2}\Big]_1^0$$
$$= \tfrac{1}{3}\left(\sqrt{8}-1\right) = \tfrac{1}{3}\left(2\sqrt{2}-1\right)$$

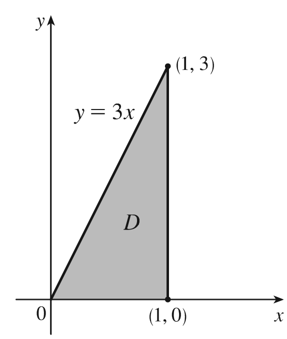
51. $D = \{(x,y) \mid 0 \le x \le 1,\ -x+1 \le y \le 1\} \cup \{(x,y) \mid -1 \le x \le 0, x+1 \le y \le 1\}$
$\cup \{(x,y) \mid 0 \le x \le 1,\ -1 \le y \le x-1\} \cup \{(x,y) \mid -1 \le x \le 0,\ -1 \le y \le -x-1\}$, all type I.

$$\iint_D x^2\,dA = \int_0^1 \int_{1-x}^1 x^2\,dy\,dx + \int_{-1}^0 \int_{x+1}^1 x^2\,dy\,dx + \int_0^1 \int_{-1}^{x-1} x^2\,dy\,dx + \int_{-1}^0 \int_{-1}^{-x-1} x^2\,dy\,dx$$
$$= 4\int_0^1 \int_{1-x}^1 x^2\,dy\,dx \qquad \text{[by symmetry of the regions and because } f(x,y)=x^2 \ge 0\text{]}$$
$$= 4\int_0^1 x^3\,dx = 4\left[\tfrac{1}{4}x^4\right]_0^1 = 1$$

53. Here $Q = \left\{(x,y) \mid x^2+y^2 \le \frac{1}{4}, x \ge 0, y \ge 0\right\}$, and $0 \le (x^2+y^2)^2 \le \left(\frac{1}{4}\right)^2 \Rightarrow -\frac{1}{16} \le -(x^2+y^2)^2 \le 0$ so $e^{-1/16} \le e^{-(x^2+y^2)^2} \le e^0 = 1$ since e^t is an increasing function. We have $A(Q) = \frac{1}{4}\pi\left(\frac{1}{2}\right)^2 = \frac{\pi}{16}$, so by Property 11, $e^{-1/16}\,A(Q) \le \iint_Q e^{-(x^2+y^2)^2}\,dA \le 1\cdot A(Q) \Rightarrow \frac{\pi}{16}e^{-1/16} \le \iint_Q e^{-(x^2+y^2)^2}\,dA \le \frac{\pi}{16}$ or we can say $0.1844 < \iint_Q e^{-(x^2+y^2)^2}\,dA < 0.1964$. (We have rounded the lower bound down and the upper bound up to preserve the inequalities.)

55. The average value of a function f of two variables defined on a rectangle R was defined in Section 16.1 [ET 15.1] as $f_{\text{ave}} = \frac{1}{A(R)}\iint_R f(x,y)dA$. Extending this definition to general regions D, we have $f_{\text{ave}} = \frac{1}{A(D)}\iint_D f(x,y)dA$.
Here $D = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le 3x\}$, so $A(D) = \frac{1}{2}(1)(3) = \frac{3}{2}$ and

$$f_{\text{ave}} = \tfrac{1}{A(D)}\iint_D f(x,y)dA = \tfrac{1}{3/2}\int_0^1\int_0^{3x} xy\,dy\,dx$$
$$= \tfrac{2}{3}\int_0^1 \left[\tfrac{1}{2}xy^2\right]_{y=0}^{y=3x} dx = \tfrac{1}{3}\int_0^1 9x^3\,dx = \tfrac{3}{4}x^4\Big]_0^1 = \tfrac{3}{4}$$



57. Since $m \le f(x,y) \le M$, $\iint_D m\,dA \le \iint_D f(x,y)\,dA \le \iint_D M\,dA$ by (8) $\Rightarrow$
$m\iint_D 1\,dA \le \iint_D f(x,y)\,dA \le M\iint_D 1\,dA$ by (7) $\Rightarrow$ $mA(D) \le \iint_D f(x,y)\,dA \le MA(D)$ by (10).

59. $\iint_D (x^2 \tan x + y^3 + 4)\, dA = \iint_D x^2 \tan x\, dA + \iint_D y^3\, dA + \iint_D 4\, dA$. But $x^2 \tan x$ is an odd function of x and D is symmetric with respect to the y-axis, so $\iint_D x^2 \tan x\, dA = 0$. Similarly, y^3 is an odd function of y and D is symmetric with respect to the x-axis, so $\iint_D y^3\, dA = 0$. Thus

$$\iint_D (x^2 \tan x + y^3 + 4)\, dA = 4 \iint_D\, dA = 4(\text{area of } D) = 4 \cdot \pi\big(\sqrt{2}\,\big)^2 = 8\pi$$

61. Since $\sqrt{1 - x^2 - y^2} \geq 0$, we can interpret $\iint_D \sqrt{1 - x^2 - y^2}\, dA$ as the volume of the solid that lies below the graph of $z = \sqrt{1 - x^2 - y^2}$ and above the region D in the xy-plane. $z = \sqrt{1 - x^2 - y^2}$ is equivalent to $x^2 + y^2 + z^2 = 1$, $z \geq 0$ which meets the xy-plane in the circle $x^2 + y^2 = 1$, the boundary of D. Thus, the solid is an upper hemisphere of radius 1 which has volume $\frac{1}{2}\left[\frac{4}{3}\pi(1)^3\right] = \frac{2}{3}\pi$.

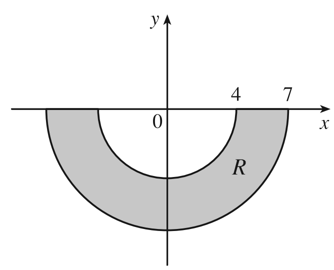
16.4 Double Integrals in Polar Coordinates — ET 15.4

1. The region R is more easily described by polar coordinates: $R = \left\{(r, \theta) \mid 0 \leq r \leq 4, 0 \leq \theta \leq \frac{3\pi}{2}\right\}$.
Thus $\iint_R f(x, y)\, dA = \int_0^{3\pi/2} \int_0^4 f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$.

3. The region R is more easily described by rectangular coordinates: $R = \left\{(x, y) \mid -1 \leq x \leq 1, 0 \leq y \leq \frac{1}{2}x + \frac{1}{2}\right\}$.
Thus $\iint_R f(x, y)\, dA = \int_{-1}^1 \int_0^{(x+1)/2} f(x, y)\, dy\, dx$.

5. The integral $\int_\pi^{2\pi} \int_4^7 r\, dr\, d\theta$ represents the area of the region $R = \{(r, \theta) \mid 4 \leq r \leq 7, \pi \leq \theta \leq 2\pi\}$, the lower half of a ring.

$$\int_\pi^{2\pi} \int_4^7 r\, dr\, d\theta = \left(\int_\pi^{2\pi} d\theta\right)\left(\int_4^7 r\, dr\right) = \big[\,\theta\,\big]_\pi^{2\pi} \left[\tfrac{1}{2}r^2\right]_4^7 = \pi \cdot \tfrac{1}{2}(49 - 16) = \tfrac{33\pi}{2}$$



7. The disk D can be described in polar coordinates as $D = \{(r, \theta) \mid 0 \leq r \leq 3, 0 \leq \theta \leq 2\pi\}$. Then

$$\iint_D xy\, dA = \int_0^{2\pi} \int_0^3 (r\cos\theta)(r\sin\theta)\, r\, dr\, d\theta = \left(\int_0^{2\pi} \sin\theta\cos\theta\, d\theta\right)\left(\int_0^3 r^3\, dr\right) = \left[\tfrac{1}{2}\sin^2\theta\right]_0^{2\pi} \left[\tfrac{1}{4}r^4\right]_0^3 = 0.$$

9. $\iint_R \cos(x^2 + y^2)\, dA = \int_0^\pi \int_0^3 \cos(r^2)\, r\, dr\, d\theta = \left(\int_0^\pi d\theta\right)\left(\int_0^3 r\cos(r^2)\, dr\right)$

$$= \big[\,\theta\,\big]_0^\pi \left[\tfrac{1}{2}\sin(r^2)\right]_0^3 = \pi \cdot \tfrac{1}{2}(\sin 9 - \sin 0) = \tfrac{\pi}{2}\sin 9$$

11. $\iint_D e^{-x^2 - y^2}\, dA = \int_{-\pi/2}^{\pi/2} \int_0^2 e^{-r^2} r\, dr\, d\theta = \left(\int_{-\pi/2}^{\pi/2} d\theta\right)\left(\int_0^2 re^{-r^2}\, dr\right)$

$$= \big[\,\theta\,\big]_{-\pi/2}^{\pi/2} \left[-\tfrac{1}{2}e^{-r^2}\right]_0^2 = \pi\left(-\tfrac{1}{2}\right)(e^{-4} - e^0) = \tfrac{\pi}{2}(1 - e^{-4})$$

13. R is the region shown in the figure, and can be described by $R = \{(r, \theta) \mid 0 \leq \theta \leq \pi/4, 1 \leq r \leq 2\}$. Thus
$\iint_R \arctan(y/x)\, dA = \int_0^{\pi/4} \int_1^2 \arctan(\tan\theta)\, r\, dr\, d\theta$ since $y/x = \tan\theta$.
Also, $\arctan(\tan\theta) = \theta$ for $0 \leq \theta \leq \pi/4$, so the integral becomes
$\int_0^{\pi/4} \int_1^2 \theta\, r\, dr\, d\theta = \int_0^{\pi/4} \theta\, d\theta \int_1^2 r\, dr = \left[\frac{1}{2}\theta^2\right]_0^{\pi/4} \left[\frac{1}{2}r^2\right]_1^2 = \frac{\pi^2}{32} \cdot \frac{3}{2} = \frac{3}{64}\pi^2$.