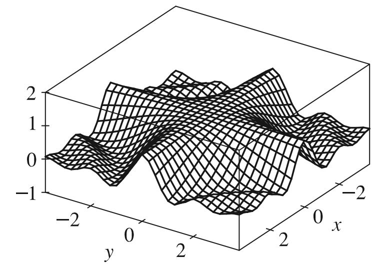
**39.** $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} = \lim_{r\to 0^+} \frac{(r\cos\theta)^3+(r\sin\theta)^3}{r^2} = \lim_{r\to 0^+} (r\cos^3\theta + r\sin^3\theta) = 0$

**41.** $\displaystyle\lim_{(x,y)\to(0,0)} \frac{e^{-x^2-y^2}-1}{x^2+y^2} = \lim_{r\to 0^+} \frac{e^{-r^2}-1}{r^2} = \lim_{r\to 0^+} \frac{e^{-r^2}(-2r)}{2r}$ [using l'Hospital's Rule]

$$= \lim_{r\to 0^+} -e^{-r^2} = -e^0 = -1$$

**43.** $f(x,y) = \begin{cases} \dfrac{\sin(xy)}{xy} & \text{if } (x,y) \neq (0,0) \\ 1 & \text{if } (x,y) = (0,0) \end{cases}$




From the graph, it appears that $f$ is continuous everywhere. We know $xy$ is continuous on $\mathbb{R}^2$ and $\sin t$ is continuous everywhere, so $\sin(xy)$ is continuous on $\mathbb{R}^2$ and $\dfrac{\sin(xy)}{xy}$ is continuous on $\mathbb{R}^2$ except possibly where $xy = 0$. To show that $f$ is continuous at those points, consider any point $(a,b)$ in $\mathbb{R}^2$ where $ab = 0$. Because $xy$ is continuous, $xy \to ab = 0$ as $(x,y)\to(a,b)$. If we let $t = xy$, then $t \to 0$ as $(x,y)\to(a,b)$ and $\displaystyle\lim_{(x,y)\to(a,b)} \frac{\sin(xy)}{xy} = \lim_{t\to 0}\frac{\sin(t)}{t} = 1$ by Equation 3.4.2 [ET 3.3.2]. Thus $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$ and $f$ is continuous on $\mathbb{R}^2$.

**45.** Since $|\mathbf{x}-\mathbf{a}|^2 = |\mathbf{x}|^2 + |\mathbf{a}|^2 - 2|\mathbf{x}||\mathbf{a}|\cos\theta \geq |\mathbf{x}|^2 + |\mathbf{a}|^2 - 2|\mathbf{x}||\mathbf{a}| = (|\mathbf{x}| - |\mathbf{a}|)^2$, we have $\big||\mathbf{x}| - |\mathbf{a}|\big| \leq |\mathbf{x}-\mathbf{a}|$. Let $\epsilon > 0$ be given and set $\delta = \epsilon$. Then if $0 < |\mathbf{x}-\mathbf{a}| < \delta$, $\big||\mathbf{x}| - |\mathbf{a}|\big| \leq |\mathbf{x}-\mathbf{a}| < \delta = \epsilon$. Hence $\lim_{\mathbf{x}\to\mathbf{a}} |\mathbf{x}| = |\mathbf{a}|$ and $f(\mathbf{x}) = |\mathbf{x}|$ is continuous on $\mathbb{R}^n$.

## 15.3 Partial Derivatives ET 14.3

**1.** (a) $\partial T/\partial x$ represents the rate of change of $T$ when we fix $y$ and $t$ and consider $T$ as a function of the single variable $x$, which describes how quickly the temperature changes when longitude changes but latitude and time are constant. $\partial T/\partial y$ represents the rate of change of $T$ when we fix $x$ and $t$ and consider $T$ as a function of $y$, which describes how quickly the temperature changes when latitude changes but longitude and time are constant. $\partial T/\partial t$ represents the rate of change of $T$ when we fix $x$ and $y$ and consider $T$ as a function of $t$, which describes how quickly the temperature changes over time for a constant longitude and latitude.

(b) $f_x(158, 21, 9)$ represents the rate of change of temperature at longitude 158°W, latitude 21°N at 9:00 AM when only longitude varies. Since the air is warmer to the west than to the east, increasing longitude results in an increased air temperature, so we would expect $f_x(158, 21, 9)$ to be positive. $f_y(158, 21, 9)$ represents the rate of change of temperature at the same time and location when only latitude varies. Since the air is warmer to the south and cooler to the north, increasing latitude results in a decreased air temperature, so we would expect $f_y(158, 21, 9)$ to be negative. $f_t(158, 21, 9)$ represents the rate of change of temperature at the same time and location when only time varies. Since typically air temperature increases from the morning to the afternoon as the sun warms it, we would expect $f_t(158, 21, 9)$ to be positive.

**3.** (a) By Definition 4, $f_T(-15, 30) = \lim\limits_{h \to 0} \dfrac{f(-15 + h, 30) - f(-15, 30)}{h}$, which we can approximate by considering $h = 5$ and $h = -5$ and using the values given in the table:

$$f_T(-15, 30) \approx \frac{f(-10, 30) - f(-15, 30)}{5} = \frac{-20 - (-26)}{5} = \frac{6}{5} = 1.2,$$

$f_T(-15, 30) \approx \dfrac{f(-20, 30) - f(-15, 30)}{-5} = \dfrac{-33 - (-26)}{-5} = \dfrac{-7}{-5} = 1.4$. Averaging these values, we estimate $f_T(-15, 30)$ to be approximately 1.3. Thus, when the actual temperature is $-15^\circ\text{C}$ and the wind speed is 30 km/h, the apparent temperature rises by about $1.3^\circ\text{C}$ for every degree that the actual temperature rises.

Similarly, $f_v(-15, 30) = \lim\limits_{h \to 0} \dfrac{f(-15, 30 + h) - f(-15, 30)}{h}$ which we can approximate by considering $h = 10$ and $h = -10$: $f_v(-15, 30) \approx \dfrac{f(-15, 40) - f(-15, 30)}{10} = \dfrac{-27 - (-26)}{10} = \dfrac{-1}{10} = -0.1$,

$f_v(-15, 30) \approx \dfrac{f(-15, 20) - f(-15, 30)}{-10} = \dfrac{-24 - (-26)}{-10} = \dfrac{2}{-10} = -0.2$. Averaging these values, we estimate $f_v(-15, 30)$ to be approximately $-0.15$. Thus, when the actual temperature is $-15^\circ\text{C}$ and the wind speed is 30 km/h, the apparent temperature decreases by about $0.15^\circ\text{C}$ for every km/h that the wind speed increases.

(b) For a fixed wind speed $v$, the values of the wind-chill index $W$ increase as temperature $T$ increases (look at a column of the table), so $\dfrac{\partial W}{\partial T}$ is positive. For a fixed temperature $T$, the values of $W$ decrease (or remain constant) as $v$ increases (look at a row of the table), so $\dfrac{\partial W}{\partial v}$ is negative (or perhaps 0).

(c) For fixed values of $T$, the function values $f(T, v)$ appear to become constant (or nearly constant) as $v$ increases, so the corresponding rate of change is 0 or near 0 as $v$ increases. This suggests that $\lim\limits_{v \to \infty} (\partial W / \partial v) = 0$.

**5.** (a) If we start at $(1, 2)$ and move in the positive $x$-direction, the graph of $f$ increases. Thus $f_x(1, 2)$ is positive.

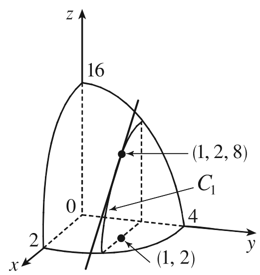
(b) If we start at $(1, 2)$ and move in the positive $y$-direction, the graph of $f$ decreases. Thus $f_y(1, 2)$ is negative.

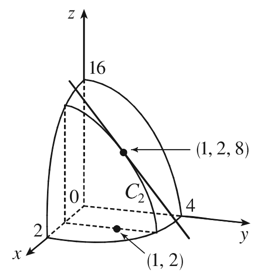
**7.** (a) $f_{xx} = \frac{\partial}{\partial x}(f_x)$, so $f_{xx}$ is the rate of change of $f_x$ in the $x$-direction. $f_x$ is negative at $(-1, 2)$ and if we move in the positive $x$-direction, the surface becomes less steep. Thus the values of $f_x$ are increasing and $f_{xx}(-1, 2)$ is positive.

(b) $f_{yy}$ is the rate of change of $f_y$ in the $y$-direction. $f_y$ is negative at $(-1, 2)$ and if we move in the positive $y$-direction, the surface becomes steeper. Thus the values of $f_y$ are decreasing, and $f_{yy}(-1, 2)$ is negative.

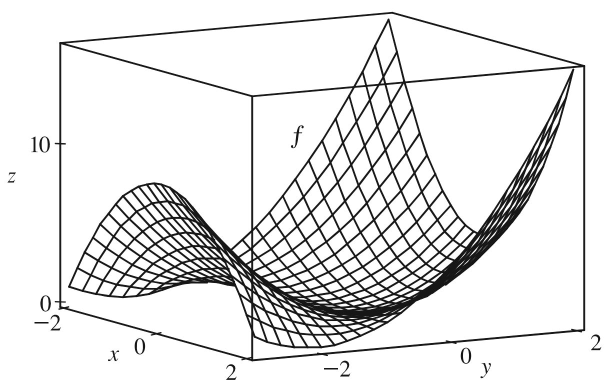
**9.** First of all, if we start at the point $(3, -3)$ and move in the positive $y$-direction, we see that both $b$ and $c$ decrease, while $a$ increases. Both $b$ and $c$ have a low point at about $(3, -1.5)$, while $a$ is 0 at this point. So $a$ is definitely the graph of $f_y$, and one of $b$ and $c$ is the graph of $f$. To see which is which, we start at the point $(-3, -1.5)$ and move in the positive $x$-direction. $b$ traces out a line with negative slope, while $c$ traces out a parabola opening downward. This tells us that $b$ is the $x$-derivative of $c$. So $c$ is the graph of $f$, $b$ is the graph of $f_x$, and $a$ is the graph of $f_y$.

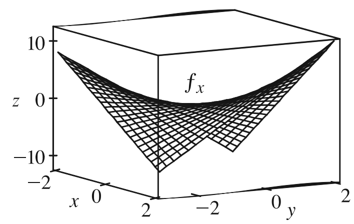
**11.** $f(x,y) = 16 - 4x^2 - y^2 \;\Rightarrow\; f_x(x,y) = -8x$ and $f_y(x,y) = -2y \;\Rightarrow\; f_x(1,2) = -8$ and $f_y(1,2) = -4$. The graph of $f$ is the paraboloid $z = 16 - 4x^2 - y^2$ and the vertical plane $y = 2$ intersects it in the parabola $z = 12 - 4x^2$, $y = 2$ (the curve $C_1$ in the first figure). The slope of the tangent line to this parabola at $(1,2,8)$ is $f_x(1,2) = -8$. Similarly the plane $x = 1$ intersects the paraboloid in the parabola $z = 12 - y^2$, $x = 1$ (the curve $C_2$ in the second figure) and the slope of the tangent line at $(1,2,8)$ is $f_y(1,2) = -4$.

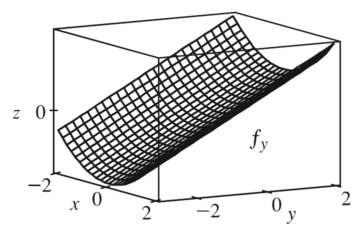



**13.** $f(x,y) = x^2 + y^2 + x^2y \;\Rightarrow\; f_x = 2x + 2xy, \quad f_y = 2y + x^2$




Note that the traces of $f$ in planes parallel to the $xz$-plane are parabolas which open downward for $y < -1$ and upward for $y > -1$, and the traces of $f_x$ in these planes are straight lines, which have negative slopes for $y < -1$ and positive slopes for $y > -1$. The traces of $f$ in planes parallel to the $yz$-plane are parabolas which always open upward, and the traces of $f_y$ in these planes are straight lines with positive slopes.

**15.** $f(x,y) = y^5 - 3xy \;\Rightarrow\; f_x(x,y) = 0 - 3y = -3y$, $f_y(x,y) = 5y^4 - 3x$

**17.** $f(x,t) = e^{-t}\cos\pi x \;\Rightarrow\; f_x(x,t) = e^{-t}(-\sin\pi x)(\pi) = -\pi e^{-t}\sin\pi x,\; f_t(x,t) = e^{-t}(-1)\cos\pi x = -e^{-t}\cos\pi x$

**19.** $z = (2x+3y)^{10} \;\Rightarrow\; \dfrac{\partial z}{\partial x} = 10(2x+3y)^9 \cdot 2 = 20(2x+3y)^9,\; \dfrac{\partial z}{\partial y} = 10(2x+3y)^9 \cdot 3 = 30(2x+3y)^9$

**21.** $f(x,y) = \dfrac{x-y}{x+y} \;\Rightarrow\; f_x(x,y) = \dfrac{(1)(x+y) - (x-y)(1)}{(x+y)^2} = \dfrac{2y}{(x+y)^2}$,

$f_y(x,y) = \dfrac{(-1)(x+y) - (x-y)(1)}{(x+y)^2} = -\dfrac{2x}{(x+y)^2}$

**23.** $w = \sin\alpha\,\cos\beta \;\Rightarrow\; \dfrac{\partial w}{\partial\alpha} = \cos\alpha\,\cos\beta,\; \dfrac{\partial w}{\partial\beta} = -\sin\alpha\,\sin\beta$

**25.** $f(r,s) = r\ln(r^2+s^2) \;\Rightarrow\; f_r(r,s) = r \cdot \dfrac{2r}{r^2+s^2} + \ln(r^2+s^2)\cdot 1 = \dfrac{2r^2}{r^2+s^2} + \ln(r^2+s^2)$,

$f_s(r,s) = r \cdot \dfrac{2s}{r^2+s^2} + 0 = \dfrac{2rs}{r^2+s^2}$

**27.** $u = te^{w/t} \quad \Rightarrow \quad \dfrac{\partial u}{\partial t} = t \cdot e^{w/t}(-wt^{-2}) + e^{w/t} \cdot 1 = e^{w/t} - \dfrac{w}{t}e^{w/t} = e^{w/t}\left(1 - \dfrac{w}{t}\right),\ \dfrac{\partial u}{\partial w} = te^{w/t} \cdot \dfrac{1}{t} = e^{w/t}$

**29.** $f(x,y,z) = xz - 5x^2y^3z^4 \quad \Rightarrow \quad f_x(x,y,z) = z - 10xy^3z^4,\ f_y(x,y,z) = -15x^2y^2z^4,\ f_z(x,y,z) = x - 20x^2y^3z^3$

**31.** $w = \ln(x + 2y + 3z) \quad \Rightarrow \quad \dfrac{\partial w}{\partial x} = \dfrac{1}{x + 2y + 3z},\ \dfrac{\partial w}{\partial y} = \dfrac{2}{x + 2y + 3z},\ \dfrac{\partial w}{\partial z} = \dfrac{3}{x + 2y + 3z}$

**33.** $u = xy\sin^{-1}(yz) \quad \Rightarrow \quad \dfrac{\partial u}{\partial x} = y\sin^{-1}(yz),\ \dfrac{\partial u}{\partial y} = xy \cdot \dfrac{1}{\sqrt{1 - (yz)^2}}\,(z) + \sin^{-1}(yz) \cdot x = \dfrac{xyz}{\sqrt{1 - y^2z^2}} + x\sin^{-1}(yz),$

$\dfrac{\partial u}{\partial z} = xy \cdot \dfrac{1}{\sqrt{1 - (yz)^2}}\,(y) = \dfrac{xy^2}{\sqrt{1 - y^2z^2}}$

**35.** $f(x,y,z,t) = xyz^2\tan(yt) \quad \Rightarrow \quad f_x(x,y,z,t) = yz^2\tan(yt),$

$f_y(x,y,z,t) = xyz^2 \cdot \sec^2(yt) \cdot t + xz^2\tan(yt) = xyz^2t\sec^2(yt) + xz^2\tan(yt),$

$f_z(x,y,z,t) = 2xyz\tan(yt),\ f_t(x,y,z,t) = xyz^2\sec^2(yt) \cdot y = xy^2z^2\sec^2(yt)$

**37.** $u = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$. For each $i = 1, \ldots, n$, $u_{x_i} = \frac{1}{2}\left(x_1^2 + x_2^2 + \cdots + x_n^2\right)^{-1/2}(2x_i) = \dfrac{x_i}{\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}}$.

**39.** $f(x,y) = \ln\left(x + \sqrt{x^2 + y^2}\right) \quad \Rightarrow$

$$f_x(x,y) = \frac{1}{x + \sqrt{x^2 + y^2}}\left[1 + \tfrac{1}{2}(x^2 + y^2)^{-1/2}(2x)\right] = \frac{1}{x + \sqrt{x^2 + y^2}}\left(1 + \frac{x}{\sqrt{x^2 + y^2}}\right),$$

so $f_x(3,4) = \dfrac{1}{3 + \sqrt{3^2 + 4^2}}\left(1 + \dfrac{3}{\sqrt{3^2 + 4^2}}\right) = \frac{1}{8}\left(1 + \frac{3}{5}\right) = \frac{1}{5}$.

**41.** $f(x,y,z) = \dfrac{y}{x + y + z} \quad \Rightarrow \quad f_y(x,y,z) = \dfrac{1(x + y + z) - y(1)}{(x + y + z)^2} = \dfrac{x + z}{(x + y + z)^2},$

so $f_y(2,1,-1) = \dfrac{2 + (-1)}{(2 + 1 + (-1))^2} = \dfrac{1}{4}$.

**43.** $f(x,y) = xy^2 - x^3y \quad \Rightarrow$

$$f_x\,(x,y) = \lim_{h \to 0}\frac{f(x + h, y) - f(x,y)}{h} = \lim_{h \to 0}\frac{(x + h)y^2 - (x + h)^3y - (xy^2 - x^3y)}{h}$$
$$= \lim_{h \to 0}\frac{h(y^2 - 3x^2y - 3xyh - yh^2)}{h} = \lim_{h \to 0}(y^2 - 3x^2y - 3xyh - yh^2) = y^2 - 3x^2y$$

$$f_y\,(x,y) = \lim_{h \to 0}\frac{f(x, y + h) - f(x,y)}{h} = \lim_{h \to 0}\frac{x(y + h)^2 - x^3(y + h) - (xy^2 - x^3y)}{h} = \lim_{h \to 0}\frac{h(2xy + xh - x^3)}{h}$$
$$= \lim_{h \to 0}(2xy + xh - x^3) = 2xy - x^3$$

**45.** $x^2 + y^2 + z^2 = 3xyz \quad \Rightarrow \quad \dfrac{\partial}{\partial x}\,(x^2 + y^2 + z^2) = \dfrac{\partial}{\partial x}\,(3xyz) \quad \Rightarrow \quad 2x + 0 + 2z\,\dfrac{\partial z}{\partial x} = 3y\left(x\,\dfrac{\partial z}{\partial x} + z \cdot 1\right) \quad \Leftrightarrow$

$2z\,\dfrac{\partial z}{\partial x} - 3xy\,\dfrac{\partial z}{\partial x} = 3yz - 2x \quad \Leftrightarrow \quad (2z - 3xy)\,\dfrac{\partial z}{\partial x} = 3yz - 2x$, so $\dfrac{\partial z}{\partial x} = \dfrac{3yz - 2x}{2z - 3xy}$.

$\dfrac{\partial}{\partial y}\,(x^2 + y^2 + z^2) = \dfrac{\partial}{\partial y}\,(3xyz) \quad \Rightarrow \quad 0 + 2y + 2z\,\dfrac{\partial z}{\partial y} = 3x\left(y\,\dfrac{\partial z}{\partial y} + z \cdot 1\right) \quad \Leftrightarrow \quad 2z\,\dfrac{\partial z}{\partial y} - 3xy\,\dfrac{\partial z}{\partial y} = 3xz - 2y \quad \Leftrightarrow$

$(2z - 3xy)\,\dfrac{\partial z}{\partial y} = 3xz - 2y$, so $\dfrac{\partial z}{\partial y} = \dfrac{3xz - 2y}{2z - 3xy}$.

**47.** $x - z = \arctan(yz) \Rightarrow \dfrac{\partial}{\partial x}(x - z) = \dfrac{\partial}{\partial x}(\arctan(yz)) \Rightarrow 1 - \dfrac{\partial z}{\partial x} = \dfrac{1}{1 + (yz)^2} \cdot y\,\dfrac{\partial z}{\partial x} \Leftrightarrow$

$1 = \left(\dfrac{y}{1 + y^2z^2} + 1\right)\dfrac{\partial z}{\partial x} \Leftrightarrow 1 = \left(\dfrac{y + 1 + y^2z^2}{1 + y^2z^2}\right)\dfrac{\partial z}{\partial x}$, so $\dfrac{\partial z}{\partial x} = \dfrac{1 + y^2z^2}{1 + y + y^2z^2}$.

$\dfrac{\partial}{\partial y}(x - z) = \dfrac{\partial}{\partial y}(\arctan(yz)) \Rightarrow 0 - \dfrac{\partial z}{\partial y} = \dfrac{1}{1 + (yz)^2} \cdot \left(y\,\dfrac{\partial z}{\partial y} + z \cdot 1\right) \Leftrightarrow$

$-\dfrac{z}{1 + y^2z^2} = \left(\dfrac{y}{1 + y^2z^2} + 1\right)\dfrac{\partial z}{\partial y} \Leftrightarrow -\dfrac{z}{1 + y^2z^2} = \left(\dfrac{y + 1 + y^2z^2}{1 + y^2z^2}\right)\dfrac{\partial z}{\partial y} \Leftrightarrow \dfrac{\partial z}{\partial y} = -\dfrac{z}{1 + y + y^2z^2}$.

**49.** (a) $z = f(x) + g(y) \Rightarrow \dfrac{\partial z}{\partial x} = f'(x), \quad \dfrac{\partial z}{\partial y} = g'(y)$

(b) $z = f(x + y)$. Let $u = x + y$. Then $\dfrac{\partial z}{\partial x} = \dfrac{df}{du}\dfrac{\partial u}{\partial x} = \dfrac{df}{du}(1) = f'(u) = f'(x + y)$,

$\dfrac{\partial z}{\partial y} = \dfrac{df}{du}\dfrac{\partial u}{\partial y} = \dfrac{df}{du}(1) = f'(u) = f'(x + y)$.

**51.** $f(x, y) = x^3y^5 + 2x^4y \Rightarrow f_x(x, y) = 3x^2y^5 + 8x^3y$, $f_y(x, y) = 5x^3y^4 + 2x^4$. Then $f_{xx}(x, y) = 6xy^5 + 24x^2y$, $f_{xy}(x, y) = 15x^2y^4 + 8x^3$, $f_{yx}(x, y) = 15x^2y^4 + 8x^3$, and $f_{yy}(x, y) = 20x^3y^3$.

**53.** $w = \sqrt{u^2 + v^2} \Rightarrow w_u = \frac{1}{2}(u^2 + v^2)^{-1/2} \cdot 2u = \dfrac{u}{\sqrt{u^2 + v^2}}$, $w_v = \frac{1}{2}(u^2 + v^2)^{-1/2} \cdot 2v = \dfrac{v}{\sqrt{u^2 + v^2}}$. Then

$$w_{uu} = \frac{1 \cdot \sqrt{u^2 + v^2} - u \cdot \frac{1}{2}(u^2 + v^2)^{-1/2}(2u)}{\left(\sqrt{u^2 + v^2}\right)^2} = \frac{\sqrt{u^2 + v^2} - u^2/\sqrt{u^2 + v^2}}{u^2 + v^2} = \frac{u^2 + v^2 - u^2}{(u^2 + v^2)^{3/2}} = \frac{v^2}{(u^2 + v^2)^{3/2}},$$

$$w_{uv} = u\left(-\tfrac{1}{2}\right)\left(u^2 + v^2\right)^{-3/2}(2v) = -\frac{uv}{(u^2 + v^2)^{3/2}},\ w_{vu} = v\left(-\tfrac{1}{2}\right)\left(u^2 + v^2\right)^{-3/2}(2u) = -\frac{uv}{(u^2 + v^2)^{3/2}},$$

$$w_{vv} = \frac{1 \cdot \sqrt{u^2 + v^2} - v \cdot \frac{1}{2}(u^2 + v^2)^{-1/2}(2v)}{\left(\sqrt{u^2 + v^2}\right)^2} = \frac{\sqrt{u^2 + v^2} - v^2/\sqrt{u^2 + v^2}}{u^2 + v^2} = \frac{u^2 + v^2 - v^2}{(u^2 + v^2)^{3/2}} = \frac{u^2}{(u^2 + v^2)^{3/2}}.$$

**55.** $z = \arctan\dfrac{x + y}{1 - xy} \Rightarrow$

$$z_x = \frac{1}{1 + \left(\frac{x+y}{1-xy}\right)^2} \cdot \frac{(1)(1 - xy) - (x + y)(-y)}{(1 - xy)^2} = \frac{1 + y^2}{(1 - xy)^2 + (x + y)^2} = \frac{1 + y^2}{1 + x^2 + y^2 + x^2y^2}$$

$$= \frac{1 + y^2}{(1 + x^2)(1 + y^2)} = \frac{1}{1 + x^2},$$

$$z_y = \frac{1}{1 + \left(\frac{x+y}{1-xy}\right)^2} \cdot \frac{(1)(1 - xy) - (x + y)(-x)}{(1 - xy)^2} = \frac{1 + x^2}{(1 - xy)^2 + (x + y)^2} = \frac{1 + x^2}{(1 + x^2)(1 + y^2)} = \frac{1}{1 + y^2}.$$

Then $z_{xx} = -(1 + x^2)^{-2} \cdot 2x = -\dfrac{2x}{(1 + x^2)^2}$, $z_{xy} = 0$, $z_{yx} = 0$, $z_{yy} = -(1 + y^2)^{-2} \cdot 2y = -\dfrac{2y}{(1 + y^2)^2}$.

**57.** $u = x\sin(x + 2y) \Rightarrow u_x = x \cdot \cos(x + 2y)(1) + \sin(x + 2y) \cdot 1 = x\cos(x + 2y) + \sin(x + 2y)$,

$u_{xy} = x(-\sin(x + 2y)(2)) + \cos(x + 2y)(2) = 2\cos(x + 2y) - 2x\sin(x + 2y)$,

$u_y = x\cos(x + 2y)(2) = 2x\cos(x + 2y)$,

$u_{yx} = 2x \cdot (-\sin(x + 2y)(1)) + \cos(x + 2y) \cdot 2 = 2\cos(x + 2y) - 2x\sin(x + 2y)$. Thus $u_{xy} = u_{yx}$.

**59.** $u = \ln\sqrt{x^2+y^2} = \ln(x^2+y^2)^{1/2} = \frac{1}{2}\ln(x^2+y^2) \Rightarrow u_x = \dfrac{1}{2}\dfrac{1}{x^2+y^2}\cdot 2x = \dfrac{x}{x^2+y^2}$,

$u_{xy} = x(-1)(x^2+y^2)^{-2}(2y) = -\dfrac{2xy}{(x^2+y^2)^2}$ and $u_y = \dfrac{1}{2}\dfrac{1}{x^2+y^2}\cdot 2y = \dfrac{y}{x^2+y^2}$,

$u_{yx} = y(-1)(x^2+y^2)^{-2}(2x) = -\dfrac{2xy}{(x^2+y^2)^2}$. Thus $u_{xy} = u_{yx}$.

**61.** $f(x,y) = 3xy^4 + x^3y^2 \Rightarrow f_x = 3y^4 + 3x^2y^2$, $f_{xx} = 6xy^2$, $f_{xxy} = 12xy$ and

$f_y = 12xy^3 + 2x^3y$, $f_{yy} = 36xy^2 + 2x^3$, $f_{yyy} = 72xy$.

**63.** $f(x,y,z) = \cos(4x+3y+2z) \Rightarrow$

$f_x = -\sin(4x+3y+2z)(4) = -4\sin(4x+3y+2z)$, $f_{xy} = -4\cos(4x+3y+2z)(3) = -12\cos(4x+3y+2z)$,

$f_{xyz} = -12(-\sin(4x+3y+2z))(2) = 24\sin(4x+3y+2z)$ and

$f_y = -\sin(4x+3y+2z)(3) = -3\sin(4x+3y+2z)$,

$f_{yz} = -3\cos(4x+3y+2z)(2) = -6\cos(4x+3y+2z)$, $f_{yzz} = -6(-\sin(4x+3y+2z))(2) = 12\sin(4x+3y+2z)$.

**65.** $u = e^{r\theta}\sin\theta \Rightarrow \dfrac{\partial u}{\partial\theta} = e^{r\theta}\cos\theta + \sin\theta\cdot e^{r\theta}(r) = e^{r\theta}(\cos\theta + r\sin\theta)$,

$\dfrac{\partial^2 u}{\partial r\,\partial\theta} = e^{r\theta}(\sin\theta) + (\cos\theta + r\sin\theta)\,e^{r\theta}(\theta) = e^{r\theta}(\sin\theta + \theta\cos\theta + r\theta\sin\theta)$,

$\dfrac{\partial^3 u}{\partial r^2\,\partial\theta} = e^{r\theta}(\theta\sin\theta) + (\sin\theta + \theta\cos\theta + r\theta\sin\theta)\cdot e^{r\theta}(\theta) = \theta e^{r\theta}(2\sin\theta + \theta\cos\theta + r\theta\sin\theta)$.

**67.** $w = \dfrac{x}{y+2z} = x(y+2z)^{-1} \Rightarrow \dfrac{\partial w}{\partial x} = (y+2z)^{-1}$, $\dfrac{\partial^2 w}{\partial y\,\partial x} = -(y+2z)^{-2}(1) = -(y+2z)^{-2}$,

$\dfrac{\partial^3 w}{\partial z\,\partial y\,\partial x} = -(-2)(y+2z)^{-3}(2) = 4(y+2z)^{-3} = \dfrac{4}{(y+2z)^3}$ and $\dfrac{\partial w}{\partial y} = x(-1)(y+2z)^{-2}(1) = -x(y+2z)^{-2}$,

$\dfrac{\partial^2 w}{\partial x\,\partial y} = -(y+2z)^{-2}$, $\dfrac{\partial^3 w}{\partial x^2\,\partial y} = 0$.

**69.** By Definition 4, $f_x(3,2) = \lim\limits_{h\to 0}\dfrac{f(3+h,2)-f(3,2)}{h}$ which we can approximate by considering $h = 0.5$ and $h = -0.5$:

$f_x(3,2) \approx \dfrac{f(3.5,2)-f(3,2)}{0.5} = \dfrac{22.4-17.5}{0.5} = 9.8$, $f_x(3,2) \approx \dfrac{f(2.5,2)-f(3,2)}{-0.5} = \dfrac{10.2-17.5}{-0.5} = 14.6$. Averaging these values, we estimate $f_x(3,2)$ to be approximately 12.2. Similarly, $f_x(3,2.2) = \lim\limits_{h\to 0}\dfrac{f(3+h,2.2)-f(3,2.2)}{h}$ which we can approximate by considering $h = 0.5$ and $h = -0.5$: $f_x(3,2.2) \approx \dfrac{f(3.5,2.2)-f(3,2.2)}{0.5} = \dfrac{26.1-15.9}{0.5} = 20.4$,

$f_x(3,2.2) \approx \dfrac{f(2.5,2.2)-f(3,2.2)}{-0.5} = \dfrac{9.3-15.9}{-0.5} = 13.2$. Averaging these values, we have $f_x(3,2.2) \approx 16.8$.

To estimate $f_{xy}(3,2)$, we first need an estimate for $f_x(3,1.8)$:

$f_x(3,1.8) \approx \dfrac{f(3.5,1.8)-f(3,1.8)}{0.5} = \dfrac{20.0-18.1}{0.5} = 3.8$, $f_x(3,1.8) \approx \dfrac{f(2.5,1.8)-f(3,1.8)}{-0.5} = \dfrac{12.5-18.1}{-0.5} = 11.2$.

Averaging these values, we get $f_x(3,1.8) \approx 7.5$. Now $f_{xy}(x,y) = \dfrac{\partial}{\partial y}[f_x(x,y)]$ and $f_x(x,y)$ is itself a function of two variables, so Definition 4 says that $f_{xy}(x,y) = \dfrac{\partial}{\partial y}\left[f_x(x,y)\right] = \lim\limits_{h\to 0}\dfrac{f_x(x,y+h)-f_x(x,y)}{h} \quad\Rightarrow$

$f_{xy}(3,2) = \lim\limits_{h\to 0}\dfrac{f_x(3,2+h)-f_x(3,2)}{h}$.

We can estimate this value using our previous work with $h = 0.2$ and $h = -0.2$:

$$f_{xy}(3,2) \approx \frac{f_x(3,2.2)-f_x(3,2)}{0.2} = \frac{16.8-12.2}{0.2} = 23,\; f_{xy}(3,2) \approx \frac{f_x(3,1.8)-f_x(3,2)}{-0.2} = \frac{7.5-12.2}{-0.2} = 23.5.$$

Averaging these values, we estimate $f_{xy}(3,2)$ to be approximately 23.25.

**71.** $u = e^{-\alpha^2k^2t}\sin kx \quad\Rightarrow\quad u_x = ke^{-\alpha^2k^2t}\cos kx$, $u_{xx} = -k^2e^{-\alpha^2k^2t}\sin kx$, and $u_t = -\alpha^2k^2e^{-\alpha^2k^2t}\sin kx$.

Thus $\alpha^2 u_{xx} = u_t$.

**73.** $u = \dfrac{1}{\sqrt{x^2+y^2+z^2}} \quad\Rightarrow\quad u_x = \left(-\frac{1}{2}\right)(x^2+y^2+z^2)^{-3/2}(2x) = -x(x^2+y^2+z^2)^{-3/2}$ and

$$u_{xx} = -(x^2+y^2+z^2)^{-3/2} - x\left(-\tfrac{3}{2}\right)(x^2+y^2+z^2)^{-5/2}(2x) = \frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}}.$$

By symmetry, $u_{yy} = \dfrac{2y^2-x^2-z^2}{(x^2+y^2+z^2)^{5/2}}$ and $u_{zz} = \dfrac{2z^2-x^2-y^2}{(x^2+y^2+z^2)^{5/2}}$.

Thus $u_{xx}+u_{yy}+u_{zz} = \dfrac{2x^2-y^2-z^2+2y^2-x^2-z^2+2z^2-x^2-y^2}{(x^2+y^2+z^2)^{5/2}} = 0$.

**75.** Let $v = x+at$, $\;w = x-at$. Then $u_t = \dfrac{\partial[f(v)+g(w)]}{\partial t} = \dfrac{df(v)}{dv}\dfrac{\partial v}{\partial t} + \dfrac{dg(w)}{dw}\dfrac{\partial w}{\partial t} = af'(v) - ag'(w)$ and

$u_{tt} = \dfrac{\partial[af'(v)-ag'(w)]}{\partial t} = a[af''(v)+ag''(w)] = a^2[f''(v)+g''(w)]$. Similarly, by using the Chain Rule we have

$u_x = f'(v)+g'(w)$ and $u_{xx} = f''(v)+g''(w)$. Thus $u_{tt} = a^2u_{xx}$.

**77.** $z = \ln(e^x+e^y) \quad\Rightarrow\quad \dfrac{\partial z}{\partial x} = \dfrac{e^x}{e^x+e^y}$ and $\dfrac{\partial z}{\partial y} = \dfrac{e^y}{e^x+e^y}$, so $\dfrac{\partial z}{\partial x}+\dfrac{\partial z}{\partial y} = \dfrac{e^x}{e^x+e^y}+\dfrac{e^y}{e^x+e^y} = \dfrac{e^x+e^y}{e^x+e^y} = 1$.

$\dfrac{\partial^2 z}{\partial x^2} = \dfrac{e^x(e^x+e^y)-e^x(e^x)}{(e^x+e^y)^2} = \dfrac{e^{x+y}}{(e^x+e^y)^2}$, $\quad\dfrac{\partial^2 z}{\partial x\,\partial y} = \dfrac{0-e^y(e^x)}{(e^x+e^y)^2} = -\dfrac{e^{x+y}}{(e^x+e^y)^2}$, and

$\dfrac{\partial^2 z}{\partial y^2} = \dfrac{e^y(e^x+e^y)-e^y(e^y)}{(e^x+e^y)^2} = \dfrac{e^{x+y}}{(e^x+e^y)^2}$. Thus

$$\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x\,\partial y}\right)^2 = \frac{e^{x+y}}{(e^x+e^y)^2}\cdot\frac{e^{x+y}}{(e^x+e^y)^2} - \left(-\frac{e^{x+y}}{(e^x+e^y)^2}\right)^2 = \frac{(e^{x+y})^2}{(e^x+e^y)^4} - \frac{(e^{x+y})^2}{(e^x+e^y)^4} = 0$$

**79.** If we fix $K = K_0$, $P(L,K_0)$ is a function of a single variable $L$, and $\dfrac{dP}{dL} = \alpha\dfrac{P}{L}$ is a separable differential equation. Then

$\dfrac{dP}{P} = \alpha\dfrac{dL}{L} \quad\Rightarrow\quad \displaystyle\int\frac{dP}{P} = \int\alpha\,\frac{dL}{L} \quad\Rightarrow\quad \ln|P| = \alpha\ln|L| + C(K_0)$, where $C(K_0)$ can depend on $K_0$. Then

$|P| = e^{\alpha\ln|L|+C(K_0)}$, and since $P>0$ and $L>0$, we have $P = e^{\alpha\ln L}e^{C(K_0)} = e^{C(K_0)}e^{\ln L^\alpha} = C_1(K_0)L^\alpha$ where

$C_1(K_0) = e^{C(K_0)}$.

**81.** By the Chain Rule, taking the partial derivative of both sides with respect to $R_1$ gives

$\dfrac{\partial R^{-1}}{\partial R}\dfrac{\partial R}{\partial R_1} = \dfrac{\partial\left[(1/R_1)+(1/R_2)+(1/R_3)\right]}{\partial R_1}$ or $-R^{-2}\dfrac{\partial R}{\partial R_1} = -R_1^{-2}$. Thus $\dfrac{\partial R}{\partial R_1} = \dfrac{R^2}{R_1^2}$.

**83.** By Exercise 82, $PV = mRT \quad \Rightarrow \quad P = \dfrac{mRT}{V}$, so $\dfrac{\partial P}{\partial T} = \dfrac{mR}{V}$. Also, $PV = mRT \quad \Rightarrow \quad V = \dfrac{mRT}{P}$ and $\dfrac{\partial V}{\partial T} = \dfrac{mR}{P}$.
Since $T = \dfrac{PV}{mR}$, we have $T\,\dfrac{\partial P}{\partial T}\,\dfrac{\partial V}{\partial T} = \dfrac{PV}{mR} \cdot \dfrac{mR}{V} \cdot \dfrac{mR}{P} = mR$.

**85.** $\dfrac{\partial K}{\partial m} = \frac{1}{2}v^2$, $\dfrac{\partial K}{\partial v} = mv$, $\dfrac{\partial^2 K}{\partial v^2} = m$. Thus $\dfrac{\partial K}{\partial m} \cdot \dfrac{\partial^2 K}{\partial v^2} = \frac{1}{2}v^2 m = K$.

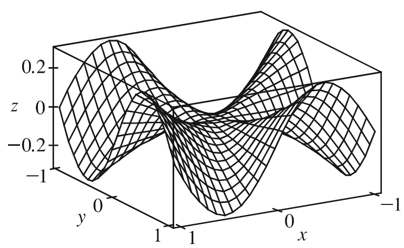
**87.** $f_x(x,y) = x + 4y \quad \Rightarrow \quad f_{xy}(x,y) = 4$ and $f_y(x,y) = 3x - y \quad \Rightarrow \quad f_{yx}(x,y) = 3$. Since $f_{xy}$ and $f_{yx}$ are continuous everywhere but $f_{xy}(x,y) \neq f_{yx}(x,y)$, Clairaut's Theorem implies that such a function $f(x,y)$ does not exist.

**89.** By the geometry of partial derivatives, the slope of the tangent line is $f_x(1,2)$. By implicit differentiation of $4x^2 + 2y^2 + z^2 = 16$, we get $8x + 2z\,(\partial z/\partial x) = 0 \quad \Rightarrow \quad \partial z/\partial x = -4x/z$, so when $x = 1$ and $z = 2$ we have $\partial z/\partial x = -2$. So the slope is $f_x(1,2) = -2$. Thus the tangent line is given by $z - 2 = -2(x-1),\ y = 2$. Taking the parameter to be $t = x - 1$, we can write parametric equations for this line: $x = 1 + t,\ y = 2,\ z = 2 - 2t$.

**91.** By Clairaut's Theorem, $f_{xyy} = (f_{xy})_y = (f_{yx})_y = f_{yxy} = (f_y)_{xy} = (f_y)_{yx} = f_{yyx}$.

**93.** Let $g(x) = f(x,0) = x(x^2)^{-3/2}e^0 = x\,|x|^{-3}$. But we are using the point $(1,0)$, so near $(1,0)$, $g(x) = x^{-2}$. Then $g'(x) = -2x^{-3}$ and $g'(1) = -2$, so using (1) we have $f_x(1,0) = g'(1) = -2$.

**95.** (a)




(b) For $(x,y) \neq (0,0)$,

$$f_x(x,y) = \frac{(3x^2y - y^3)(x^2+y^2) - (x^3y - xy^3)(2x)}{(x^2+y^2)^2} = \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2}$$

and by symmetry $f_y(x,y) = \dfrac{x^5 - 4x^3y^2 - xy^4}{(x^2+y^2)^2}$.

(c) $f_x(0,0) = \lim\limits_{h\to 0} \dfrac{f(h,0) - f(0,0)}{h} = \lim\limits_{h\to 0} \dfrac{(0/h^2) - 0}{h} = 0$ and $f_y(0,0) = \lim\limits_{h\to 0} \dfrac{f(0,h) - f(0,0)}{h} = 0$.

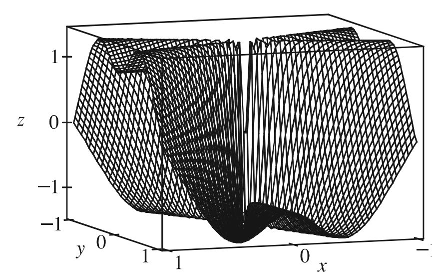
(d) By (3), $f_{xy}(0,0) = \dfrac{\partial f_x}{\partial y} = \lim\limits_{h\to 0} \dfrac{f_x(0,h) - f_x(0,0)}{h} = \lim\limits_{h\to 0} \dfrac{(-h^5 - 0)/h^4}{h} = -1$ while by (2),

$$f_{yx}(0,0) = \frac{\partial f_y}{\partial x} = \lim_{h\to 0} \frac{f_y(h,0) - f_y(0,0)}{h} = \lim_{h\to 0} \frac{h^5/h^4}{h} = 1.$$

(e) For $(x,y) \neq (0,0)$, we use a CAS to compute

$$f_{xy}(x,y) = \frac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{(x^2+y^2)^3}$$

Now as $(x,y) \to (0,0)$ along the $x$-axis, $f_{xy}(x,y) \to 1$ while as $(x,y) \to (0,0)$ along the $y$-axis, $f_{xy}(x,y) \to -1$. Thus $f_{xy}$ isn't continuous at $(0,0)$ and Clairaut's Theorem doesn't apply, so there is no contradiction. The graphs of $f_{xy}$ and $f_{yx}$ are identical except at the origin, where we observe the discontinuity.



## 15.4 Tangent Planes and Linear Approximations ET 14.4

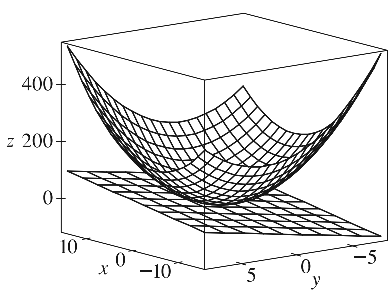
**1.** $z = f(x,y) = 4x^2 - y^2 + 2y \quad \Rightarrow \quad f_x(x,y) = 8x$, $f_y(x,y) = -2y + 2$, so $f_x(-1,2) = -8$, $f_y(-1,2) = -2$. By Equation 2, an equation of the tangent plane is $z - 4 = f_x(-1,2)[x - (-1)] + f_y(-1,2)(y-2) \quad \Rightarrow$ $z - 4 = -8(x+1) - 2(y-2)$ or $z = -8x - 2y$.

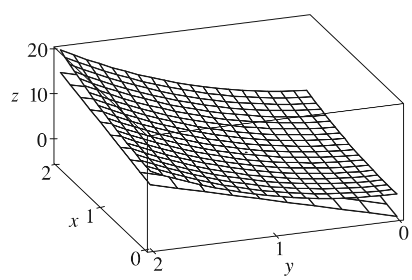
**3.** $z = f(x,y) = \sqrt{xy} \quad \Rightarrow \quad f_x(x,y) = \frac{1}{2}(xy)^{-1/2} \cdot y = \frac{1}{2}\sqrt{y/x}$, $f_y(x,y) = \frac{1}{2}(xy)^{-1/2} \cdot x = \frac{1}{2}\sqrt{x/y}$, so $f_x(1,1) = \frac{1}{2}$ and $f_y(1,1) = \frac{1}{2}$. Thus an equation of the tangent plane is $z - 1 = f_x(1,1)(x-1) + f_y(1,1)(y-1) \quad \Rightarrow$ $z - 1 = \frac{1}{2}(x-1) + \frac{1}{2}(y-1)$ or $x + y - 2z = 0$.

**5.** $z = f(x,y) = y\cos(x-y) \quad \Rightarrow \quad f_x = y(-\sin(x-y)(1)) = -y\sin(x-y)$, $f_y = y(-\sin(x-y)(-1)) + \cos(x-y) = y\sin(x-y) + \cos(x-y)$, so $f_x(2,2) = -2\sin(0) = 0$, $f_y(2,2) = 2\sin(0) + \cos(0) = 1$ and an equation of the tangent plane is $z - 2 = 0(x-2) + 1(y-2)$ or $z = y$.

**6.** $z = f(x,y) = e^{x^2-y^2} \quad \Rightarrow \quad f_x(x,y) = 2xe^{x^2-y^2}$, $f_y(x,y) = -2ye^{x^2-y^2}$, so $f_x(1,-1) = 2$, $f_y(1,-1) = 2$. By Equation 2, an equation of the tangent plane is $z - 1 = f_x(1,-1)(x-1) + f_y(1,-1)[y - (-1)] \quad \Rightarrow$ $z - 1 = 2(x-1) + 2(y+1)$ or $z = 2x + 2y + 1$.

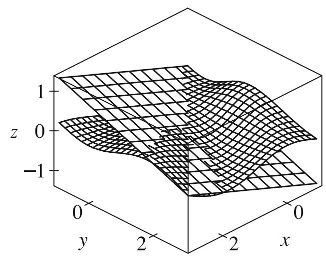
**7.** $z = f(x,y) = x^2 + xy + 3y^2$, so $f_x(x,y) = 2x + y \quad \Rightarrow \quad f_x(1,1) = 3$, $f_y(x,y) = x + 6y \quad \Rightarrow \quad f_y(1,1) = 7$ and an equation of the tangent plane is $z - 5 = 3(x-1) + 7(y-1)$ or $z = 3x + 7y - 5$. After zooming in, the surface and the tangent plane become almost indistinguishable. (Here, the tangent plane is below the surface.) If we zoom in farther, the surface and the tangent plane will appear to coincide.

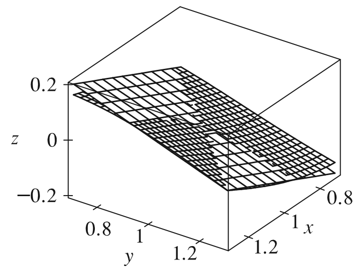



**9.** $f(x,y) = \dfrac{xy\sin(x-y)}{1+x^2+y^2}$. A CAS gives $f_x(x,y) = \dfrac{y\sin(x-y) + xy\cos(x-y)}{1+x^2+y^2} - \dfrac{2x^2y\sin(x-y)}{(1+x^2+y^2)^2}$ and

$f_y(x,y) = \dfrac{x\sin(x-y) - xy\cos(x-y)}{1+x^2+y^2} - \dfrac{2xy^2\sin(x-y)}{(1+x^2+y^2)^2}$. We use the CAS to evaluate these at $(1,1)$, and then substitute the results into Equation 2 to compute an equation of the tangent plane: $z = \frac{1}{3}x - \frac{1}{3}y$. The surface and tangent plane are shown in the first graph below. After zooming in, the surface and the tangent plane become almost indistinguishable, as shown in the second graph. (Here, the tangent plane is shown with fewer traces than the surface.) If we zoom in farther, the surface and the tangent plane will appear to coincide.




**11.** $f(x,y)=x\sqrt{y}$. The partial derivatives are $f_x(x,y)=\sqrt{y}$ and $f_y(x,y)=\dfrac{x}{2\sqrt{y}}$, so $f_x(1,4)=2$ and $f_y(1,4)=\frac{1}{4}$. Both $f_x$ and $f_y$ are continuous functions for $y>0$, so by Theorem 8, $f$ is differentiable at $(1,4)$. By Equation 3, the linearization of $f$ at $(1,4)$ is given by $L(x,y)=f(1,4)+f_x(1,4)(x-1)+f_y(1,4)(y-4)=2+2(x-1)+\frac{1}{4}(y-4)=2x+\frac{1}{4}y-1$.

**13.** $f(x,y)=\dfrac{x}{x+y}$. The partial derivatives are $f_x(x,y)=\dfrac{1(x+y)-x(1)}{(x+y)^2}=y/(x+y)^2$ and $f_y(x,y)=x(-1)(x+y)^{-2}\cdot 1=-x/(x+y)^2$, so $f_x(2,1)=\frac{1}{9}$ and $f_y(2,1)=-\frac{2}{9}$. Both $f_x$ and $f_y$ are continuous functions for $y\neq -x$, so $f$ is differentiable at $(2,1)$ by Theorem 8. The linearization of $f$ at $(2,1)$ is given by $L\,(x,y)=f(2,1)+f_x(2,1)(x-2)+f_y(2,1)(y-1)=\frac{2}{3}+\frac{1}{9}(x-2)-\frac{2}{9}(y-1)=\frac{1}{9}x-\frac{2}{9}y+\frac{2}{3}$.

**15.** $f(x,y)=e^{-xy}\cos y$. The partial derivatives are $f_x(x,y)=e^{-xy}(-y)\cos y=-ye^{-xy}\cos y$ and $f_y(x,y)=e^{-xy}(-\sin y)+(\cos y)e^{-xy}(-x)=-e^{-xy}(\sin y+x\cos y)$, so $f_x(\pi,0)=0$ and $f_y(\pi,0)=-\pi$. Both $f_x$ and $f_y$ are continuous functions, so $f$ is differentiable at $(\pi,0)$, and the linearization of $f$ at $(\pi,0)$ is $L(x,y)=f(\pi,0)+f_x(\pi,0)(x-\pi)+f_y(\pi,0)(y-0)=1+0(x-\pi)-\pi(y-0)=1-\pi y$.

**17.** Let $f(x,y)=\dfrac{2x+3}{4y+1}$. Then $f_x(x,y)=\dfrac{2}{4y+1}$ and $f_y(x,y)=(2x+3)(-1)(4y+1)^{-2}(4)=\dfrac{-8x-12}{(4y+1)^2}$. Both $f_x$ and $f_y$ are continuous functions for $y\neq-\frac{1}{4}$, so by Theorem 8, $f$ is differentiable at $(0,0)$. We have $f_x(0,0)=2$, $f_y(0,0)=-12$ and the linear approximation of $f$ at $(0,0)$ is $f(x,y)\approx f(0,0)+f_x(0,0)(x-0)+f_y(0,0)(y-0)=3+2x-12y$.

**19.** $f(x,y)=\sqrt{20-x^2-7y^2}\;\Rightarrow\; f_x(x,y)=-\dfrac{x}{\sqrt{20-x^2-7y^2}}$ and $f_y(x,y)=-\dfrac{7y}{\sqrt{20-x^2-7y^2}}$, so $f_x(2,1)=-\frac{2}{3}$ and $f_y(2,1)=-\frac{7}{3}$. Then the linear approximation of $f$ at $(2,1)$ is given by $f\,(x,y)\approx f(2,1)+f_x(2,1)(x-2)+f_y(2,1)(y-1)=3-\frac{2}{3}(x-2)-\frac{7}{3}(y-1)=-\frac{2}{3}x-\frac{7}{3}y+\frac{20}{3}$. Thus $f(1.95,1.08)\approx-\frac{2}{3}(1.95)-\frac{7}{3}(1.08)+\frac{20}{3}=2.84\overline{6}$.

**21.** $f(x,y,z)=\sqrt{x^2+y^2+z^2}\;\Rightarrow\; f_x(x,y,z)=\dfrac{x}{\sqrt{x^2+y^2+z^2}}$, $f_y(x,y,z)=\dfrac{y}{\sqrt{x^2+y^2+z^2}}$, and $f_z(x,y,z)=\dfrac{z}{\sqrt{x^2+y^2+z^2}}$, so $f_x(3,2,6)=\frac{3}{7}$, $f_y(3,2,6)=\frac{2}{7}$, $f_z(3,2,6)=\frac{6}{7}$. Then the linear approximation of $f$ at

$(3, 2, 6)$ is given by

$$f(x, y, z) \approx f(3, 2, 6) + f_x(3, 2, 6)(x - 3) + f_y(3, 2, 6)(y - 2) + f_z(3, 2, 6)(z - 6)$$
$$= 7 + \tfrac{3}{7}(x - 3) + \tfrac{2}{7}(y - 2) + \tfrac{6}{7}(z - 6) = \tfrac{3}{7}x + \tfrac{2}{7}y + \tfrac{6}{7}z$$

Thus $\sqrt{(3.02)^2 + (1.97)^2 + (5.99)^2} = f(3.02, 1.97, 5.99) \approx \frac{3}{7}(3.02) + \frac{2}{7}(1.97) + \frac{6}{7}(5.99) \approx 6.9914$.

**23.** From the table, $f(94, 80) = 127$. To estimate $f_T(94, 80)$ and $f_H(94, 80)$ we follow the procedure used in Section 15.3 [ET 14.3]. Since $f_T(94, 80) = \lim\limits_{h \to 0} \dfrac{f(94 + h, 80) - f(94, 80)}{h}$, we approximate this quantity with $h = \pm 2$ and use the values given in the table:

$$f_T(94, 80) \approx \frac{f(96, 80) - f(94, 80)}{2} = \frac{135 - 127}{2} = 4, \qquad f_T(94, 80) \approx \frac{f(92, 80) - f(94, 80)}{-2} = \frac{119 - 127}{-2} = 4$$

Averaging these values gives $f_T(94, 80) \approx 4$. Similarly, $f_H(94, 80) = \lim\limits_{h \to 0} \dfrac{f(94, 80 + h) - f(94, 80)}{h}$, so we use $h = \pm 5$:

$$f_H(94, 80) \approx \frac{f(94, 85) - f(94, 80)}{5} = \frac{132 - 127}{5} = 1, \qquad f_H(94, 80) \approx \frac{f(94, 75) - f(94, 80)}{-5} = \frac{122 - 127}{-5} = 1$$

Averaging these values gives $f_H(94, 80) \approx 1$. The linear approximation, then, is

$$f(T, H) \approx f(94, 80) + f_T(94, 80)(T - 94) + f_H(94, 80)(H - 80)$$
$$\approx 127 + 4(T - 94) + 1(H - 80) \qquad [\text{or } 4T + H - 329]$$

Thus when $T = 95$ and $H = 78$, $f(95, 78) \approx 127 + 4(95 - 94) + 1(78 - 80) = 129$, so we estimate the heat index to be approximately $129^\circ\text{F}$.

**25.** $z = x^3 \ln(y^2) \quad \Rightarrow \quad dz = \dfrac{\partial z}{\partial x} dx + \dfrac{\partial z}{\partial y} dy = 3x^2 \ln(y^2)\, dx + x^3 \cdot \dfrac{1}{y^2}(2y)\, dy = 3x^2 \ln(y^2)\, dx + \dfrac{2x^3}{y} dy$

**27.** $m = p^5 q^3 \quad \Rightarrow \quad dm = \dfrac{\partial m}{\partial p}\, dp + \dfrac{\partial m}{\partial q}\, dq = 5p^4 q^3\, dp + 3p^5 q^2\, dq$

**29.** $R = \alpha\beta^2 \cos\gamma \quad \Rightarrow \quad dR = \dfrac{\partial R}{\partial \alpha}\, d\alpha + \dfrac{\partial R}{\partial \beta}\, d\beta + \dfrac{\partial R}{\partial \gamma}\, d\gamma = \beta^2 \cos\gamma\, d\alpha + 2\alpha\beta\, \cos\gamma\, d\beta - \alpha\beta^2 \sin\gamma\, d\gamma$

**31.** $dx = \Delta x = 0.05$, $dy = \Delta y = 0.1$, $z = 5x^2 + y^2$, $z_x = 10x$, $z_y = 2y$. Thus when $x = 1$ and $y = 2$, $dz = z_x(1, 2)\, dx + z_y(1, 2)\, dy = (10)(0.05) + (4)(0.1) = 0.9$ while $\Delta z = f(1.05, 2.1) - f(1, 2) = 5(1.05)^2 + (2.1)^2 - 5 - 4 = 0.9225$.

**33.** $dA = \dfrac{\partial A}{\partial x}\, dx + \dfrac{\partial A}{\partial y}\, dy = y\, dx + x\, dy$ and $|\Delta x| \le 0.1$, $|\Delta y| \le 0.1$. We use $dx = 0.1$, $dy = 0.1$ with $x = 30$, $y = 24$; then the maximum error in the area is about $dA = 24(0.1) + 30(0.1) = 5.4\ \text{cm}^2$.

**35.** The volume of a can is $V = \pi r^2 h$ and $\Delta V \approx dV$ is an estimate of the amount of tin. Here $dV = 2\pi r h\, dr + \pi r^2\, dh$, so put $dr = 0.04$, $dh = 0.08$ (0.04 on top, 0.04 on bottom) and then $\Delta V \approx dV = 2\pi(48)(0.04) + \pi(16)(0.08) \approx 16.08\ \text{cm}^3$. Thus the amount of tin is about $16\ \text{cm}^3$.

**37.** The area of the rectangle is $A = xy$, and $\Delta A \approx dA$ is an estimate of the area of paint in the stripe. Here $dA = y\,dx + x\,dy$, so with $dx = dy = \frac{3+3}{12} = \frac{1}{2}$, $\Delta A \approx dA = (100)\left(\frac{1}{2}\right) + (200)\left(\frac{1}{2}\right) = 150 \text{ ft}^2$. Thus there are approximately 150 $\text{ft}^2$ of paint in the stripe.

**39.** First we find $\dfrac{\partial R}{\partial R_1}$ implicitly by taking partial derivatives of both sides with respect to $R_1$:

$\dfrac{\partial}{\partial R_1}\left(\dfrac{1}{R}\right) = \dfrac{\partial\,[(1/R_1) + (1/R_2) + (1/R_3)]}{\partial R_1} \quad\Rightarrow\quad -R^{-2}\dfrac{\partial R}{\partial R_1} = -R_1^{-2} \quad\Rightarrow\quad \dfrac{\partial R}{\partial R_1} = \dfrac{R^2}{R_1^2}$. Then by symmetry,

$\dfrac{\partial R}{\partial R_2} = \dfrac{R^2}{R_2^2}$, $\dfrac{\partial R}{\partial R_3} = \dfrac{R^2}{R_3^2}$. When $R_1 = 25$, $R_2 = 40$ and $R_3 = 50$, $\dfrac{1}{R} = \dfrac{17}{200} \quad\Leftrightarrow\quad R = \frac{200}{17}\ \Omega$.

Since the possible error for each $R_i$ is 0.5%, the maximum error of $R$ is attained by setting $\Delta R_i = 0.005R_i$. So

$$\Delta R \approx dR = \frac{\partial R}{\partial R_1}\,\Delta R_1 + \frac{\partial R}{\partial R_2}\,\Delta R_2 + \frac{\partial R}{\partial R_3}\,\Delta R_3 = (0.005)R^2\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right) = (0.005)R = \tfrac{1}{17} \approx 0.059\ \Omega.$$

**41.** The errors in measurement are at most 2%, so $\left|\dfrac{\Delta w}{w}\right| \le 0.02$ and $\left|\dfrac{\Delta h}{h}\right| \le 0.02$. The relative error in the calculated surface area is

$$\frac{\Delta S}{S} \approx \frac{dS}{S} = \frac{0.1091(0.425w^{0.425-1})h^{0.725}\,dw + 0.1091w^{0.425}(0.725h^{0.725-1})\,dh}{0.1091w^{0.425}h^{0.725}} = 0.425\frac{dw}{w} + 0.725\frac{dh}{h}$$

To estimate the maximum relative error, we use $\dfrac{dw}{w} = \left|\dfrac{\Delta w}{w}\right| = 0.02$ and $\dfrac{dh}{h} = \left|\dfrac{\Delta h}{h}\right| = 0.02 \quad\Rightarrow$

$\dfrac{dS}{S} = 0.425\,(0.02) + 0.725\,(0.02) = 0.023$. Thus the maximum percentage error is approximately 2.3%.

**43.** $\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b) = (a + \Delta x)^2 + (b + \Delta y)^2 - (a^2 + b^2)$

$= a^2 + 2a\,\Delta x + (\Delta x)^2 + b^2 + 2b\,\Delta y + (\Delta y)^2 - a^2 - b^2 = 2a\,\Delta x + (\Delta x)^2 + 2b\,\Delta y + (\Delta y)^2$

But $f_x(a, b) = 2a$ and $f_y(a, b) = 2b$ and so $\Delta z = f_x(a, b)\,\Delta x + f_y(a, b)\,\Delta y + \Delta x\,\Delta x + \Delta y\,\Delta y$, which is Definition 7 with $\varepsilon_1 = \Delta x$ and $\varepsilon_2 = \Delta y$. Hence $f$ is differentiable.

**45.** To show that $f$ is continuous at $(a, b)$ we need to show that $\displaystyle\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$ or equivalently $\displaystyle\lim_{(\Delta x,\Delta y)\to(0,0)} f(a + \Delta x, b + \Delta y) = f(a, b)$. Since $f$ is differentiable at $(a, b)$,

$f(a + \Delta x, b + \Delta y) - f(a, b) = \Delta z = f_x(a, b)\,\Delta x + f_y(a, b)\,\Delta y + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y$, where $\epsilon_1$ and $\epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$. Thus $f(a + \Delta x, b + \Delta y) = f(a, b) + f_x(a, b)\,\Delta x + f_y(a, b)\,\Delta y + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y$. Taking the limit of both sides as $(\Delta x, \Delta y) \to (0, 0)$ gives $\displaystyle\lim_{(\Delta x,\Delta y)\to(0,0)} f(a + \Delta x, b + \Delta y) = f(a, b)$. Thus $f$ is continuous at $(a, b)$.

## 15.5 The Chain Rule ET 14.5

**1.** $z = x^2 + y^2 + xy,\ x = \sin t,\ y = e^t \quad\Rightarrow\quad \dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt} = (2x + y)\cos t + (2y + x)e^t$

**3.** $z = \sqrt{1 + x^2 + y^2},\quad x = \ln t,\quad y = \cos t \quad\Rightarrow$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \tfrac{1}{2}(1+x^2+y^2)^{-1/2}(2x)\cdot\frac{1}{t} + \tfrac{1}{2}(1+x^2+y^2)^{-1/2}(2y)(-\sin t) = \frac{1}{\sqrt{1+x^2+y^2}}\left(\frac{x}{t} - y\sin t\right)$$