15 □ PARTIAL DERIVATIVES □ ET 14

15.1 Functions of Several Variables ET 14.1

1. (a) From Table 1, $f(-15, 40) = -27$, which means that if the temperature is $-15^\circ\mathrm{C}$ and the wind speed is $40\ \mathrm{km/h}$, then the air would feel equivalent to approximately $-27^\circ\mathrm{C}$ without wind.

(b) The question is asking: when the temperature is $-20^\circ\mathrm{C}$, what wind speed gives a wind-chill index of $-30^\circ\mathrm{C}$? From Table 1, the speed is $20\ \mathrm{km/h}$.

(c) The question is asking: when the wind speed is $20\ \mathrm{km/h}$, what temperature gives a wind-chill index of $-49^\circ\mathrm{C}$? From Table 1, the temperature is $-35^\circ\mathrm{C}$.

(d) The function $W = f(-5, v)$ means that we fix T at -5 and allow v to vary, resulting in a function of one variable. In other words, the function gives wind-chill index values for different wind speeds when the temperature is $-5^\circ\mathrm{C}$. From Table 1 (look at the row corresponding to $T = -5$), the function decreases and appears to approach a constant value as v increases.

(e) The function $W = f(T, 50)$ means that we fix v at 50 and allow T to vary, again giving a function of one variable. In other words, the function gives wind-chill index values for different temperatures when the wind speed is $50\ \mathrm{km/h}$. From Table 1 (look at the column corresponding to $v = 50$), the function increases almost linearly as T increases.

3. If the amounts of labor and capital are both doubled, we replace L, K in the function with $2L, 2K$, giving

$$P(2L, 2K) = 1.01(2L)^{0.75}(2K)^{0.25} = 1.01(2^{0.75})(2^{0.25})L^{0.75}K^{0.25} = (2^1)1.01L^{0.75}K^{0.25} = 2P(L, K)$$

Thus, the production is doubled. It is also true for the general case $P(L, K) = bL^\alpha K^{1-\alpha}$:
$P(2L, 2K) = b(2L)^\alpha(2K)^{1-\alpha} = b(2^\alpha)(2^{1-\alpha})L^\alpha K^{1-\alpha} = (2^{\alpha+1-\alpha})bL^\alpha K^{1-\alpha} = 2P(L, K)$.

5. (a) According to Table 4, $f(40, 15) = 25$, which means that if a 40-knot wind has been blowing in the open sea for 15 hours, it will create waves with estimated heights of 25 feet.

(b) $h = f(30, t)$ means we fix v at 30 and allow t to vary, resulting in a function of one variable. Thus here, $h = f(30, t)$ gives the wave heights produced by 30-knot winds blowing for t hours. From the table (look at the row corresponding to $v = 30$), the function increases but at a declining rate as t increases. In fact, the function values appear to be approaching a limiting value of approximately 19, which suggests that 30-knot winds cannot produce waves higher than about 19 feet.

(c) $h = f(v, 30)$ means we fix t at 30, again giving a function of one variable. So, $h = f(v, 30)$ gives the wave heights produced by winds of speed v blowing for 30 hours. From the table (look at the column corresponding to $t = 30$), the function appears to increase at an increasing rate, with no apparent limiting value. This suggests that faster winds (lasting 30 hours) always create higher waves.

7. (a) $f(2, 0) = 2^2 e^{3(2)(0)} = 4(1) = 4$

(b) Since both x^2 and the exponential function are defined everywhere, $x^2 e^{3xy}$ is defined for all choices of values for x and y. Thus the domain of f is $\mathbb{R}^2$.

(c) Because the range of $g(x, y) = 3xy$ is $\mathbb{R}$, and the range of e^x is $(0, \infty)$, the range of $e^{g(x,y)} = e^{3xy}$ is $(0, \infty)$.

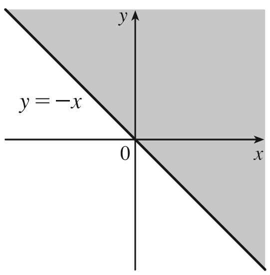
The range of x^2 is $[0, \infty)$, so the range of the product $x^2 e^{3xy}$ is $[0, \infty)$.

9. (a) $f(2, -1, 6) = e^{\sqrt{6-2^2-(-1)^2}} = e^{\sqrt{1}} = e$.

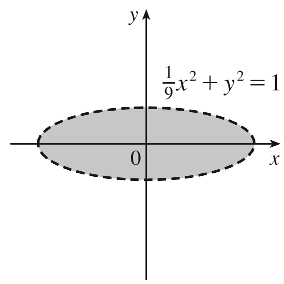
(b) $e^{\sqrt{z-x^2-y^2}}$ is defined when $z - x^2 - y^2 \geq 0 \quad \Rightarrow \quad z \geq x^2 + y^2$. Thus the domain of f is $\{(x, y, z) \mid z \geq x^2 + y^2\}$.

(c) Since $\sqrt{z - x^2 - y^2} \geq 0$, we have $e^{\sqrt{z-x^2-y^2}} \geq 1$. Thus the range of f is $[1, \infty)$.

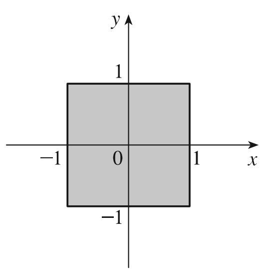
11. $\sqrt{x + y}$ is defined only when $x + y \geq 0$, or $y \geq -x$. So the domain of f is $\{(x, y) \mid y \geq -x\}$.



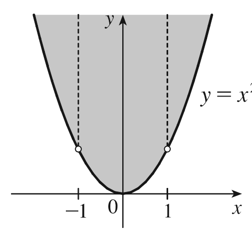
13. $\ln(9 - x^2 - 9y^2)$ is defined only when $9 - x^2 - 9y^2 > 0$, or $\frac{1}{9}x^2 + y^2 < 1$. So the domain of f is $\{(x, y) \mid \frac{1}{9}x^2 + y^2 < 1\}$, the interior of an ellipse.



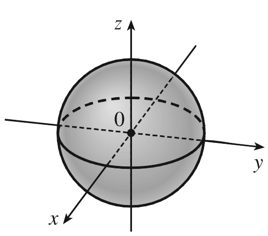
15. $\sqrt{1 - x^2}$ is defined only when $1 - x^2 \geq 0$, or $x^2 \leq 1 \quad \Leftrightarrow \quad -1 \leq x \leq 1$, and $\sqrt{1 - y^2}$ is defined only when $1 - y^2 \geq 0$, or $y^2 \leq 1 \quad \Leftrightarrow \quad -1 \leq y \leq 1$. Thus the domain of f is $\{(x, y) \mid -1 \leq x \leq 1, \ -1 \leq y \leq 1\}$.



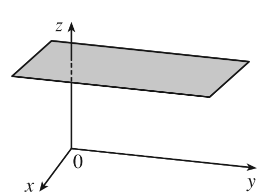
17. $\sqrt{y - x^2}$ is defined only when $y - x^2 \geq 0$, or $y \geq x^2$. In addition, f is not defined if $1 - x^2 = 0 \quad \Rightarrow \quad x = \pm 1$. Thus the domain of f is $\{(x, y) \mid y \geq x^2, \ x \neq \pm 1\}$.



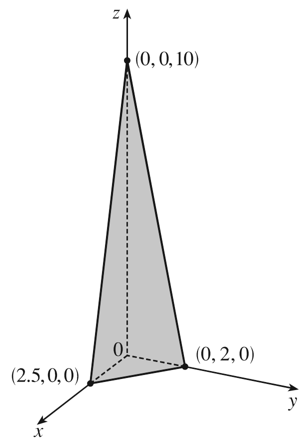
19. We need $1 - x^2 - y^2 - z^2 \geq 0$ or $x^2 + y^2 + z^2 \leq 1$, so $D = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\}$ (the points inside or on the sphere of radius 1, center the origin).



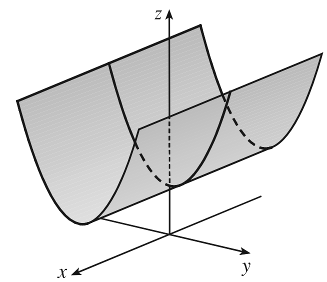
21. $z = 3$, a horizontal plane through the point $(0, 0, 3)$.



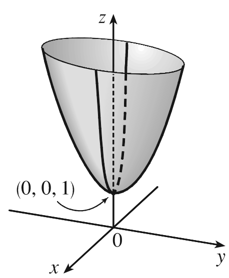
23. $z = 10 - 4x - 5y$ or $4x + 5y + z = 10$, a plane with intercepts 2.5, 2, and 10.



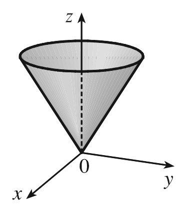
25. $z = y^2 + 1$, a parabolic cylinder



27. $z = 4x^2 + y^2 + 1$, an elliptic paraboloid with vertex at $(0, 0, 1)$.



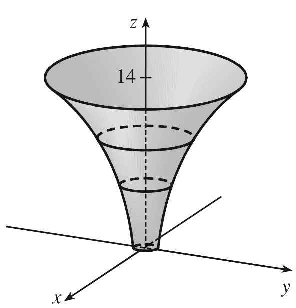
29. $z = \sqrt{x^2 + y^2}$ so $x^2 + y^2 = z^2$ and $z \geq 0$, the top half of a right circular cone.



31. The point $(-3, 3)$ lies between the level curves with z-values 50 and 60. Since the point is a little closer to the level curve with $z = 60$, we estimate that $f(-3, 3) \approx 56$. The point $(3, -2)$ appears to be just about halfway between the level curves with z-values 30 and 40, so we estimate $f(3, -2) \approx 35$. The graph rises as we approach the origin, gradually from above, steeply from below.

33. Near A, the level curves are very close together, indicating that the terrain is quite steep. At B, the level curves are much farther apart, so we would expect the terrain to be much less steep than near A, perhaps almost flat.

35.



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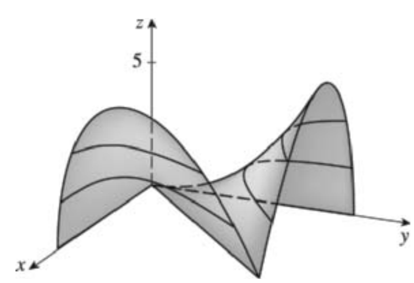
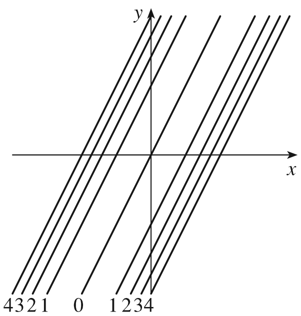
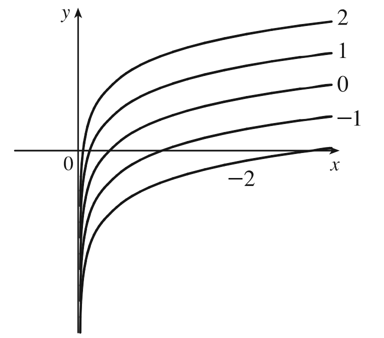


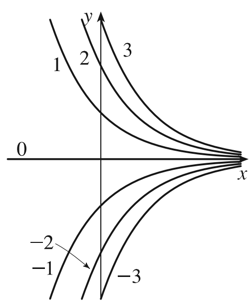
39. The level curves are $(y - 2x)^2 = k$ or $y = 2x \pm \sqrt{k}$, $k \geq 0$, a family of pairs of parallel lines.



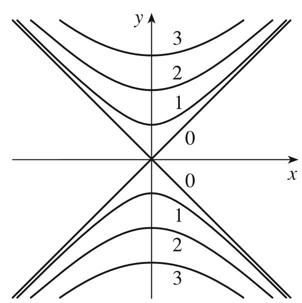
41. The level curves are $y - \ln x = k$ or $y = \ln x + k$.



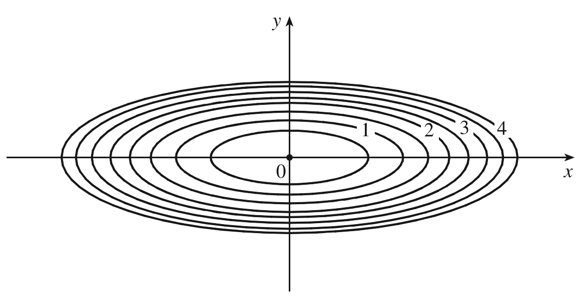
43. The level curves are $ye^x = k$ or $y = ke^{-x}$, a family of exponential curves.

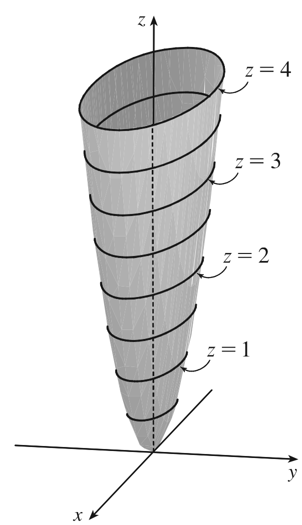


45. The level curves are $\sqrt{y^2 - x^2} = k$ or $y^2 - x^2 = k^2$, $k \geq 0$. When $k = 0$ the level curve is the pair of lines $y = \pm x$. For $k > 0$, the level curves are hyperbolas with axis the y-axis.



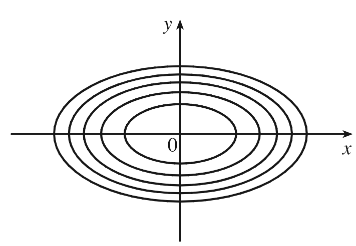
47. The contour map consists of the level curves $k = x^2 + 9y^2$, a family of ellipses with major axis the x-axis. (Or, if $k = 0$, the origin.)
The graph of $f(x, y)$ is the surface $z = x^2 + 9y^2$, an elliptic paraboloid.



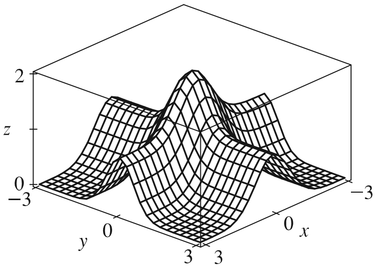


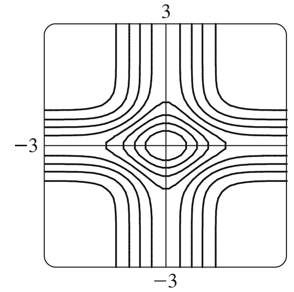
If we visualize lifting each ellipse $k = x^2 + 9y^2$ of the contour map to the plane $z = k$, we have horizontal traces that indicate the shape of the graph of f.

49. The isothermals are given by $k = 100/(1 + x^2 + 2y^2)$ or $x^2 + 2y^2 = (100 - k)/k$ $[0 < k \leq 100]$, a family of ellipses.

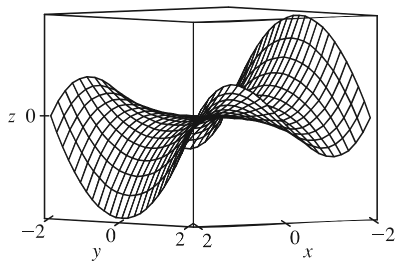


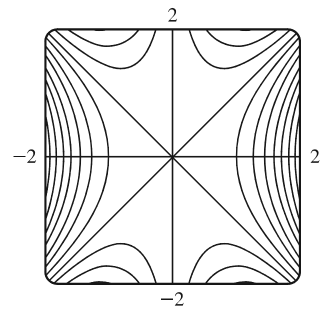
51. $f(x, y) = e^{-x^2} + e^{-2y^2}$





53. $f(x, y) = xy^2 - x^3$





The traces parallel to the yz-plane (such as the left-front trace in the graph above) are parabolas; those parallel to the xz-plane (such as the right-front trace) are cubic curves. The surface is called a monkey saddle because a monkey sitting on the surface near the origin has places for both legs and tail to rest.

55. (a) C (b) II

Reasons: This function is periodic in both x and y, and the function is the same when x is interchanged with y, so its graph is symmetric about the plane $y = x$. In addition, the function is 0 along the x- and y-axes. These conditions are satisfied only by C and II.

57. (a) F (b) I

Reasons: This function is periodic in both x and y but is constant along the lines $y = x + k$, a condition satisfied only by F and I.

59. (a) B (b) VI

Reasons: This function is 0 along the lines $x = \pm 1$ and $y = \pm 1$. The only contour map in which this could occur is VI. Also note that the trace in the xz-plane is the parabola $z = 1 - x^2$ and the trace in the yz-plane is the parabola $z = 1 - y^2$, so the graph is B.

61. $k = x + 3y + 5z$ is a family of parallel planes with normal vector $\langle 1, 3, 5 \rangle$.

63. $k = x^2 - y^2 + z^2$ are the equations of the level surfaces. For $k = 0$, the surface is a right circular cone with vertex the origin and axis the y-axis. For $k > 0$, we have a family of hyperboloids of one sheet with axis the y-axis. For $k < 0$, we have a family of hyperboloids of two sheets with axis the y-axis.

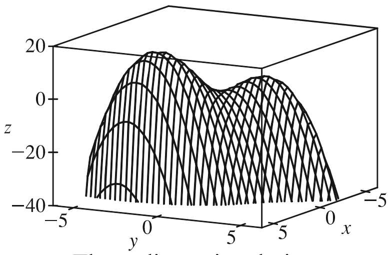
65. (a) The graph of g is the graph of f shifted upward 2 units.

(b) The graph of g is the graph of f stretched vertically by a factor of 2.

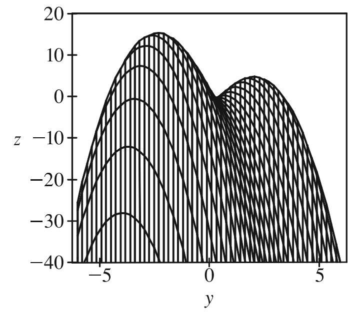
(c) The graph of g is the graph of f reflected about the xy-plane.

(d) The graph of $g(x, y) = -f(x, y) + 2$ is the graph of f reflected about the xy-plane and then shifted upward 2 units.

67. $f(x, y) = 3x - x^4 - 4y^2 - 10xy$



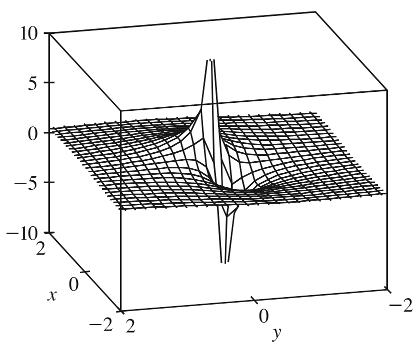
Three-dimensional view



Front view

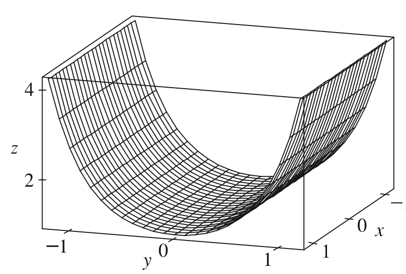
It does appear that the function has a maximum value, at the higher of the two "hilltops." From the front view graph, the maximum value appears to be approximately 15. Both hilltops could be considered local maximum points, as the values of f there are larger than at the neighboring points. There does not appear to be any local minimum point; although the valley shape between the two peaks looks like a minimum of some kind, some neighboring points have lower function values.

69.



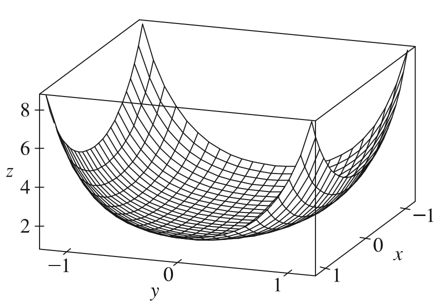
$f(x, y) = \dfrac{x + y}{x^2 + y^2}$. As both x and y become large, the function values appear to approach 0, regardless of which direction is considered. As (x, y) approaches the origin, the graph exhibits asymptotic behavior. From some directions, $f(x, y) \to \infty$, while in others $f(x, y) \to -\infty$. (These are the vertical spikes visible in the graph.) If the graph is examined carefully, however, one can see that $f(x, y)$ approaches 0 along the line $y = -x$.

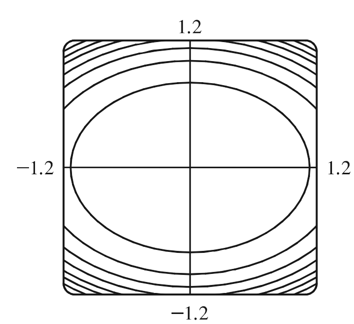
71. $f(x, y) = e^{cx^2 + y^2}$. First, if $c = 0$, the graph is the cylindrical surface $z = e^{y^2}$ (whose level curves are parallel lines). When $c > 0$, the vertical trace above the y-axis remains fixed while the sides of the surface in the x-direction "curl" upward, giving the graph a shape resembling an elliptic paraboloid. The level curves of the surface are ellipses centered at the origin.



$c = 0$

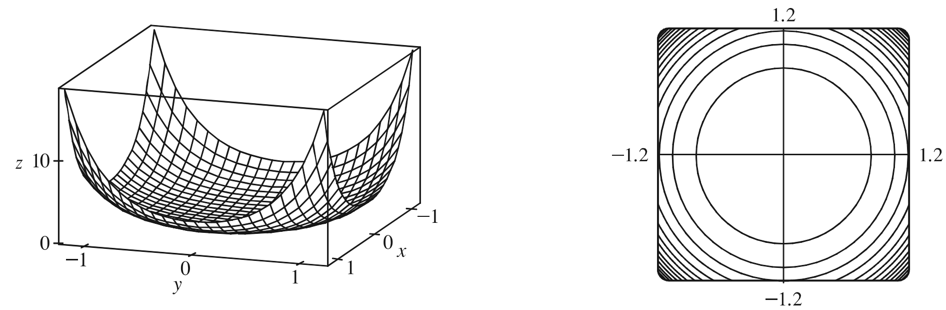
For $0 < c < 1$, the ellipses have major axis the x-axis and the eccentricity increases as $c \to 0$.





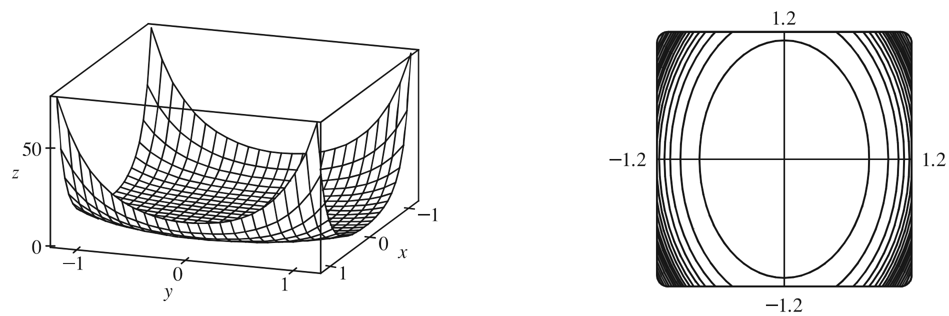
$c = 0.5$ (level curves in increments of 1)

For $c = 1$ the level curves are circles centered at the origin.



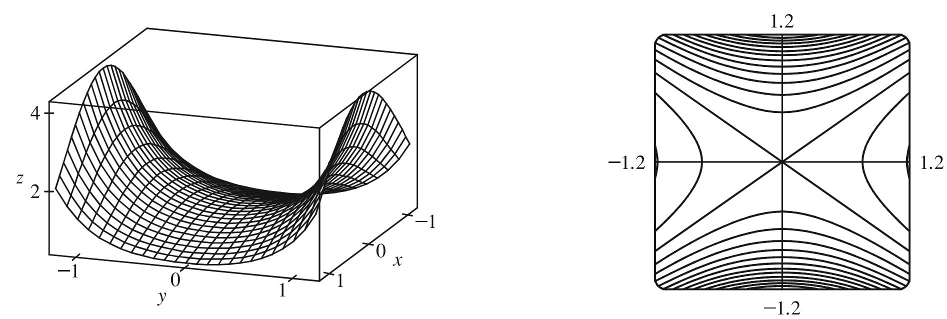
$c = 1$ (level curves in increments of 1)

When $c > 1$, the level curves are ellipses with major axis the y-axis, and the eccentricity increases as c increases.

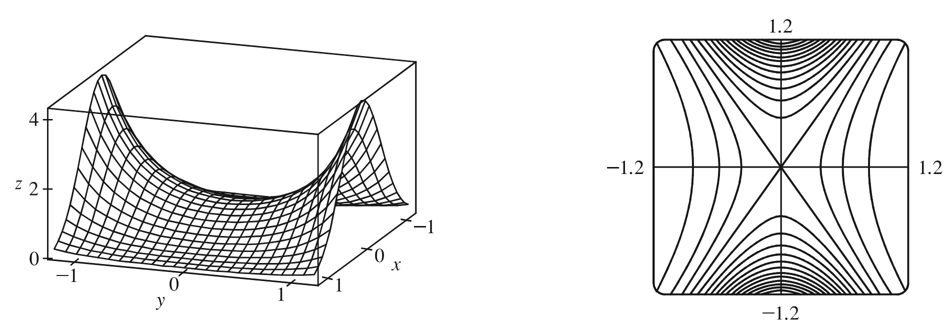


$c = 2$ (level curves in increments of 4)

For values of $c < 0$, the sides of the surface in the x-direction curl downward and approach the xy-plane (while the vertical trace $x = 0$ remains fixed), giving a saddle-shaped appearance to the graph near the point $(0, 0, 1)$. The level curves consist of a family of hyperbolas. As c decreases, the surface becomes flatter in the x-direction and the surface's approach to the curve in the trace $x = 0$ becomes steeper, as the graphs demonstrate.



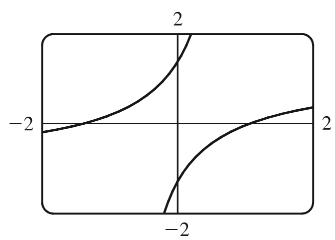
$c = -0.5$ (level curves in increments of 0.25)



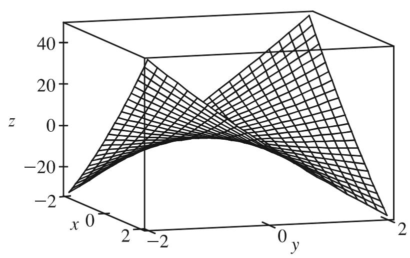
$c = -2$ (level curves in increments of 0.25)

73. $z = x^2 + y^2 + cxy$. When $c < -2$, the surface intersects the plane $z = k \neq 0$ in a hyperbola. (See graph below.) It intersects the plane $x = y$ in the parabola $z = (2 + c)x^2$, and the plane $x = -y$ in the parabola $z = (2 - c)x^2$. These parabolas open in opposite directions, so the surface is a hyperbolic paraboloid.

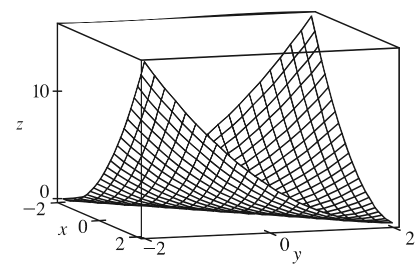
When $c = -2$ the surface is $z = x^2 + y^2 - 2xy = (x - y)^2$. So the surface is constant along each line $x - y = k$. That is, the surface is a cylinder with axis $x - y = 0$, $z = 0$. The shape of the cylinder is determined by its intersection with the plane $x + y = 0$, where $z = 4x^2$, and hence the cylinder is parabolic with minima of 0 on the line $y = x$.



$c = -5$, $z = 2$



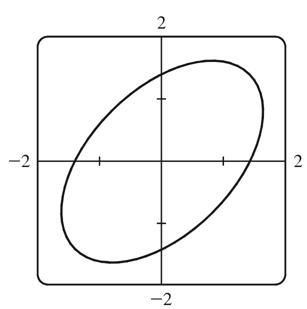
$c = -10$



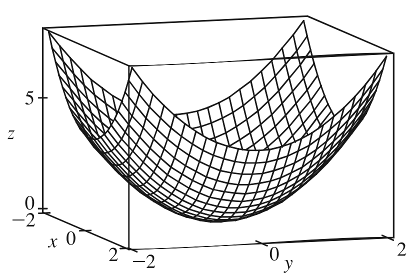
$c = -2$

When $-2 < c \leq 0$, $z \geq 0$ for all x and y. If x and y have the same sign, then $x^2 + y^2 + cxy \geq x^2 + y^2 - 2xy = (x - y)^2 \geq 0$. If they have opposite signs, then $cxy \geq 0$. The intersection with the surface and the plane $z = k > 0$ is an ellipse (see graph below). The intersection with the surface and the planes $x = 0$ and $y = 0$ are parabolas $z = y^2$ and $z = x^2$ respectively, so the surface is an elliptic paraboloid.

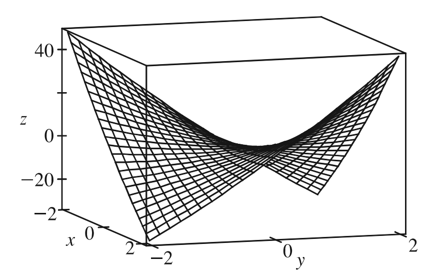
When $c > 0$ the graphs have the same shape, but are reflected in the plane $x = 0$, because $x^2 + y^2 + cxy = (-x)^2 + y^2 + (-c)(-x)y$. That is, the value of z is the same for c at (x, y) as it is for $-c$ at $(-x, y)$.



$c = -1$, $z = 2$



$c = 0$



$c = 10$

So the surface is an elliptic paraboloid for $0 < c < 2$, a parabolic cylinder for $c = 2$, and a hyperbolic paraboloid for $c > 2$.

75. (a) $P = bL^\alpha K^{1-\alpha} \quad \Rightarrow \quad \dfrac{P}{K} = bL^\alpha K^{-\alpha} \quad \Rightarrow \quad \dfrac{P}{K} = b\left(\dfrac{L}{K}\right)^\alpha \quad \Rightarrow \quad \ln \dfrac{P}{K} = \ln\left(b\left(\dfrac{L}{K}\right)^\alpha\right) \quad \Rightarrow$

$$\ln \frac{P}{K} = \ln b + \alpha \ln\left(\frac{L}{K}\right)$$

(b) We list the values for $\ln(L/K)$ and $\ln(P/K)$ for the years 1899–1922. (Historically, these values were rounded to 2 decimal places.)

53. $\dfrac{\partial z}{\partial r} = \dfrac{\partial z}{\partial x}\cos\theta + \dfrac{\partial z}{\partial y}\sin\theta$ and $\dfrac{\partial z}{\partial \theta} = -\dfrac{\partial z}{\partial x}r\sin\theta + \dfrac{\partial z}{\partial y}r\cos\theta$. Then

$$\begin{aligned}\frac{\partial^2 z}{\partial r^2} &= \cos\theta\left(\frac{\partial^2 z}{\partial x^2}\cos\theta + \frac{\partial^2 z}{\partial y\,\partial x}\sin\theta\right) + \sin\theta\left(\frac{\partial^2 z}{\partial y^2}\sin\theta + \frac{\partial^2 z}{\partial x\,\partial y}\cos\theta\right)\\ &= \cos^2\theta\,\frac{\partial^2 z}{\partial x^2} + 2\cos\theta\,\sin\theta\,\frac{\partial^2 z}{\partial x\,\partial y} + \sin^2\theta\,\frac{\partial^2 z}{\partial y^2}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2 z}{\partial \theta^2} &= -r\cos\theta\,\frac{\partial z}{\partial x} + (-r\sin\theta)\left(\frac{\partial^2 z}{\partial x^2}(-r\sin\theta) + \frac{\partial^2 z}{\partial y\,\partial x}r\cos\theta\right)\\ &\qquad -r\sin\theta\,\frac{\partial z}{\partial y} + r\cos\theta\left(\frac{\partial^2 z}{\partial y^2}r\cos\theta + \frac{\partial^2 z}{\partial x\,\partial y}(-r\sin\theta)\right)\\ &= -r\cos\theta\,\frac{\partial z}{\partial x} - r\sin\theta\,\frac{\partial z}{\partial y} + r^2\sin^2\theta\,\frac{\partial^2 z}{\partial x^2} - 2r^2\cos\theta\,\sin\theta\,\frac{\partial^2 z}{\partial x\,\partial y} + r^2\cos^2\theta\,\frac{\partial^2 z}{\partial y^2}\end{aligned}$$

Thus

$$\begin{aligned}\frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} + \frac{1}{r}\frac{\partial z}{\partial r} &= (\cos^2\theta + \sin^2\theta)\,\frac{\partial^2 z}{\partial x^2} + \left(\sin^2\theta + \cos^2\theta\right)\frac{\partial^2 z}{\partial y^2}\\ &\qquad -\frac{1}{r}\cos\theta\,\frac{\partial z}{\partial x} - \frac{1}{r}\sin\theta\,\frac{\partial z}{\partial y} + \frac{1}{r}\left(\cos\theta\,\frac{\partial z}{\partial x} + \sin\theta\,\frac{\partial z}{\partial y}\right)\\ &= \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \text{ as desired.}\end{aligned}$$

55. (a) Since f is a polynomial, it has continuous second-order partial derivatives, and

$f(tx,ty) = (tx)^2(ty) + 2(tx)(ty)^2 + 5(ty)^3 = t^3x^2y + 2t^3xy^2 + 5t^3y^3 = t^3(x^2y + 2xy^2 + 5y^3) = t^3 f(x,y)$.

Thus, f is homogeneous of degree 3.

(b) Differentiating both sides of $f(tx,ty) = t^n f(x,y)$ with respect to t using the Chain Rule, we get

$\dfrac{\partial}{\partial t}f(tx,ty) = \dfrac{\partial}{\partial t}[t^n f(x,y)] \quad\Leftrightarrow$

$\dfrac{\partial}{\partial(tx)}f(tx,ty)\cdot\dfrac{\partial(tx)}{\partial t} + \dfrac{\partial}{\partial(ty)}f(tx,ty)\cdot\dfrac{\partial(ty)}{\partial t} = x\dfrac{\partial}{\partial(tx)}f(tx,ty) + y\dfrac{\partial}{\partial(ty)}f(tx,ty) = nt^{n-1}f(x,y)$.

Setting $t = 1$: $x\dfrac{\partial}{\partial x}f(x,y) + y\dfrac{\partial}{\partial y}f(x,y) = nf(x,y)$.

57. Differentiating both sides of $f(tx,ty) = t^n f(x,y)$ with respect to x using the Chain Rule, we get

$\dfrac{\partial}{\partial x}f(tx,ty) = \dfrac{\partial}{\partial x}[t^n f(x,y)] \quad\Leftrightarrow$

$\dfrac{\partial}{\partial\,(tx)}f(tx,ty)\cdot\dfrac{\partial\,(tx)}{\partial x} + \dfrac{\partial}{\partial\,(ty)}f(tx,ty)\cdot\dfrac{\partial\,(ty)}{\partial x} = t^n\dfrac{\partial}{\partial x}f(x,y) \quad\Leftrightarrow\quad tf_x(tx,ty) = t^n f_x(x,y)$.

Thus $f_x(tx,ty) = t^{n-1}f_x(x,y)$.

15.6 Directional Derivatives and the Gradient Vector — ET 14.6

1. We can approximate the directional derivative of the pressure function at K in the direction of S by the average rate of change of pressure between the points where the red line intersects the contour lines closest to K (extend the red line slightly at the left). In the direction of S, the pressure changes from 1000 millibars to 996 millibars and we estimate the distance between these two points to be approximately 50 km (using the fact that the distance from K to S is 300 km). Then the rate of change of pressure in the direction given is approximately $\frac{996-1000}{50} = -0.08$ millibar/km.

3. $D_{\mathbf{u}}\, f(-20,30) = \nabla f\,(-20,30) \cdot \mathbf{u} = f_T(-20,30)\left(\frac{1}{\sqrt{2}}\right) + f_v(-20,30)\left(\frac{1}{\sqrt{2}}\right)$.

$f_T(-20,30) = \lim\limits_{h\to 0} \dfrac{f(-20+h,30) - f(-20,30)}{h}$, so we can approximate $f_T(-20,30)$ by considering $h = \pm 5$ and using the values given in the table: $f_T(-20,30) \approx \dfrac{f(-15,30) - f(-20,30)}{5} = \dfrac{-26-(-33)}{5} = 1.4$,

$f_T(-20,30) \approx \dfrac{f(-25,30) - f(-20,30)}{-5} = \dfrac{-39-(-33)}{-5} = 1.2$. Averaging these values gives $f_T(-20,30) \approx 1.3$.

Similarly, $f_v(-20,30) = \lim\limits_{h\to 0} \dfrac{f(-20,30+h) - f(-20,30)}{h}$, so we can approximate $f_v(-20,30)$ with $h = \pm 10$:

$f_v(-20,30) \approx \dfrac{f(-20,40) - f(-20,30)}{10} = \dfrac{-34-(-33)}{10} = -0.1$,

$f_v(-20,30) \approx \dfrac{f(-20,20) - f(-20,30)}{-10} = \dfrac{-30-(-33)}{-10} = -0.3$. Averaging these values gives $f_v(-20,30) \approx -0.2$.

Then $D_{\mathbf{u}} f(-20,30) \approx 1.3\left(\frac{1}{\sqrt{2}}\right) + (-0.2)\left(\frac{1}{\sqrt{2}}\right) \approx 0.778$.

5. $f(x,y) = ye^{-x} \quad\Rightarrow\quad f_x(x,y) = -ye^{-x}$ and $f_y(x,y) = e^{-x}$. If $\mathbf{u}$ is a unit vector in the direction of $\theta = 2\pi/3$, then from Equation 6, $D_{\mathbf{u}}\, f(0,4) = f_x(0,4)\cos\left(\frac{2\pi}{3}\right) + f_y(0,4)\sin\left(\frac{2\pi}{3}\right) = -4\cdot\left(-\frac{1}{2}\right) + 1\cdot\frac{\sqrt{3}}{2} = 2 + \frac{\sqrt{3}}{2}$.

7. $f(x,y) = \sin(2x+3y)$

(a) $\nabla f(x,y) = \dfrac{\partial f}{\partial x}\,\mathbf{i} + \dfrac{\partial f}{\partial y}\,\mathbf{j} = [\cos(2x+3y)\cdot 2]\,\mathbf{i} + [\cos(2x+3y)\cdot 3]\,\mathbf{j} = 2\cos(2x+3y)\,\mathbf{i} + 3\cos(2x+3y)\,\mathbf{j}$

(b) $\nabla f(-6,4) = (2\cos 0)\mathbf{i} + (3\cos 0)\mathbf{j} = 2\,\mathbf{i} + 3\,\mathbf{j}$

(c) By Equation 9, $D_{\mathbf{u}}\, f(-6,4) = \nabla f(-6,4)\cdot\mathbf{u} = (2\,\mathbf{i} + 3\,\mathbf{j})\cdot\frac{1}{2}\left(\sqrt{3}\,\mathbf{i} - \mathbf{j}\right) = \frac{1}{2}\left(2\sqrt{3} - 3\right) = \sqrt{3} - \frac{3}{2}$.

9. $f(x,y,z) = xe^{2yz}$

(a) $\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle = \left\langle e^{2yz}, 2xze^{2yz}, 2xye^{2yz}\right\rangle$

(b) $\nabla f(3,0,2) = \langle 1,12,0\rangle$

(c) By Equation 14, $D_{\mathbf{u}} f(3,0,2) = \nabla f(3,0,2)\cdot\mathbf{u} = \langle 1,12,0\rangle\cdot\left\langle\frac{2}{3},-\frac{2}{3},\frac{1}{3}\right\rangle = \frac{2}{3} - \frac{24}{3} + 0 = -\frac{22}{3}$.

11. $f(x,y) = 1 + 2x\sqrt{y} \quad\Rightarrow\quad \nabla f(x,y) = \left\langle 2\sqrt{y}, 2x\cdot\frac{1}{2}y^{-1/2}\right\rangle = \left\langle 2\sqrt{y}, x/\sqrt{y}\right\rangle$, $\nabla f(3,4) = \left\langle 4,\frac{3}{2}\right\rangle$, and a unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \dfrac{1}{\sqrt{4^2+(-3)^2}}\langle 4,-3\rangle = \left\langle\frac{4}{5},-\frac{3}{5}\right\rangle$, so $D_{\mathbf{u}}\, f(3,4) = \nabla f(3,4)\cdot\mathbf{u} = \left\langle 4,\frac{3}{2}\right\rangle\cdot\left\langle\frac{4}{5},-\frac{3}{5}\right\rangle = \frac{23}{10}$.

13. $g(p,q) = p^4 - p^2q^3 \quad\Rightarrow\quad \nabla g(p,q) = \left(4p^3 - 2pq^3\right)\mathbf{i} + \left(-3p^2q^2\right)\mathbf{j}$, $\nabla g(2,1) = 28\,\mathbf{i} - 12\,\mathbf{j}$, and a unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{1}{\sqrt{1^2+3^2}}(\mathbf{i} + 3\,\mathbf{j}) = \frac{1}{\sqrt{10}}(\mathbf{i} + 3\,\mathbf{j})$, so

$D_{\mathbf{u}}\, g(2,1) = \nabla g(2,1)\cdot\mathbf{u} = (28\,\mathbf{i} - 12\,\mathbf{j})\cdot\frac{1}{\sqrt{10}}(\mathbf{i} + 3\,\mathbf{j}) = \frac{1}{\sqrt{10}}(28 - 36) = -\frac{8}{\sqrt{10}}$ or $-\frac{4\sqrt{10}}{5}$.

15. $f(x,y,z) = xe^y + ye^z + ze^x \quad\Rightarrow\quad \nabla f(x,y,z) = \langle e^y + ze^x, xe^y + e^z, ye^z + e^x\rangle$, $\nabla f(0,0,0) = \langle 1,1,1\rangle$, and a unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{1}{\sqrt{25+1+4}}\langle 5,1,-2\rangle = \frac{1}{\sqrt{30}}\langle 5,1,-2\rangle$, so

$D_{\mathbf{u}}\, f(0,0,0) = \nabla f(0,0,0)\cdot\mathbf{u} = \langle 1,1,1\rangle\cdot\frac{1}{\sqrt{30}}\langle 5,1,-2\rangle = \frac{4}{\sqrt{30}}$.

17. $g(x,y,z)=(x+2y+3z)^{3/2} \quad\Rightarrow$

$$\nabla g(x,y,z)=\left\langle \tfrac{3}{2}(x+2y+3z)^{1/2}(1), \tfrac{3}{2}(x+2y+3z)^{1/2}(2), \tfrac{3}{2}(x+2y+3z)^{1/2}(3)\right\rangle$$
$$=\left\langle \tfrac{3}{2}\sqrt{x+2y+3z}, 3\sqrt{x+2y+3z}, \tfrac{9}{2}\sqrt{x+2y+3z}\right\rangle, \nabla g(1,1,2)=\left\langle \tfrac{9}{2}, 9, \tfrac{27}{2}\right\rangle,$$

and a unit vector in the direction of $\mathbf{v}=2\,\mathbf{j}-\mathbf{k}$ is $\mathbf{u}=\frac{2}{\sqrt{5}}\,\mathbf{j}-\frac{1}{\sqrt{5}}\,\mathbf{k}$, so

$$D_{\mathbf{u}}\,g(1,1,2)=\left\langle \tfrac{9}{2}, 9, \tfrac{27}{2}\right\rangle\cdot\left\langle 0, \tfrac{2}{\sqrt{5}}, -\tfrac{1}{\sqrt{5}}\right\rangle=\tfrac{18}{\sqrt{5}}-\tfrac{27}{2\sqrt{5}}=\tfrac{9}{2\sqrt{5}}.$$

19. $f(x,y)=\sqrt{xy} \quad\Rightarrow\quad \nabla f(x,y)=\left\langle \frac{1}{2}(xy)^{-1/2}(y), \frac{1}{2}(xy)^{-1/2}(x)\right\rangle=\left\langle \dfrac{y}{2\sqrt{xy}}, \dfrac{x}{2\sqrt{xy}}\right\rangle$, so $\nabla f(2,8)=\left\langle 1, \frac{1}{4}\right\rangle$.

The unit vector in the direction of $\overrightarrow{PQ}=\langle 5-2, 4-8\rangle=\langle 3,-4\rangle$ is $\mathbf{u}=\left\langle \frac{3}{5}, -\frac{4}{5}\right\rangle$, so

$D_{\mathbf{u}}\,f(2,8)=\nabla f(2,8)\cdot\mathbf{u}=\left\langle 1,\frac{1}{4}\right\rangle\cdot\left\langle \frac{3}{5},-\frac{4}{5}\right\rangle=\frac{2}{5}$.

21. $f(x,y)=y^2/x=y^2x^{-1} \quad\Rightarrow\quad \nabla f(x,y)=\left\langle -y^2x^{-2}, 2yx^{-1}\right\rangle=\left\langle -y^2/x^2, 2y/x\right\rangle$.

$\nabla f(2,4)=\langle -4,4\rangle$, or equivalently $\langle -1,1\rangle$, is the direction of maximum rate of change, and the maximum rate is $|\nabla f(2,4)|=\sqrt{16+16}=4\sqrt{2}$.

23. $f(x,y)=\sin(xy) \quad\Rightarrow\quad \nabla f(x,y)=\langle y\cos(xy), x\cos(xy)\rangle$, $\nabla f(1,0)=\langle 0,1\rangle$. Thus the maximum rate of change is $|\nabla f(1,0)|=1$ in the direction $\langle 0,1\rangle$.

25. $f(x,y,z)=\sqrt{x^2+y^2+z^2} \quad\Rightarrow$

$$\nabla f(x,y,z)=\left\langle \tfrac{1}{2}(x^2+y^2+z^2)^{-1/2}\cdot 2x, \tfrac{1}{2}(x^2+y^2+z^2)^{-1/2}\cdot 2y, \tfrac{1}{2}(x^2+y^2+z^2)^{-1/2}\cdot 2z\right\rangle$$
$$=\left\langle \frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}}\right\rangle,$$

$\nabla f(3,6,-2)=\left\langle \frac{3}{\sqrt{49}}, \frac{6}{\sqrt{49}}, \frac{-2}{\sqrt{49}}\right\rangle=\left\langle \frac{3}{7}, \frac{6}{7}, -\frac{2}{7}\right\rangle$. Thus the maximum rate of change is

$|\nabla f(3,6,-2)|=\sqrt{\left(\frac{3}{7}\right)^2+\left(\frac{6}{7}\right)^2+\left(-\frac{2}{7}\right)^2}=\sqrt{\frac{9+36+4}{49}}=1$ in the direction $\left\langle \frac{3}{7}, \frac{6}{7}, -\frac{2}{7}\right\rangle$ or equivalently $\langle 3,6,-2\rangle$.

27. (a) As in the proof of Theorem 15, $D_{\mathbf{u}}\,f=|\nabla f|\cos\theta$. Since the minimum value of $\cos\theta$ is -1 occurring when $\theta=\pi$, the minimum value of $D_{\mathbf{u}}\,f$ is $-|\nabla f|$ occurring when $\theta=\pi$, that is when $\mathbf{u}$ is in the opposite direction of ∇f (assuming $\nabla f\neq\mathbf{0}$).

(b) $f(x,y)=x^4y-x^2y^3 \quad\Rightarrow\quad \nabla f(x,y)=\left\langle 4x^3y-2xy^3, x^4-3x^2y^2\right\rangle$, so f decreases fastest at the point $(2,-3)$ in the direction $-\nabla f(2,-3)=-\langle 12,-92\rangle=\langle -12,92\rangle$.

29. The direction of fastest change is $\nabla f(x,y)=(2x-2)\,\mathbf{i}+(2y-4)\,\mathbf{j}$, so we need to find all points (x,y) where $\nabla f(x,y)$ is parallel to $\mathbf{i}+\mathbf{j} \quad\Leftrightarrow\quad (2x-2)\,\mathbf{i}+(2y-4)\,\mathbf{j}=k\,(\mathbf{i}+\mathbf{j}) \quad\Leftrightarrow\quad k=2x-2$ and $k=2y-4$. Then $2x-2=2y-4 \quad\Rightarrow$ $y=x+1$, so the direction of fastest change is $\mathbf{i}+\mathbf{j}$ at all points on the line $y=x+1$.

31. $T = \dfrac{k}{\sqrt{x^2+y^2+z^2}}$ and $120 = T(1,2,2) = \dfrac{k}{3}$ so $k = 360$.

(a) $\mathbf{u} = \dfrac{\langle 1,-1,1\rangle}{\sqrt{3}}$,

$$D_{\mathbf{u}}T(1,2,2) = \nabla T(1,2,2)\cdot\mathbf{u} = \left[-360\left(x^2+y^2+z^2\right)^{-3/2}\langle x,y,z\rangle\right]_{(1,2,2)}\cdot\mathbf{u} = -\tfrac{40}{3}\langle 1,2,2\rangle\cdot\tfrac{1}{\sqrt{3}}\langle 1,-1,1\rangle = -\tfrac{40}{3\sqrt{3}}$$

(b) From (a), $\nabla T = -360\left(x^2+y^2+z^2\right)^{-3/2}\langle x,y,z\rangle$, and since $\langle x,y,z\rangle$ is the position vector of the point (x,y,z), the vector $-\langle x,y,z\rangle$, and thus ∇T, always points toward the origin.

33. $\nabla V(x,y,z) = \langle 10x-3y+yz, xz-3x, xy\rangle$, $\nabla V(3,4,5) = \langle 38,6,12\rangle$

(a) $D_{\mathbf{u}}\,V(3,4,5) = \langle 38,6,12\rangle\cdot\frac{1}{\sqrt{3}}\langle 1,1,-1\rangle = \frac{32}{\sqrt{3}}$

(b) $\nabla V(3,4,5) = \langle 38,6,12\rangle$, or equivalently, $\langle 19,3,6\rangle$.

(c) $|\nabla V(3,4,5)| = \sqrt{38^2+6^2+12^2} = \sqrt{1624} = 2\sqrt{406}$

35. A unit vector in the direction of $\overrightarrow{AB}$ is $\mathbf{i}$ and a unit vector in the direction of $\overrightarrow{AC}$ is $\mathbf{j}$. Thus $D_{\overrightarrow{AB}}\,f(1,3) = f_x(1,3) = 3$ and $D_{\overrightarrow{AC}}\,f(1,3) = f_y(1,3) = 26$. Therefore $\nabla f(1,3) = \langle f_x(1,3), f_y(1,3)\rangle = \langle 3,26\rangle$, and by definition, $D_{\overrightarrow{AD}}\,f(1,3) = \nabla f\cdot\mathbf{u}$ where $\mathbf{u}$ is a unit vector in the direction of $\overrightarrow{AD}$, which is $\left\langle\frac{5}{13},\frac{12}{13}\right\rangle$. Therefore, $D_{\overrightarrow{AD}}\,f(1,3) = \langle 3,26\rangle\cdot\left\langle\frac{5}{13},\frac{12}{13}\right\rangle = 3\cdot\frac{5}{13} + 26\cdot\frac{12}{13} = \frac{327}{13}$.

37. (a) $$\nabla(au+bv) = \left\langle\frac{\partial(au+bv)}{\partial x},\frac{\partial(au+bv)}{\partial y}\right\rangle = \left\langle a\,\frac{\partial u}{\partial x}+b\,\frac{\partial v}{\partial x}, a\,\frac{\partial u}{\partial y}+b\,\frac{\partial v}{\partial y}\right\rangle = a\left\langle\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right\rangle + b\left\langle\frac{\partial v}{\partial x},\frac{\partial v}{\partial y}\right\rangle = a\,\nabla u + b\,\nabla v$$

(b) $$\nabla(uv) = \left\langle v\,\frac{\partial u}{\partial x}+u\,\frac{\partial v}{\partial x}, v\,\frac{\partial u}{\partial y}+u\,\frac{\partial v}{\partial y}\right\rangle = v\left\langle\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right\rangle + u\left\langle\frac{\partial v}{\partial x},\frac{\partial v}{\partial y}\right\rangle = v\,\nabla u + u\,\nabla v$$

(c) $$\nabla\left(\frac{u}{v}\right) = \left\langle\frac{v\,\frac{\partial u}{\partial x}-u\,\frac{\partial v}{\partial x}}{v^2},\frac{v\,\frac{\partial u}{\partial y}-u\,\frac{\partial v}{\partial y}}{v^2}\right\rangle = \frac{v\left\langle\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right\rangle - u\left\langle\frac{\partial v}{\partial x},\frac{\partial v}{\partial y}\right\rangle}{v^2} = \frac{v\,\nabla u - u\,\nabla v}{v^2}$$

(d) $$\nabla u^n = \left\langle\frac{\partial(u^n)}{\partial x},\frac{\partial(u^n)}{\partial y}\right\rangle = \left\langle nu^{n-1}\,\frac{\partial u}{\partial x}, nu^{n-1}\,\frac{\partial u}{\partial y}\right\rangle = nu^{n-1}\,\nabla u$$

39. Let $F(x,y,z) = 2(x-2)^2+(y-1)^2+(z-3)^2$. Then $2(x-2)^2+(y-1)^2+(z-3)^2 = 10$ is a level surface of F. $F_x(x,y,z) = 4(x-2) \;\Rightarrow\; F_x(3,3,5) = 4$, $F_y(x,y,z) = 2(y-1) \;\Rightarrow\; F_y(3,3,5) = 4$, and $F_z(x,y,z) = 2(z-3) \;\Rightarrow\; F_z(3,3,5) = 4$.

(a) Equation 19 gives an equation of the tangent plane at $(3,3,5)$ as $4(x-3)+4(y-3)+4(z-5) = 0 \;\Leftrightarrow\; 4x+4y+4z = 44$ or equivalently $x+y+z = 11$.

(b) By Equation 20, the normal line has symmetric equations $\dfrac{x-3}{4} = \dfrac{y-3}{4} = \dfrac{z-5}{4}$ or equivalently $x-3 = y-3 = z-5$. Corresponding parametric equations are $x = 3+t$, $y = 3+t$, $z = 5+t$.

41. Let $F(x, y, z) = x^2 - 2y^2 + z^2 + yz$. Then $x^2 - 2y^2 + z^2 + yz = 2$ is a level surface of F and $\nabla F(x, y, z) = \langle 2x, -4y + z, 2z + y \rangle$.

(a) $\nabla F(2, 1, -1) = \langle 4, -5, -1 \rangle$ is a normal vector for the tangent plane at $(2, 1, -1)$, so an equation of the tangent plane is $4(x - 2) - 5(y - 1) - 1(z + 1) = 0$ or $4x - 5y - z = 4$.

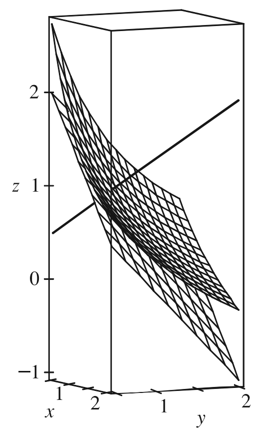
(b) The normal line has direction $\langle 4, -5, -1 \rangle$, so parametric equations are $x = 2 + 4t,\ y = 1 - 5t,\ z = -1 - t$, and symmetric equations are $\dfrac{x - 2}{4} = \dfrac{y - 1}{-5} = \dfrac{z + 1}{-1}$.

43. $F(x, y, z) = -z + xe^y \cos z \quad \Rightarrow \quad \nabla F(x, y, z) = \langle e^y \cos z, xe^y \cos z, -1 - xe^y \sin z \rangle$ and $\nabla F(1, 0, 0) = \langle 1, 1, -1 \rangle$.

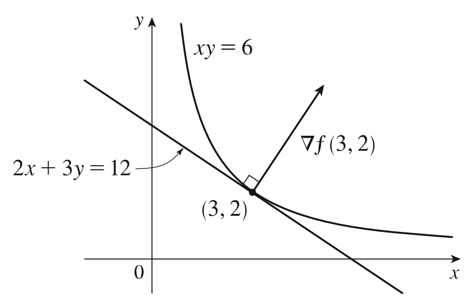
(a) $1(x - 1) + 1(y - 0) - 1(z - 0) = 0$ or $x + y - z = 1$

(b) $x - 1 = y = -z$

45. $F(x, y, z) = xy + yz + zx$, $\nabla F(x, y, z) = \langle y + z, x + z, y + x \rangle$, $\nabla F(1, 1, 1) = \langle 2, 2, 2 \rangle$, so an equation of the tangent plane is $2x + 2y + 2z = 6$ or $x + y + z = 3$, and the normal line is given by $x - 1 = y - 1 = z - 1$ or $x = y = z$. To graph the surface we solve for z: $z = \dfrac{3 - xy}{x + y}$.



47. $f(x, y) = xy \quad \Rightarrow \quad \nabla f(x, y) = \langle y, x \rangle$, $\nabla f(3, 2) = \langle 2, 3 \rangle$. $\nabla f(3, 2)$ is perpendicular to the tangent line, so the tangent line has equation $\nabla f(3, 2) \cdot \langle x - 3, y - 2 \rangle = 0 \quad \Rightarrow \quad \langle 2, 3 \rangle \cdot \langle x - 3, x - 2 \rangle = 0 \quad \Rightarrow$ $2(x - 3) + 3(y - 2) = 0$ or $2x + 3y = 12$.



49. $\nabla F(x_0, y_0, z_0) = \left\langle \dfrac{2x_0}{a^2}, \dfrac{2y_0}{b^2}, \dfrac{2z_0}{c^2} \right\rangle$. Thus an equation of the tangent plane at (x_0, y_0, z_0) is

$\dfrac{2x_0}{a^2}\, x + \dfrac{2y_0}{b^2}\, y + \dfrac{2z_0}{c^2}\, z = 2\left(\dfrac{x_0^2}{a^2} + \dfrac{y_0^2}{b^2} + \dfrac{z_0^2}{c^2} \right) = 2(1) = 2$ since (x_0, y_0, z_0) is a point on the ellipsoid. Hence $\dfrac{x_0}{a^2}\, x + \dfrac{y_0}{b^2}\, y + \dfrac{z_0}{c^2}\, z = 1$ is an equation of the tangent plane.

51. $\nabla F(x_0, y_0, z_0) = \left\langle \dfrac{2x_0}{a^2}, \dfrac{2y_0}{b^2}, \dfrac{-1}{c} \right\rangle$, so an equation of the tangent plane is $\dfrac{2x_0}{a^2}\,x + \dfrac{2y_0}{b^2}\,y - \dfrac{1}{c}\,z = \dfrac{2x_0^2}{a^2} + \dfrac{2y_0^2}{b^2} - \dfrac{z_0}{c}$

or $\dfrac{2x_0}{a^2}\,x + \dfrac{2y_0}{b^2}\,y = \dfrac{z}{c} + 2\left(\dfrac{x_0^2}{a^2} + \dfrac{y_0^2}{b^2}\right) - \dfrac{z_0}{c}$. But $\dfrac{z_0}{c} = \dfrac{x_0^2}{a^2} + \dfrac{y_0^2}{b^2}$, so the equation can be written as

$\dfrac{2x_0}{a^2}\,x + \dfrac{2y_0}{b^2}\,y = \dfrac{z + z_0}{c}$.

53. The hyperboloid $x^2 - y^2 - z^2 = 1$ is a level surface of $F(x,y,z) = x^2 - y^2 - z^2$ and $\nabla F(x,y,z) = \langle 2x, -2y, -2z \rangle$ is a normal vector to the surface and hence a normal vector for the tangent plane at (x,y,z). The tangent plane is parallel to the plane $z = x + y$ or $x + y - z = 0$ if and only if the corresponding normal vectors are parallel, so we need a point (x_0, y_0, z_0) on the hyperboloid where $\langle 2x_0, -2y_0, -2z_0 \rangle = c\,\langle 1, 1, -1 \rangle$ or equivalently $\langle x_0, -y_0, -z_0 \rangle = k\,\langle 1, 1, -1 \rangle$ for some $k \neq 0$. Then we must have $x_0 = k$, $y_0 = -k$, $z_0 = k$ and substituting into the equation of the hyperboloid gives $k^2 - (-k)^2 - k^2 = 1 \quad\Leftrightarrow\quad -k^2 = 1$, an impossibility. Thus there is no such point on the hyperboloid.

55. Let (x_0, y_0, z_0) be a point on the cone [other than $(0,0,0)$]. Then an equation of the tangent plane to the cone at this point is $2x_0x + 2y_0y - 2z_0z = 2\left(x_0^2 + y_0^2 - z_0^2\right)$. But $x_0^2 + y_0^2 = z_0^2$ so the tangent plane is given by $x_0x + y_0y - z_0z = 0$, a plane which always contains the origin.

57. Let (x_0, y_0, z_0) be a point on the surface. Then an equation of the tangent plane at the point is

$\dfrac{x}{2\sqrt{x_0}} + \dfrac{y}{2\sqrt{y_0}} + \dfrac{z}{2\sqrt{z_0}} = \dfrac{\sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0}}{2}$. But $\sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0} = \sqrt{c}$, so the equation is

$\dfrac{x}{\sqrt{x_0}} + \dfrac{y}{\sqrt{y_0}} + \dfrac{z}{\sqrt{z_0}} = \sqrt{c}$. The x-, y-, and z-intercepts are $\sqrt{cx_0}$, $\sqrt{cy_0}$ and $\sqrt{cz_0}$ respectively. (The x-intercept is found by setting $y = z = 0$ and solving the resulting equation for x, and the y- and z-intercepts are found similarly.) So the sum of the intercepts is $\sqrt{c}\left(\sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0}\right) = c$, a constant.

59. If $f(x,y,z) = z - x^2 - y^2$ and $g(x,y,z) = 4x^2 + y^2 + z^2$, then the tangent line is perpendicular to both ∇f and ∇g at $(-1, 1, 2)$. The vector $\mathbf{v} = \nabla f \times \nabla g$ will therefore be parallel to the tangent line.

We have $\nabla f(x,y,z) = \langle -2x, -2y, 1 \rangle \quad\Rightarrow\quad \nabla f(-1,1,2) = \langle 2, -2, 1 \rangle$, and $\nabla g(x,y,z) = \langle 8x, 2y, 2z \rangle \quad\Rightarrow$

$\nabla g(-1,1,2) = \langle -8, 2, 4 \rangle$. Hence $\mathbf{v} = \nabla f \times \nabla g = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -2 & 1 \\ -8 & 2 & 4 \end{vmatrix} = -10\,\mathbf{i} - 16\,\mathbf{j} - 12\,\mathbf{k}$.

Parametric equations are: $x = -1 - 10t,\; y = 1 - 16t,\; z = 2 - 12t$.

61. (a) The direction of the normal line of F is given by ∇F, and that of G by ∇G. Assuming that $\nabla F \neq 0 \neq \nabla G$, the two normal lines are perpendicular at P if $\nabla F \cdot \nabla G = 0$ at $P \quad\Leftrightarrow$ $\langle \partial F/\partial x, \partial F/\partial y, \partial F/\partial z \rangle \cdot \langle \partial G/\partial x, \partial G/\partial y, \partial G/\partial z \rangle = 0$ at $P \quad\Leftrightarrow\quad F_xG_x + F_yG_y + F_zG_z = 0$ at P.

(b) Here $F = x^2 + y^2 - z^2$ and $G = x^2 + y^2 + z^2 - r^2$, so

$\nabla F \cdot \nabla G = \langle 2x, 2y, -2z \rangle \cdot \langle 2x, 2y, 2z \rangle = 4x^2 + 4y^2 - 4z^2 = 4F = 0$, since the point (x,y,z) lies on the graph of

$F = 0$. To see that this is true without using calculus, note that $G = 0$ is the equation of a sphere centered at the origin and $F = 0$ is the equation of a right circular cone with vertex at the origin (which is generated by lines through the origin). At any point of intersection, the sphere's normal line (which passes through the origin) lies on the cone, and thus is perpendicular to the cone's normal line. So the surfaces with equations $F = 0$ and $G = 0$ are everywhere orthogonal.

63. Let $\mathbf{u} = \langle a, b\rangle$ and $\mathbf{v} = \langle c, d\rangle$. Then we know that at the given point, $D_{\mathbf{u}}\, f = \nabla f \cdot \mathbf{u} = af_x + bf_y$ and $D_{\mathbf{v}}\, f = \nabla f \cdot \mathbf{v} = cf_x + df_y$. But these are just two linear equations in the two unknowns f_x and f_y, and since $\mathbf{u}$ and $\mathbf{v}$ are not parallel, we can solve the equations to find $\nabla f = \langle f_x, f_y\rangle$ at the given point. In fact,

$$\nabla f = \left\langle \frac{d\,D_{\mathbf{u}}\, f - b\,D_{\mathbf{v}}\, f}{ad - bc}, \frac{a\,D_{\mathbf{v}}\, f - c\,D_{\mathbf{u}}\, f}{ad - bc} \right\rangle.$$

15.7 Maximum and Minimum Values ET 14.7

1. (a) First we compute $D(1,1) = f_{xx}(1,1)\, f_{yy}(1,1) - [f_{xy}(1,1)]^2 = (4)(2) - (1)^2 = 7$. Since $D(1,1) > 0$ and $f_{xx}(1,1) > 0$, f has a local minimum at $(1,1)$ by the Second Derivatives Test.

(b) $D(1,1) = f_{xx}(1,1)\, f_{yy}(1,1) - [f_{xy}(1,1)]^2 = (4)(2) - (3)^2 = -1$. Since $D(1,1) < 0$, f has a saddle point at $(1,1)$ by the Second Derivatives Test.

3. In the figure, a point at approximately $(1,1)$ is enclosed by level curves which are oval in shape and indicate that as we move away from the point in any direction the values of f are increasing. Hence we would expect a local minimum at or near $(1,1)$. The level curves near $(0,0)$ resemble hyperbolas, and as we move away from the origin, the values of f increase in some directions and decrease in others, so we would expect to find a saddle point there.

To verify our predictions, we have $f(x,y) = 4 + x^3 + y^3 - 3xy \;\Rightarrow\; f_x(x,y) = 3x^2 - 3y$, $f_y(x,y) = 3y^2 - 3x$. We have critical points where these partial derivatives are equal to 0: $3x^2 - 3y = 0$, $3y^2 - 3x = 0$. Substituting $y = x^2$ from the first equation into the second equation gives $3(x^2)^2 - 3x = 0 \;\Rightarrow\; 3x(x^3 - 1) = 0 \;\Rightarrow\; x = 0$ or $x = 1$. Then we have two critical points, $(0,0)$ and $(1,1)$. The second partial derivatives are $f_{xx}(x,y) = 6x$, $f_{xy}(x,y) = -3$, and $f_{yy}(x,y) = 6y$, so $D(x,y) = f_{xx}(x,y)\, f_{yy}(x,y) - [f_{xy}(x,y)]^2 = (6x)(6y) - (-3)^2 = 36xy - 9$. Then $D(0,0) = 36(0)(0) - 9 = -9$, and $D(1,1) = 36(1)(1) - 9 = 27$. Since $D(0,0) < 0$, f has a saddle point at $(0,0)$ by the Second Derivatives Test. Since $D(1,1) > 0$ and $f_{xx}(1,1) > 0$, f has a local minimum at $(1,1)$.

5. $f(x,y) = 9 - 2x + 4y - x^2 - 4y^2 \;\Rightarrow\; f_x = -2 - 2x$, $f_y = 4 - 8y$, $f_{xx} = -2$, $f_{xy} = 0$, $f_{yy} = -8$. Then $f_x = 0$ and $f_y = 0$ imply $x = -1$ and $y = \frac{1}{2}$, and the only critical point is $\left(-1, \frac{1}{2}\right)$. $D(x,y) = f_{xx}f_{yy} - (f_{xy})^2 = (-2)(-8) - 0^2 = 16$, and since $D\left(-1, \frac{1}{2}\right) = 16 > 0$ and $f_{xx}\left(-1, \frac{1}{2}\right) = -2 < 0$, $f\left(-1, \frac{1}{2}\right) = 11$ is a local maximum by the Second Derivatives Test.