37. The area of the rectangle is $A = xy$, and $\Delta A \approx dA$ is an estimate of the area of paint in the stripe. Here $dA = y\,dx + x\,dy$, so with $dx = dy = \frac{3+3}{12} = \frac{1}{2}$, $\Delta A \approx dA = (100)\left(\frac{1}{2}\right) + (200)\left(\frac{1}{2}\right) = 150$ ft^2. Thus there are approximately 150 ft^2 of paint in the stripe.

39. First we find $\dfrac{\partial R}{\partial R_1}$ implicitly by taking partial derivatives of both sides with respect to R_1:

$$\frac{\partial}{\partial R_1}\left(\frac{1}{R}\right) = \frac{\partial\left[(1/R_1) + (1/R_2) + (1/R_3)\right]}{\partial R_1} \quad\Rightarrow\quad -R^{-2}\frac{\partial R}{\partial R_1} = -R_1^{-2} \quad\Rightarrow\quad \frac{\partial R}{\partial R_1} = \frac{R^2}{R_1^2}.$$ Then by symmetry,

$\dfrac{\partial R}{\partial R_2} = \dfrac{R^2}{R_2^2}$, $\dfrac{\partial R}{\partial R_3} = \dfrac{R^2}{R_3^2}$. When $R_1 = 25$, $R_2 = 40$ and $R_3 = 50$, $\dfrac{1}{R} = \dfrac{17}{200} \quad\Leftrightarrow\quad R = \frac{200}{17}\ \Omega$.

Since the possible error for each R_i is 0.5%, the maximum error of R is attained by setting $\Delta R_i = 0.005R_i$. So

$$\Delta R \approx dR = \frac{\partial R}{\partial R_1}\Delta R_1 + \frac{\partial R}{\partial R_2}\Delta R_2 + \frac{\partial R}{\partial R_3}\Delta R_3 = (0.005)R^2\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right) = (0.005)R = \tfrac{1}{17} \approx 0.059\ \Omega.$$

41. The errors in measurement are at most 2%, so $\left|\dfrac{\Delta w}{w}\right| \le 0.02$ and $\left|\dfrac{\Delta h}{h}\right| \le 0.02$. The relative error in the calculated surface area is

$$\frac{\Delta S}{S} \approx \frac{dS}{S} = \frac{0.1091(0.425w^{0.425-1})h^{0.725}\,dw + 0.1091w^{0.425}(0.725h^{0.725-1})\,dh}{0.1091w^{0.425}h^{0.725}} = 0.425\frac{dw}{w} + 0.725\frac{dh}{h}$$

To estimate the maximum relative error, we use $\dfrac{dw}{w} = \left|\dfrac{\Delta w}{w}\right| = 0.02$ and $\dfrac{dh}{h} = \left|\dfrac{\Delta h}{h}\right| = 0.02 \quad\Rightarrow$

$\dfrac{dS}{S} = 0.425\,(0.02) + 0.725\,(0.02) = 0.023$. Thus the maximum percentage error is approximately 2.3%.

43.
$$\begin{aligned}\Delta z &= f(a + \Delta x, b + \Delta y) - f(a, b) = (a + \Delta x)^2 + (b + \Delta y)^2 - (a^2 + b^2) \\ &= a^2 + 2a\,\Delta x + (\Delta x)^2 + b^2 + 2b\,\Delta y + (\Delta y)^2 - a^2 - b^2 = 2a\,\Delta x + (\Delta x)^2 + 2b\,\Delta y + (\Delta y)^2\end{aligned}$$

But $f_x(a, b) = 2a$ and $f_y(a, b) = 2b$ and so $\Delta z = f_x(a, b)\,\Delta x + f_y(a, b)\,\Delta y + \Delta x\,\Delta x + \Delta y\,\Delta y$, which is Definition 7 with $\varepsilon_1 = \Delta x$ and $\varepsilon_2 = \Delta y$. Hence f is differentiable.

45. To show that f is continuous at (a, b) we need to show that $\displaystyle\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$ or

equivalently $\displaystyle\lim_{(\Delta x,\Delta y)\to(0,0)} f(a + \Delta x, b + \Delta y) = f(a, b)$. Since f is differentiable at (a, b),

$f(a + \Delta x, b + \Delta y) - f(a, b) = \Delta z = f_x(a, b)\,\Delta x + f_y(a, b)\,\Delta y + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y$, where ϵ_1 and $\epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$. Thus $f(a + \Delta x, b + \Delta y) = f(a, b) + f_x(a, b)\,\Delta x + f_y(a, b)\,\Delta y + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y$. Taking the limit of both sides as $(\Delta x, \Delta y) \to (0, 0)$ gives $\displaystyle\lim_{(\Delta x,\Delta y)\to(0,0)} f(a + \Delta x, b + \Delta y) = f(a, b)$. Thus f is continuous at (a, b).

15.5 The Chain Rule ET 14.5

1. $z = x^2 + y^2 + xy,\ x = \sin t,\ y = e^t \quad\Rightarrow\quad \dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt} = (2x + y)\cos t + (2y + x)e^t$

3. $z = \sqrt{1 + x^2 + y^2},\ x = \ln t,\ y = \cos t \quad\Rightarrow$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \tfrac{1}{2}(1 + x^2 + y^2)^{-1/2}(2x)\cdot\frac{1}{t} + \tfrac{1}{2}(1 + x^2 + y^2)^{-1/2}(2y)(-\sin t) = \frac{1}{\sqrt{1 + x^2 + y^2}}\left(\frac{x}{t} - y\sin t\right)$$

5. $w = xe^{y/z},\ x = t^2,\ y = 1 - t,\ z = 1 + 2t \quad \Rightarrow$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} = e^{y/z} \cdot 2t + xe^{y/z}\left(\frac{1}{z}\right) \cdot (-1) + xe^{y/z}\left(-\frac{y}{z^2}\right) \cdot 2 = e^{y/z}\left(2t - \frac{x}{z} - \frac{2xy}{z^2}\right)$$

7. $z = x^2y^3,\ x = s\cos t,\ y = s\sin t \quad \Rightarrow$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = 2xy^3\cos t + 3x^2y^2\sin t$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = (2xy^3)(-s\sin t) + (3x^2y^2)(s\cos t) = -2sxy^3\sin t + 3sx^2y^2\cos t$$

9. $z = \sin\theta\cos\phi,\ \ \theta = st^2,\ \ \phi = s^2t \quad \Rightarrow$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial s} + \frac{\partial z}{\partial \phi}\frac{\partial \phi}{\partial s} = (\cos\theta\ \cos\phi)(t^2) + (-\sin\theta\ \sin\phi)(2st) = t^2\cos\theta\ \cos\phi - 2st\sin\theta\ \sin\phi$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial t} + \frac{\partial z}{\partial \phi}\frac{\partial \phi}{\partial t} = (\cos\theta\ \cos\phi)(2st) + (-\sin\theta\ \sin\phi)(s^2) = 2st\cos\theta\ \cos\phi - s^2\sin\theta\ \sin\phi$$

11. $z = e^r\cos\theta,\ r = st,\ \theta = \sqrt{s^2 + t^2} \quad \Rightarrow$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial s} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial s} = e^r\cos\theta \cdot t + e^r(-\sin\theta) \cdot \tfrac{1}{2}(s^2 + t^2)^{-1/2}(2s) = te^r\cos\theta - e^r\sin\theta \cdot \frac{s}{\sqrt{s^2 + t^2}}$$

$$= e^r\left(t\cos\theta - \frac{s}{\sqrt{s^2 + t^2}}\sin\theta\right)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial t} = e^r\cos\theta \cdot s + e^r(-\sin\theta) \cdot \tfrac{1}{2}(s^2 + t^2)^{-1/2}(2t) = se^r\cos\theta - e^r\sin\theta \cdot \frac{t}{\sqrt{s^2 + t^2}}$$

$$= e^r\left(s\cos\theta - \frac{t}{\sqrt{s^2 + t^2}}\sin\theta\right)$$

13. When $t = 3$, $x = g(3) = 2$ and $y = h(3) = 7$. By the Chain Rule (2),

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = f_x(2,7)g'(3) + f_y(2,7)\,h'(3) = (6)(5) + (-8)(-4) = 62.$$

15. $g(u,v) = f(x(u,v), y(u,v))$ where $x = e^u + \sin v,\ y = e^u + \cos v \quad \Rightarrow$

$\dfrac{\partial x}{\partial u} = e^u,\ \dfrac{\partial x}{\partial v} = \cos v,\ \dfrac{\partial y}{\partial u} = e^u,\ \dfrac{\partial y}{\partial v} = -\sin v$. By the Chain Rule (3), $\dfrac{\partial g}{\partial u} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial u}$. Then

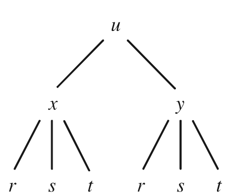
$$g_u(0,0) = f_x(x(0,0), y(0,0))\,x_u(0,0) + f_y(x(0,0), y(0,0))\,y_u(0,0) = f_x(1,2)(e^0) + f_y(1,2)(e^0) = 2(1) + 5(1) = 7.$$

Similarly, $\dfrac{\partial g}{\partial v} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial v} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial v}$. Then

$$g_v(0,0) = f_x(x(0,0), y(0,0))\,x_v(0,0) + f_y(x(0,0), y(0,0))\,y_v(0,0) = f_x(1,2)(\cos 0) + f_y(1,2)(-\sin 0)$$

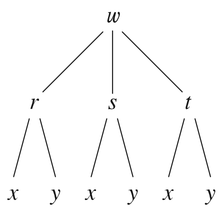
$$= 2(1) + 5(0) = 2$$

17.



$u = f(x,y),\ x = x(r,s,t),\ y = y(r,s,t) \quad \Rightarrow$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r},\ \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s},\ \frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}$$

19. 

$w = f(r, s, t),\ \ r = r(x, y),\ \ s = s(x, y),\ t = t(x, y) \ \ \Rightarrow$

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial w}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial x}, \quad \frac{\partial w}{\partial y} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial w}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial y}$$

21. $z = x^2 + xy^3,\ x = uv^2 + w^3,\ y = u + ve^w \quad \Rightarrow$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = (2x + y^3)(v^2) + (3xy^2)(1),$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = (2x + y^3)(2uv) + (3xy^2)(e^w),$$

$$\frac{\partial z}{\partial w} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial w} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial w} = (2x + y^3)(3w^2) + (3xy^2)(ve^w).$$

When $u = 2$, $v = 1$, and $w = 0$, we have $x = 2$, $y = 3$,

so $\dfrac{\partial z}{\partial u} = (31)(1) + (54)(1) = 85,\ \dfrac{\partial z}{\partial v} = (31)\,(4) + (54)(1) = 178,\ \dfrac{\partial z}{\partial w} = (31)(0) + (54)(1) = 54.$

23. $R = \ln(u^2 + v^2 + w^2),\ u = x + 2y,\ v = 2x - y,\ w = 2xy \quad \Rightarrow$

$$\frac{\partial R}{\partial x} = \frac{\partial R}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial R}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial R}{\partial w}\frac{\partial w}{\partial x} = \frac{2u}{u^2 + v^2 + w^2}\,(1) + \frac{2v}{u^2 + v^2 + w^2}\,(2) + \frac{2w}{u^2 + v^2 + w^2}\,(2y)$$
$$= \frac{2u + 4v + 4wy}{u^2 + v^2 + w^2},$$

$$\frac{\partial R}{\partial y} = \frac{\partial R}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial R}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial R}{\partial w}\frac{\partial w}{\partial y} = \frac{2u}{u^2 + v^2 + w^2}\,(2) + \frac{2v}{u^2 + v^2 + w^2}\,(-1) + \frac{2w}{u^2 + v^2 + w^2}\,(2x)$$
$$= \frac{4u - 2v + 4wx}{u^2 + v^2 + w^2}.$$

When $x = y = 1$ we have $u = 3$, $v = 1$, and $w = 2$, so $\dfrac{\partial R}{\partial x} = \dfrac{9}{7}$ and $\dfrac{\partial R}{\partial y} = \dfrac{9}{7}$.

25. $u = x^2 + yz,\ x = pr\cos\theta,\ y = pr\sin\theta,\ z = p + r \quad \Rightarrow$

$$\frac{\partial u}{\partial p} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial p} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial p} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial p} = (2x)(r\cos\theta) + (z)(r\sin\theta) + (y)(1) = 2xr\cos\theta + zr\sin\theta + y,$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial r} = (2x)(p\cos\theta) + (z)(p\sin\theta) + (y)(1) = 2xp\cos\theta + zp\sin\theta + y,$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial \theta} = (2x)(-pr\sin\theta) + (z)(pr\cos\theta) + (y)(0) = -2xpr\sin\theta + zpr\cos\theta.$$

When $p = 2$, $r = 3$, and $\theta = 0$ we have $x = 6$, $y = 0$, and $z = 5$, so $\dfrac{\partial u}{\partial p} = 36$, $\dfrac{\partial u}{\partial r} = 24$, and $\dfrac{\partial u}{\partial \theta} = 30$.

27. $\sqrt{xy} = 1 + x^2y$, so let $F(x, y) = (xy)^{1/2} - 1 - x^2y = 0$. Then by Equation 6

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{\frac{1}{2}(xy)^{-1/2}(y) - 2xy}{\frac{1}{2}(xy)^{-1/2}(x) - x^2} = -\frac{y - 4xy\sqrt{xy}}{x - 2x^2\sqrt{xy}} = \frac{4(xy)^{3/2} - y}{x - 2x^2\sqrt{xy}}.$$

29. $\cos(x - y) = xe^y$, so let $F(x, y) = \cos(x - y) - xe^y = 0$.

Then $\dfrac{dy}{dx} = -\dfrac{F_x}{F_y} = -\dfrac{-\sin(x - y) - e^y}{-\sin(x - y)(-1) - xe^y} = \dfrac{\sin(x - y) + e^y}{\sin(x - y) - xe^y}.$

31. $x^2 + y^2 + z^2 = 3xyz$, so let $F(x, y, z) = x^2 + y^2 + z^2 - 3xyz = 0$. Then by Equations 7

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{2x - 3yz}{2z - 3xy} = \frac{3yz - 2x}{2z - 3xy} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{2y - 3xz}{2z - 3xy} = \frac{3xz - 2y}{2z - 3xy}.$$

33. $x - z = \arctan(yz)$, so let $F(x, y, z) = x - z - \arctan(yz) = 0$. Then

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{1}{-1 - \dfrac{1}{1 + (yz)^2}(y)} = \frac{1 + y^2z^2}{1 + y + y^2z^2}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{-\dfrac{1}{1 + (yz)^2}(z)}{-1 - \dfrac{1}{1 + (yz)^2}(y)} = -\frac{\dfrac{z}{1 + y^2z^2}}{\dfrac{1 + y^2z^2 + y}{1 + y^2z^2}} = -\frac{z}{1 + y + y^2z^2}$$

35. Since x and y are each functions of t, $T(x, y)$ is a function of t, so by the Chain Rule, $\dfrac{dT}{dt} = \dfrac{\partial T}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial T}{\partial y}\dfrac{dy}{dt}$. After 3 seconds, $x = \sqrt{1 + t} = \sqrt{1 + 3} = 2$, $y = 2 + \frac{1}{3}t = 2 + \frac{1}{3}(3) = 3$, $\dfrac{dx}{dt} = \dfrac{1}{2\sqrt{1 + t}} = \dfrac{1}{2\sqrt{1 + 3}} = \dfrac{1}{4}$, and $\dfrac{dy}{dt} = \dfrac{1}{3}$.

Then $\dfrac{dT}{dt} = T_x(2, 3)\dfrac{dx}{dt} + T_y(2, 3)\dfrac{dy}{dt} = 4\left(\frac{1}{4}\right) + 3\left(\frac{1}{3}\right) = 2$. Thus the temperature is rising at a rate of 2°C/s.

37. $C = 1449.2 + 4.6T - 0.055T^2 + 0.00029T^3 + 0.016D$, so $\dfrac{\partial C}{\partial T} = 4.6 - 0.11T + 0.00087T^2$ and $\dfrac{\partial C}{\partial D} = 0.016$.

According to the graph, the diver is experiencing a temperature of approximately 12.5°C at $t = 20$ minutes, so $\dfrac{\partial C}{\partial T} = 4.6 - 0.11(12.5) + 0.00087(12.5)^2 \approx 3.36$. By sketching tangent lines at $t = 20$ to the graphs given, we estimate $\dfrac{dD}{dt} \approx \dfrac{1}{2}$ and $\dfrac{dT}{dt} \approx -\dfrac{1}{10}$. Then, by the Chain Rule, $\dfrac{dC}{dt} = \dfrac{\partial C}{\partial T}\dfrac{dT}{dt} + \dfrac{\partial C}{\partial D}\dfrac{dD}{dt} \approx (3.36)\left(-\frac{1}{10}\right) + (0.016)\left(\frac{1}{2}\right) \approx -0.33$.

Thus the speed of sound experienced by the diver is decreasing at a rate of approximately 0.33 m/s per minute.

39. (a) $V = \ell wh$, so by the Chain Rule,

$$\frac{dV}{dt} = \frac{\partial V}{\partial \ell}\frac{d\ell}{dt} + \frac{\partial V}{\partial w}\frac{dw}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt} = wh\frac{d\ell}{dt} + \ell h\frac{dw}{dt} + \ell w\frac{dh}{dt} = 2 \cdot 2 \cdot 2 + 1 \cdot 2 \cdot 2 + 1 \cdot 2 \cdot (-3) = 6 \text{ m}^3\text{/s}.$$

(b) $S = 2(\ell w + \ell h + wh)$, so by the Chain Rule,

$$\begin{aligned}\frac{dS}{dt} &= \frac{\partial S}{\partial \ell}\frac{d\ell}{dt} + \frac{\partial S}{\partial w}\frac{dw}{dt} + \frac{\partial S}{\partial h}\frac{dh}{dt} = 2(w + h)\frac{d\ell}{dt} + 2(\ell + h)\frac{dw}{dt} + 2(\ell + w)\frac{dh}{dt} \\ &= 2(2 + 2)2 + 2(1 + 2)2 + 2(1 + 2)(-3) = 10 \text{ m}^2\text{/s}\end{aligned}$$

(c) $L^2 = \ell^2 + w^2 + h^2 \quad \Rightarrow \quad 2L\dfrac{dL}{dt} = 2\ell\dfrac{d\ell}{dt} + 2w\dfrac{dw}{dt} + 2h\dfrac{dh}{dt} = 2(1)(2) + 2(2)(2) + 2(2)(-3) = 0 \quad \Rightarrow$

$dL/dt = 0$ m/s.

41. $\dfrac{dP}{dt} = 0.05$, $\dfrac{dT}{dt} = 0.15$, $V = 8.31\dfrac{T}{P}$ and $\dfrac{dV}{dt} = \dfrac{8.31}{P}\dfrac{dT}{dt} - 8.31\dfrac{T}{P^2}\dfrac{dP}{dt}$. Thus when $P = 20$ and $T = 320$,

$$\frac{dV}{dt} = 8.31\left[\frac{0.15}{20} - \frac{(0.05)(320)}{400}\right] \approx -0.27 \text{ L/s}.$$

43. Let x be the length of the first side of the triangle and y the length of the second side. The area A of the triangle is given by $A = \frac{1}{2}xy\sin\theta$ where θ is the angle between the two sides. Thus A is a function of x, y, and θ, and x, y, and θ are each in turn functions of time t. We are given that $\dfrac{dx}{dt} = 3$, $\dfrac{dy}{dt} = -2$, and because A is constant, $\dfrac{dA}{dt} = 0$. By the Chain Rule, $\dfrac{dA}{dt} = \dfrac{\partial A}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial A}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial A}{\partial \theta}\dfrac{d\theta}{dt} \Rightarrow \dfrac{dA}{dt} = \frac{1}{2}y\sin\theta \cdot \dfrac{dx}{dt} + \frac{1}{2}x\sin\theta \cdot \dfrac{dy}{dt} + \frac{1}{2}xy\cos\theta \cdot \dfrac{d\theta}{dt}$. When $x = 20$, $y = 30$, and $\theta = \pi/6$ we have

$$0 = \tfrac{1}{2}(30)\left(\sin\tfrac{\pi}{6}\right)(3) + \tfrac{1}{2}(20)\left(\sin\tfrac{\pi}{6}\right)(-2) + \tfrac{1}{2}(20)(30)\left(\cos\tfrac{\pi}{6}\right)\frac{d\theta}{dt}$$

$$= 45 \cdot \tfrac{1}{2} - 20 \cdot \tfrac{1}{2} + 300 \cdot \frac{\sqrt{3}}{2} \cdot \frac{d\theta}{dt} = \tfrac{25}{2} + 150\sqrt{3}\,\frac{d\theta}{dt}$$

Solving for $\dfrac{d\theta}{dt}$ gives $\dfrac{d\theta}{dt} = \dfrac{-25/2}{150\sqrt{3}} = -\dfrac{1}{12\sqrt{3}}$, so the angle between the sides is decreasing at a rate of $1/\left(12\sqrt{3}\right) \approx 0.048\ \text{rad/s}$.

45. (a) By the Chain Rule, $\dfrac{\partial z}{\partial r} = \dfrac{\partial z}{\partial x}\cos\theta + \dfrac{\partial z}{\partial y}\sin\theta$, $\dfrac{\partial z}{\partial \theta} = \dfrac{\partial z}{\partial x}(-r\sin\theta) + \dfrac{\partial z}{\partial y}r\cos\theta$.

(b) $\left(\dfrac{\partial z}{\partial r}\right)^2 = \left(\dfrac{\partial z}{\partial x}\right)^2\cos^2\theta + 2\dfrac{\partial z}{\partial x}\dfrac{\partial z}{\partial y}\cos\theta\,\sin\theta + \left(\dfrac{\partial z}{\partial y}\right)^2\sin^2\theta$,

$\left(\dfrac{\partial z}{\partial \theta}\right)^2 = \left(\dfrac{\partial z}{\partial x}\right)^2 r^2\sin^2\theta - 2\dfrac{\partial z}{\partial x}\dfrac{\partial z}{\partial y}r^2\cos\theta\,\sin\theta + \left(\dfrac{\partial z}{\partial y}\right)^2 r^2\cos^2\theta$. Thus

$\left(\dfrac{\partial z}{\partial r}\right)^2 + \dfrac{1}{r^2}\left(\dfrac{\partial z}{\partial \theta}\right)^2 = \left[\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2\right](\cos^2\theta + \sin^2\theta) = \left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2$.

47. Let $u = x - y$. Then $\dfrac{\partial z}{\partial x} = \dfrac{dz}{du}\dfrac{\partial u}{\partial x} = \dfrac{dz}{du}$ and $\dfrac{\partial z}{\partial y} = \dfrac{dz}{du}(-1)$. Thus $\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = 0$.

49. Let $u = x + at,\ v = x - at$. Then $z = f(u) + g(v)$, so $\partial z/\partial u = f'(u)$ and $\partial z/\partial v = g'(v)$.

Thus $\dfrac{\partial z}{\partial t} = \dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial t} + \dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial t} = af'(u) - ag'(v)$ and

$\dfrac{\partial^2 z}{\partial t^2} = a\dfrac{\partial}{\partial t}[f'(u) - g'(v)] = a\left(\dfrac{df'(u)}{du}\dfrac{\partial u}{\partial t} - \dfrac{dg'(v)}{dv}\dfrac{\partial v}{\partial t}\right) = a^2 f''(u) + a^2 g''(v)$.

Similarly $\dfrac{\partial z}{\partial x} = f'(u) + g'(v)$ and $\dfrac{\partial^2 z}{\partial x^2} = f''(u) + g''(v)$. Thus $\dfrac{\partial^2 z}{\partial t^2} = a^2\dfrac{\partial^2 z}{\partial x^2}$.

51. $\dfrac{\partial z}{\partial s} = \dfrac{\partial z}{\partial x}2s + \dfrac{\partial z}{\partial y}2r$. Then

$$\frac{\partial^2 z}{\partial r\,\partial s} = \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}2s\right) + \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}2r\right)$$

$$= \frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial r}2s + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\frac{\partial y}{\partial r}2s + \frac{\partial z}{\partial x}\frac{\partial}{\partial r}2s + \frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial r}2r + \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)\frac{\partial x}{\partial r}2r + \frac{\partial z}{\partial y}2$$

$$= 4rs\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y\,\partial x}4s^2 + 0 + 4rs\frac{\partial^2 z}{\partial y^2} + \frac{\partial^2 z}{\partial x\,\partial y}4r^2 + 2\frac{\partial z}{\partial y}$$

By the continuity of the partials, $\dfrac{\partial^2 z}{\partial r\partial s} = 4rs\dfrac{\partial^2 z}{\partial x^2} + 4rs\dfrac{\partial^2 z}{\partial y^2} + (4r^2 + 4s^2)\dfrac{\partial^2 z}{\partial x\,\partial y} + 2\dfrac{\partial z}{\partial y}$.

53. $\dfrac{\partial z}{\partial r} = \dfrac{\partial z}{\partial x}\cos\theta + \dfrac{\partial z}{\partial y}\sin\theta$ and $\dfrac{\partial z}{\partial \theta} = -\dfrac{\partial z}{\partial x} r\sin\theta + \dfrac{\partial z}{\partial y} r\cos\theta$. Then

$$\begin{aligned}\frac{\partial^2 z}{\partial r^2} &= \cos\theta\left(\frac{\partial^2 z}{\partial x^2}\cos\theta + \frac{\partial^2 z}{\partial y\,\partial x}\sin\theta\right) + \sin\theta\left(\frac{\partial^2 z}{\partial y^2}\sin\theta + \frac{\partial^2 z}{\partial x\,\partial y}\cos\theta\right)\\ &= \cos^2\theta\,\frac{\partial^2 z}{\partial x^2} + 2\cos\theta\,\sin\theta\,\frac{\partial^2 z}{\partial x\,\partial y} + \sin^2\theta\,\frac{\partial^2 z}{\partial y^2}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2 z}{\partial \theta^2} &= -r\cos\theta\,\frac{\partial z}{\partial x} + (-r\sin\theta)\left(\frac{\partial^2 z}{\partial x^2}(-r\sin\theta) + \frac{\partial^2 z}{\partial y\,\partial x} r\cos\theta\right)\\ &\qquad -r\sin\theta\,\frac{\partial z}{\partial y} + r\cos\theta\left(\frac{\partial^2 z}{\partial y^2} r\cos\theta + \frac{\partial^2 z}{\partial x\,\partial y}(-r\sin\theta)\right)\\ &= -r\cos\theta\,\frac{\partial z}{\partial x} - r\sin\theta\,\frac{\partial z}{\partial y} + r^2\sin^2\theta\,\frac{\partial^2 z}{\partial x^2} - 2r^2\cos\theta\,\sin\theta\,\frac{\partial^2 z}{\partial x\,\partial y} + r^2\cos^2\theta\,\frac{\partial^2 z}{\partial y^2}\end{aligned}$$

Thus

$$\begin{aligned}\frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} + \frac{1}{r}\frac{\partial z}{\partial r} &= (\cos^2\theta + \sin^2\theta)\,\frac{\partial^2 z}{\partial x^2} + \left(\sin^2\theta + \cos^2\theta\right)\frac{\partial^2 z}{\partial y^2}\\ &\qquad -\frac{1}{r}\cos\theta\,\frac{\partial z}{\partial x} - \frac{1}{r}\sin\theta\,\frac{\partial z}{\partial y} + \frac{1}{r}\left(\cos\theta\,\frac{\partial z}{\partial x} + \sin\theta\,\frac{\partial z}{\partial y}\right)\\ &= \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \text{ as desired.}\end{aligned}$$

55. (a) Since f is a polynomial, it has continuous second-order partial derivatives, and

$f(tx, ty) = (tx)^2(ty) + 2(tx)(ty)^2 + 5(ty)^3 = t^3x^2y + 2t^3xy^2 + 5t^3y^3 = t^3(x^2y + 2xy^2 + 5y^3) = t^3 f(x, y)$.

Thus, f is homogeneous of degree 3.

(b) Differentiating both sides of $f(tx, ty) = t^n f(x, y)$ with respect to t using the Chain Rule, we get

$$\frac{\partial}{\partial t} f(tx, ty) = \frac{\partial}{\partial t}\left[t^n f(x, y)\right] \quad\Leftrightarrow$$

$$\frac{\partial}{\partial (tx)} f(tx, ty)\cdot\frac{\partial (tx)}{\partial t} + \frac{\partial}{\partial (ty)} f(tx, ty)\cdot\frac{\partial (ty)}{\partial t} = x\frac{\partial}{\partial (tx)} f(tx, ty) + y\frac{\partial}{\partial (ty)} f(tx, ty) = nt^{n-1} f(x, y).$$

Setting $t = 1$: $x\dfrac{\partial}{\partial x} f(x, y) + y\dfrac{\partial}{\partial y} f(x, y) = nf(x, y)$.

57. Differentiating both sides of $f(tx, ty) = t^n f(x, y)$ with respect to x using the Chain Rule, we get

$$\frac{\partial}{\partial x} f(tx, ty) = \frac{\partial}{\partial x}\left[t^n f(x, y)\right] \quad\Leftrightarrow$$

$$\frac{\partial}{\partial\,(tx)} f(tx, ty)\cdot\frac{\partial\,(tx)}{\partial x} + \frac{\partial}{\partial\,(ty)} f(tx, ty)\cdot\frac{\partial\,(ty)}{\partial x} = t^n\frac{\partial}{\partial x} f(x, y) \quad\Leftrightarrow\quad tf_x(tx, ty) = t^n f_x(x, y).$$

Thus $f_x(tx, ty) = t^{n-1} f_x(x, y)$.

15.6 Directional Derivatives and the Gradient Vector **ET 14.6**

1. We can approximate the directional derivative of the pressure function at K in the direction of S by the average rate of change of pressure between the points where the red line intersects the contour lines closest to K (extend the red line slightly at the left). In the direction of S, the pressure changes from 1000 millibars to 996 millibars and we estimate the distance between these two points to be approximately 50 km (using the fact that the distance from K to S is 300 km). Then the rate of change of pressure in the direction given is approximately $\frac{996 - 1000}{50} = -0.08$ millibar/km.

3. $D_{\mathbf{u}}\, f(-20, 30) = \nabla f\,(-20, 30) \cdot \mathbf{u} = f_T(-20, 30)\left(\frac{1}{\sqrt{2}}\right) + f_v(-20, 30)\left(\frac{1}{\sqrt{2}}\right)$.

$f_T(-20, 30) = \lim\limits_{h \to 0} \dfrac{f(-20 + h, 30) - f(-20, 30)}{h}$, so we can approximate $f_T(-20, 30)$ by considering $h = \pm 5$ and using the values given in the table: $f_T(-20, 30) \approx \dfrac{f(-15, 30) - f(-20, 30)}{5} = \dfrac{-26 - (-33)}{5} = 1.4$,

$f_T(-20, 30) \approx \dfrac{f(-25, 30) - f(-20, 30)}{-5} = \dfrac{-39 - (-33)}{-5} = 1.2$. Averaging these values gives $f_T(-20, 30) \approx 1.3$.

Similarly, $f_v(-20, 30) = \lim\limits_{h \to 0} \dfrac{f(-20, 30 + h) - f(-20, 30)}{h}$, so we can approximate $f_v(-20, 30)$ with $h = \pm 10$:

$f_v(-20, 30) \approx \dfrac{f(-20, 40) - f(-20, 30)}{10} = \dfrac{-34 - (-33)}{10} = -0.1$,

$f_v(-20, 30) \approx \dfrac{f(-20, 20) - f(-20, 30)}{-10} = \dfrac{-30 - (-33)}{-10} = -0.3$. Averaging these values gives $f_v(-20, 30) \approx -0.2$.

Then $D_{\mathbf{u}} f(-20, 30) \approx 1.3\left(\frac{1}{\sqrt{2}}\right) + (-0.2)\left(\frac{1}{\sqrt{2}}\right) \approx 0.778$.

5. $f(x, y) = ye^{-x} \quad \Rightarrow \quad f_x(x, y) = -ye^{-x}$ and $f_y(x, y) = e^{-x}$. If $\mathbf{u}$ is a unit vector in the direction of $\theta = 2\pi/3$, then from Equation 6, $D_{\mathbf{u}}\, f(0, 4) = f_x(0, 4)\cos\left(\frac{2\pi}{3}\right) + f_y(0, 4)\sin\left(\frac{2\pi}{3}\right) = -4 \cdot \left(-\frac{1}{2}\right) + 1 \cdot \frac{\sqrt{3}}{2} = 2 + \frac{\sqrt{3}}{2}$.

7. $f(x, y) = \sin(2x + 3y)$

(a) $\nabla f(x, y) = \dfrac{\partial f}{\partial x}\,\mathbf{i} + \dfrac{\partial f}{\partial y}\,\mathbf{j} = [\cos(2x + 3y) \cdot 2]\,\mathbf{i} + [\cos(2x + 3y) \cdot 3]\,\mathbf{j} = 2\cos(2x + 3y)\,\mathbf{i} + 3\cos(2x + 3y)\,\mathbf{j}$

(b) $\nabla f(-6, 4) = (2\cos 0)\mathbf{i} + (3\cos 0)\mathbf{j} = 2\,\mathbf{i} + 3\,\mathbf{j}$

(c) By Equation 9, $D_{\mathbf{u}}\, f(-6, 4) = \nabla f(-6, 4) \cdot \mathbf{u} = (2\,\mathbf{i} + 3\,\mathbf{j}) \cdot \frac{1}{2}\left(\sqrt{3}\,\mathbf{i} - \mathbf{j}\right) = \frac{1}{2}\left(2\sqrt{3} - 3\right) = \sqrt{3} - \frac{3}{2}$.

9. $f(x, y, z) = xe^{2yz}$

(a) $\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)\rangle = \left\langle e^{2yz}, 2xze^{2yz}, 2xye^{2yz}\right\rangle$

(b) $\nabla f(3, 0, 2) = \langle 1, 12, 0\rangle$

(c) By Equation 14, $D_{\mathbf{u}} f(3, 0, 2) = \nabla f(3, 0, 2) \cdot \mathbf{u} = \langle 1, 12, 0\rangle \cdot \left\langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right\rangle = \frac{2}{3} - \frac{24}{3} + 0 = -\frac{22}{3}$.

11. $f(x, y) = 1 + 2x\sqrt{y} \quad \Rightarrow \quad \nabla f(x, y) = \left\langle 2\sqrt{y}, 2x \cdot \frac{1}{2}y^{-1/2}\right\rangle = \left\langle 2\sqrt{y}, x/\sqrt{y}\right\rangle$, $\nabla f(3, 4) = \left\langle 4, \frac{3}{2}\right\rangle$, and a unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \dfrac{1}{\sqrt{4^2 + (-3)^2}}\langle 4, -3\rangle = \left\langle \frac{4}{5}, -\frac{3}{5}\right\rangle$, so $D_{\mathbf{u}}\, f(3, 4) = \nabla f(3, 4) \cdot \mathbf{u} = \left\langle 4, \frac{3}{2}\right\rangle \cdot \left\langle \frac{4}{5}, -\frac{3}{5}\right\rangle = \frac{23}{10}$.

13. $g(p, q) = p^4 - p^2q^3 \quad \Rightarrow \quad \nabla g(p, q) = \left(4p^3 - 2pq^3\right)\mathbf{i} + \left(-3p^2q^2\right)\mathbf{j}$, $\nabla g(2, 1) = 28\,\mathbf{i} - 12\,\mathbf{j}$, and a unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{1}{\sqrt{1^2 + 3^2}}(\mathbf{i} + 3\,\mathbf{j}) = \frac{1}{\sqrt{10}}(\mathbf{i} + 3\,\mathbf{j})$, so

$D_{\mathbf{u}}\, g(2, 1) = \nabla g(2, 1) \cdot \mathbf{u} = (28\,\mathbf{i} - 12\,\mathbf{j}) \cdot \frac{1}{\sqrt{10}}(\mathbf{i} + 3\,\mathbf{j}) = \frac{1}{\sqrt{10}}\,(28 - 36) = -\frac{8}{\sqrt{10}}$ or $-\frac{4\sqrt{10}}{5}$.

15. $f(x, y, z) = xe^y + ye^z + ze^x \quad \Rightarrow \quad \nabla f(x, y, z) = \langle e^y + ze^x, xe^y + e^z, ye^z + e^x\rangle$, $\nabla f(0, 0, 0) = \langle 1, 1, 1\rangle$, and a unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{1}{\sqrt{25+1+4}}\langle 5, 1, -2\rangle = \frac{1}{\sqrt{30}}\langle 5, 1, -2\rangle$, so

$D_{\mathbf{u}}\, f(0, 0, 0) = \nabla f(0, 0, 0) \cdot \mathbf{u} = \langle 1, 1, 1\rangle \cdot \frac{1}{\sqrt{30}}\langle 5, 1, -2\rangle = \frac{4}{\sqrt{30}}$.

17. $g(x,y,z) = (x+2y+3z)^{3/2} \quad \Rightarrow$

$$\nabla g(x,y,z) = \left\langle \tfrac{3}{2}(x+2y+3z)^{1/2}(1), \tfrac{3}{2}(x+2y+3z)^{1/2}(2), \tfrac{3}{2}(x+2y+3z)^{1/2}(3) \right\rangle$$

$$= \left\langle \tfrac{3}{2}\sqrt{x+2y+3z}, 3\sqrt{x+2y+3z}, \tfrac{9}{2}\sqrt{x+2y+3z} \right\rangle, \nabla g(1,1,2) = \left\langle \tfrac{9}{2}, 9, \tfrac{27}{2} \right\rangle,$$

and a unit vector in the direction of $\mathbf{v} = 2\mathbf{j} - \mathbf{k}$ is $\mathbf{u} = \frac{2}{\sqrt{5}}\mathbf{j} - \frac{1}{\sqrt{5}}\mathbf{k}$, so

$$D_{\mathbf{u}}\, g(1,1,2) = \left\langle \tfrac{9}{2}, 9, \tfrac{27}{2} \right\rangle \cdot \left\langle 0, \tfrac{2}{\sqrt{5}}, -\tfrac{1}{\sqrt{5}} \right\rangle = \tfrac{18}{\sqrt{5}} - \tfrac{27}{2\sqrt{5}} = \tfrac{9}{2\sqrt{5}}.$$

19. $f(x,y) = \sqrt{xy} \quad \Rightarrow \quad \nabla f(x,y) = \left\langle \frac{1}{2}(xy)^{-1/2}(y), \frac{1}{2}(xy)^{-1/2}(x) \right\rangle = \left\langle \dfrac{y}{2\sqrt{xy}}, \dfrac{x}{2\sqrt{xy}} \right\rangle$, so $\nabla f(2,8) = \left\langle 1, \frac{1}{4} \right\rangle$.

The unit vector in the direction of $\overrightarrow{PQ} = \langle 5-2, 4-8 \rangle = \langle 3, -4 \rangle$ is $\mathbf{u} = \left\langle \frac{3}{5}, -\frac{4}{5} \right\rangle$, so

$D_{\mathbf{u}}\, f(2,8) = \nabla f(2,8) \cdot \mathbf{u} = \left\langle 1, \frac{1}{4} \right\rangle \cdot \left\langle \frac{3}{5}, -\frac{4}{5} \right\rangle = \frac{2}{5}$.

21. $f(x,y) = y^2/x = y^2 x^{-1} \quad \Rightarrow \quad \nabla f(x,y) = \langle -y^2 x^{-2}, 2yx^{-1} \rangle = \langle -y^2/x^2, 2y/x \rangle$.

$\nabla f(2,4) = \langle -4, 4 \rangle$, or equivalently $\langle -1, 1 \rangle$, is the direction of maximum rate of change, and the maximum rate is $|\nabla f(2,4)| = \sqrt{16+16} = 4\sqrt{2}$.

23. $f(x,y) = \sin(xy) \quad \Rightarrow \quad \nabla f(x,y) = \langle y\cos(xy), x\cos(xy) \rangle$, $\nabla f(1,0) = \langle 0, 1 \rangle$. Thus the maximum rate of change is $|\nabla f(1,0)| = 1$ in the direction $\langle 0, 1 \rangle$.

25. $f(x,y,z) = \sqrt{x^2+y^2+z^2} \quad \Rightarrow$

$$\nabla f(x,y,z) = \left\langle \tfrac{1}{2}(x^2+y^2+z^2)^{-1/2} \cdot 2x, \tfrac{1}{2}(x^2+y^2+z^2)^{-1/2} \cdot 2y, \tfrac{1}{2}(x^2+y^2+z^2)^{-1/2} \cdot 2z \right\rangle$$

$$= \left\langle \frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}} \right\rangle,$$

$\nabla f(3,6,-2) = \left\langle \frac{3}{\sqrt{49}}, \frac{6}{\sqrt{49}}, \frac{-2}{\sqrt{49}} \right\rangle = \left\langle \frac{3}{7}, \frac{6}{7}, -\frac{2}{7} \right\rangle$. Thus the maximum rate of change is

$|\nabla f(3,6,-2)| = \sqrt{\left(\frac{3}{7}\right)^2 + \left(\frac{6}{7}\right)^2 + \left(-\frac{2}{7}\right)^2} = \sqrt{\frac{9+36+4}{49}} = 1$ in the direction $\left\langle \frac{3}{7}, \frac{6}{7}, -\frac{2}{7} \right\rangle$ or equivalently $\langle 3, 6, -2 \rangle$.

27. (a) As in the proof of Theorem 15, $D_{\mathbf{u}}\, f = |\nabla f| \cos\theta$. Since the minimum value of $\cos\theta$ is -1 occurring when $\theta = \pi$, the minimum value of $D_{\mathbf{u}}\, f$ is $-|\nabla f|$ occurring when $\theta = \pi$, that is when $\mathbf{u}$ is in the opposite direction of ∇f (assuming $\nabla f \neq \mathbf{0}$).

(b) $f(x,y) = x^4 y - x^2 y^3 \quad \Rightarrow \quad \nabla f(x,y) = \langle 4x^3 y - 2xy^3, x^4 - 3x^2 y^2 \rangle$, so f decreases fastest at the point $(2,-3)$ in the direction $-\nabla f(2,-3) = -\langle 12, -92 \rangle = \langle -12, 92 \rangle$.

29. The direction of fastest change is $\nabla f(x,y) = (2x-2)\,\mathbf{i} + (2y-4)\,\mathbf{j}$, so we need to find all points (x,y) where $\nabla f(x,y)$ is parallel to $\mathbf{i} + \mathbf{j} \quad \Leftrightarrow \quad (2x-2)\,\mathbf{i} + (2y-4)\,\mathbf{j} = k\,(\mathbf{i} + \mathbf{j}) \quad \Leftrightarrow \quad k = 2x-2$ and $k = 2y-4$. Then $2x-2 = 2y-4 \quad \Rightarrow$ $y = x+1$, so the direction of fastest change is $\mathbf{i} + \mathbf{j}$ at all points on the line $y = x+1$.

31. $T = \dfrac{k}{\sqrt{x^2 + y^2 + z^2}}$ and $120 = T(1, 2, 2) = \dfrac{k}{3}$ so $k = 360$.

(a) $\mathbf{u} = \dfrac{\langle 1, -1, 1\rangle}{\sqrt{3}}$,

$D_{\mathbf{u}}T(1, 2, 2) = \nabla T(1, 2, 2) \cdot \mathbf{u} = \left[-360\left(x^2 + y^2 + z^2\right)^{-3/2} \langle x, y, z\rangle\right]_{(1,2,2)} \cdot \mathbf{u} = -\frac{40}{3} \langle 1, 2, 2\rangle \cdot \frac{1}{\sqrt{3}} \langle 1, -1, 1\rangle = -\frac{40}{3\sqrt{3}}$

(b) From (a), $\nabla T = -360\left(x^2 + y^2 + z^2\right)^{-3/2} \langle x, y, z\rangle$, and since $\langle x, y, z\rangle$ is the position vector of the point (x, y, z), the vector $-\langle x, y, z\rangle$, and thus ∇T, always points toward the origin.

33. $\nabla V(x, y, z) = \langle 10x - 3y + yz, xz - 3x, xy\rangle$, $\nabla V(3, 4, 5) = \langle 38, 6, 12\rangle$

(a) $D_{\mathbf{u}}\, V(3, 4, 5) = \langle 38, 6, 12\rangle \cdot \frac{1}{\sqrt{3}} \langle 1, 1, -1\rangle = \frac{32}{\sqrt{3}}$

(b) $\nabla V(3, 4, 5) = \langle 38, 6, 12\rangle$, or equivalently, $\langle 19, 3, 6\rangle$.

(c) $|\nabla V(3, 4, 5)| = \sqrt{38^2 + 6^2 + 12^2} = \sqrt{1624} = 2\sqrt{406}$

35. A unit vector in the direction of $\overrightarrow{AB}$ is $\mathbf{i}$ and a unit vector in the direction of $\overrightarrow{AC}$ is $\mathbf{j}$. Thus $D_{\overrightarrow{AB}}\, f(1, 3) = f_x(1, 3) = 3$ and $D_{\overrightarrow{AC}}\, f(1, 3) = f_y(1, 3) = 26$. Therefore $\nabla f(1, 3) = \langle f_x(1, 3), f_y(1, 3)\rangle = \langle 3, 26\rangle$, and by definition, $D_{\overrightarrow{AD}}\, f(1, 3) = \nabla f \cdot \mathbf{u}$ where $\mathbf{u}$ is a unit vector in the direction of $\overrightarrow{AD}$, which is $\left\langle \frac{5}{13}, \frac{12}{13}\right\rangle$. Therefore,

$D_{\overrightarrow{AD}}\, f\,(1, 3) = \langle 3, 26\rangle \cdot \left\langle \frac{5}{13}, \frac{12}{13}\right\rangle = 3 \cdot \frac{5}{13} + 26 \cdot \frac{12}{13} = \frac{327}{13}$.

37. (a) $\nabla(au + bv) = \left\langle \dfrac{\partial(au + bv)}{\partial x}, \dfrac{\partial(au + bv)}{\partial y}\right\rangle = \left\langle a\,\dfrac{\partial u}{\partial x} + b\,\dfrac{\partial v}{\partial x}, a\,\dfrac{\partial u}{\partial y} + b\,\dfrac{\partial v}{\partial y}\right\rangle = a\left\langle \dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}\right\rangle + b\left\langle \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}\right\rangle$

$= a\,\nabla u + b\,\nabla v$

(b) $\nabla(uv) = \left\langle v\,\dfrac{\partial u}{\partial x} + u\,\dfrac{\partial v}{\partial x}, v\,\dfrac{\partial u}{\partial y} + u\,\dfrac{\partial v}{\partial y}\right\rangle = v\left\langle \dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}\right\rangle + u\left\langle \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}\right\rangle = v\,\nabla u + u\,\nabla v$

(c) $\nabla\left(\dfrac{u}{v}\right) = \left\langle \dfrac{v\,\dfrac{\partial u}{\partial x} - u\,\dfrac{\partial v}{\partial x}}{v^2}, \dfrac{v\,\dfrac{\partial u}{\partial y} - u\,\dfrac{\partial v}{\partial y}}{v^2}\right\rangle = \dfrac{v\left\langle \dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}\right\rangle - u\left\langle \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}\right\rangle}{v^2} = \dfrac{v\,\nabla u - u\,\nabla v}{v^2}$

(d) $\nabla u^n = \left\langle \dfrac{\partial(u^n)}{\partial x}, \dfrac{\partial(u^n)}{\partial y}\right\rangle = \left\langle nu^{n-1}\,\dfrac{\partial u}{\partial x}, nu^{n-1}\,\dfrac{\partial u}{\partial y}\right\rangle = nu^{n-1}\,\nabla u$

39. Let $F(x, y, z) = 2(x - 2)^2 + (y - 1)^2 + (z - 3)^2$. Then $2(x - 2)^2 + (y - 1)^2 + (z - 3)^2 = 10$ is a level surface of F.

$F_x(x, y, z) = 4(x - 2) \quad\Rightarrow\quad F_x(3, 3, 5) = 4$, $F_y(x, y, z) = 2(y - 1) \quad\Rightarrow\quad F_y(3, 3, 5) = 4$, and

$F_z(x, y, z) = 2(z - 3) \quad\Rightarrow\quad F_z(3, 3, 5) = 4$.

(a) Equation 19 gives an equation of the tangent plane at $(3, 3, 5)$ as $4(x - 3) + 4(y - 3) + 4(z - 5) = 0 \quad\Leftrightarrow$ $4x + 4y + 4z = 44$ or equivalently $x + y + z = 11$.

(b) By Equation 20, the normal line has symmetric equations $\dfrac{x - 3}{4} = \dfrac{y - 3}{4} = \dfrac{z - 5}{4}$ or equivalently $x - 3 = y - 3 = z - 5$. Corresponding parametric equations are $x = 3 + t$, $y = 3 + t$, $z = 5 + t$.

41. Let $F(x, y, z) = x^2 - 2y^2 + z^2 + yz$. Then $x^2 - 2y^2 + z^2 + yz = 2$ is a level surface of F and $\nabla F(x, y, z) = \langle 2x, -4y + z, 2z + y \rangle$.

(a) $\nabla F(2, 1, -1) = \langle 4, -5, -1 \rangle$ is a normal vector for the tangent plane at $(2, 1, -1)$, so an equation of the tangent plane is $4(x - 2) - 5(y - 1) - 1(z + 1) = 0$ or $4x - 5y - z = 4$.

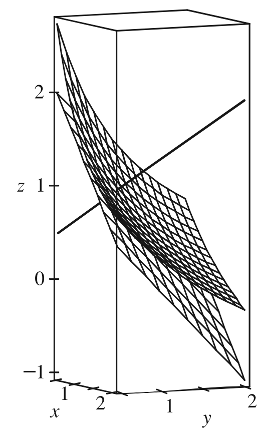
(b) The normal line has direction $\langle 4, -5, -1 \rangle$, so parametric equations are $x = 2 + 4t,\ y = 1 - 5t,\ z = -1 - t$, and symmetric equations are $\dfrac{x - 2}{4} = \dfrac{y - 1}{-5} = \dfrac{z + 1}{-1}$.

43. $F(x, y, z) = -z + xe^y \cos z \quad \Rightarrow \quad \nabla F(x, y, z) = \langle e^y \cos z, xe^y \cos z, -1 - xe^y \sin z \rangle$ and $\nabla F(1, 0, 0) = \langle 1, 1, -1 \rangle$.

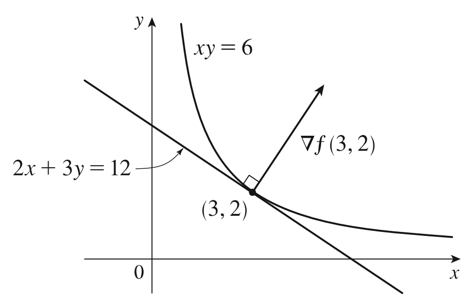
(a) $1(x - 1) + 1(y - 0) - 1(z - 0) = 0$ or $x + y - z = 1$

(b) $x - 1 = y = -z$

45. $F(x, y, z) = xy + yz + zx$, $\nabla F(x, y, z) = \langle y + z, x + z, y + x \rangle$, $\nabla F(1, 1, 1) = \langle 2, 2, 2 \rangle$, so an equation of the tangent plane is $2x + 2y + 2z = 6$ or $x + y + z = 3$, and the normal line is given by $x - 1 = y - 1 = z - 1$ or $x = y = z$. To graph the surface we solve for z: $z = \dfrac{3 - xy}{x + y}$.



47. $f(x, y) = xy \quad \Rightarrow \quad \nabla f(x, y) = \langle y, x \rangle$, $\nabla f(3, 2) = \langle 2, 3 \rangle$. $\nabla f(3, 2)$ is perpendicular to the tangent line, so the tangent line has equation $\nabla f(3, 2) \cdot \langle x - 3, y - 2 \rangle = 0 \quad \Rightarrow \quad \langle 2, 3 \rangle \cdot \langle x - 3, x - 2 \rangle = 0 \quad \Rightarrow$ $2(x - 3) + 3(y - 2) = 0$ or $2x + 3y = 12$.



49. $\nabla F(x_0, y_0, z_0) = \left\langle \dfrac{2x_0}{a^2}, \dfrac{2y_0}{b^2}, \dfrac{2z_0}{c^2} \right\rangle$. Thus an equation of the tangent plane at (x_0, y_0, z_0) is

$\dfrac{2x_0}{a^2}\,x + \dfrac{2y_0}{b^2}\,y + \dfrac{2z_0}{c^2}\,z = 2\left(\dfrac{x_0^2}{a^2} + \dfrac{y_0^2}{b^2} + \dfrac{z_0^2}{c^2}\right) = 2(1) = 2$ since (x_0, y_0, z_0) is a point on the ellipsoid. Hence

$\dfrac{x_0}{a^2}\,x + \dfrac{y_0}{b^2}\,y + \dfrac{z_0}{c^2}\,z = 1$ is an equation of the tangent plane.

51. $\nabla F(x_0, y_0, z_0) = \left\langle \dfrac{2x_0}{a^2}, \dfrac{2y_0}{b^2}, \dfrac{-1}{c} \right\rangle$, so an equation of the tangent plane is $\dfrac{2x_0}{a^2}\,x + \dfrac{2y_0}{b^2}\,y - \dfrac{1}{c}\,z = \dfrac{2x_0^2}{a^2} + \dfrac{2y_0^2}{b^2} - \dfrac{z_0}{c}$

or $\dfrac{2x_0}{a^2}\,x + \dfrac{2y_0}{b^2}\,y = \dfrac{z}{c} + 2\left(\dfrac{x_0^2}{a^2} + \dfrac{y_0^2}{b^2}\right) - \dfrac{z_0}{c}$. But $\dfrac{z_0}{c} = \dfrac{x_0^2}{a^2} + \dfrac{y_0^2}{b^2}$, so the equation can be written as

$\dfrac{2x_0}{a^2}\,x + \dfrac{2y_0}{b^2}\,y = \dfrac{z + z_0}{c}$.

53. The hyperboloid $x^2 - y^2 - z^2 = 1$ is a level surface of $F(x,y,z) = x^2 - y^2 - z^2$ and $\nabla F\,(x,y,z) = \langle 2x, -2y, -2z \rangle$ is a normal vector to the surface and hence a normal vector for the tangent plane at (x,y,z). The tangent plane is parallel to the plane $z = x + y$ or $x + y - z = 0$ if and only if the corresponding normal vectors are parallel, so we need a point (x_0, y_0, z_0) on the hyperboloid where $\langle 2x_0, -2y_0, -2z_0 \rangle = c\,\langle 1, 1, -1 \rangle$ or equivalently $\langle x_0, -y_0, -z_0 \rangle = k\,\langle 1, 1, -1 \rangle$ for some $k \neq 0$. Then we must have $x_0 = k$, $y_0 = -k$, $z_0 = k$ and substituting into the equation of the hyperboloid gives $k^2 - (-k)^2 - k^2 = 1 \quad\Leftrightarrow\quad -k^2 = 1$, an impossibility. Thus there is no such point on the hyperboloid.

55. Let (x_0, y_0, z_0) be a point on the cone [other than $(0,0,0)$]. Then an equation of the tangent plane to the cone at this point is $2x_0x + 2y_0y - 2z_0z = 2\left(x_0^2 + y_0^2 - z_0^2\right)$. But $x_0^2 + y_0^2 = z_0^2$ so the tangent plane is given by $x_0x + y_0y - z_0z = 0$, a plane which always contains the origin.

57. Let (x_0, y_0, z_0) be a point on the surface. Then an equation of the tangent plane at the point is

$\dfrac{x}{2\sqrt{x_0}} + \dfrac{y}{2\sqrt{y_0}} + \dfrac{z}{2\sqrt{z_0}} = \dfrac{\sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0}}{2}$. But $\sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0} = \sqrt{c}$, so the equation is

$\dfrac{x}{\sqrt{x_0}} + \dfrac{y}{\sqrt{y_0}} + \dfrac{z}{\sqrt{z_0}} = \sqrt{c}$. The x-, y-, and z-intercepts are $\sqrt{cx_0}$, $\sqrt{cy_0}$ and $\sqrt{cz_0}$ respectively. (The x-intercept is found by setting $y = z = 0$ and solving the resulting equation for x, and the y- and z-intercepts are found similarly.) So the sum of the intercepts is $\sqrt{c}\left(\sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0}\right) = c$, a constant.

59. If $f(x,y,z) = z - x^2 - y^2$ and $g(x,y,z) = 4x^2 + y^2 + z^2$, then the tangent line is perpendicular to both ∇f and ∇g at $(-1, 1, 2)$. The vector $\mathbf{v} = \nabla f \times \nabla g$ will therefore be parallel to the tangent line.

We have $\nabla f(x,y,z) = \langle -2x, -2y, 1 \rangle \quad\Rightarrow\quad \nabla f(-1,1,2) = \langle 2, -2, 1 \rangle$, and $\nabla g(x,y,z) = \langle 8x, 2y, 2z \rangle \quad\Rightarrow$

$\nabla g(-1,1,2) = \langle -8, 2, 4 \rangle$. Hence $\mathbf{v} = \nabla f \times \nabla g = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -2 & 1 \\ -8 & 2 & 4 \end{vmatrix} = -10\,\mathbf{i} - 16\,\mathbf{j} - 12\,\mathbf{k}$.

Parametric equations are: $x = -1 - 10t,\;\; y = 1 - 16t,\;\; z = 2 - 12t$.

61. (a) The direction of the normal line of F is given by ∇F, and that of G by ∇G. Assuming that $\nabla F \neq 0 \neq \nabla G$, the two normal lines are perpendicular at P if $\nabla F \cdot \nabla G = 0$ at $P \quad\Leftrightarrow$ $\langle \partial F/\partial x, \partial F/\partial y, \partial F/\partial z \rangle \cdot \langle \partial G/\partial x, \partial G/\partial y, \partial G/\partial z \rangle = 0$ at $P \quad\Leftrightarrow\quad F_xG_x + F_yG_y + F_zG_z = 0$ at P.

(b) Here $F = x^2 + y^2 - z^2$ and $G = x^2 + y^2 + z^2 - r^2$, so

$\nabla F \cdot \nabla G = \langle 2x, 2y, -2z \rangle \cdot \langle 2x, 2y, 2z \rangle = 4x^2 + 4y^2 - 4z^2 = 4F = 0$, since the point (x,y,z) lies on the graph of

$F = 0$. To see that this is true without using calculus, note that $G = 0$ is the equation of a sphere centered at the origin and $F = 0$ is the equation of a right circular cone with vertex at the origin (which is generated by lines through the origin). At any point of intersection, the sphere's normal line (which passes through the origin) lies on the cone, and thus is perpendicular to the cone's normal line. So the surfaces with equations $F = 0$ and $G = 0$ are everywhere orthogonal.

63. Let $\mathbf{u} = \langle a, b \rangle$ and $\mathbf{v} = \langle c, d \rangle$. Then we know that at the given point, $D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = af_x + bf_y$ and $D_{\mathbf{v}} f = \nabla f \cdot \mathbf{v} = cf_x + df_y$. But these are just two linear equations in the two unknowns f_x and f_y, and since $\mathbf{u}$ and $\mathbf{v}$ are not parallel, we can solve the equations to find $\nabla f = \langle f_x, f_y \rangle$ at the given point. In fact,

$$\nabla f = \left\langle \frac{d\, D_{\mathbf{u}} f - b\, D_{\mathbf{v}} f}{ad - bc}, \frac{a\, D_{\mathbf{v}} f - c\, D_{\mathbf{u}} f}{ad - bc} \right\rangle.$$

15.7 Maximum and Minimum Values ET 14.7

1. (a) First we compute $D(1,1) = f_{xx}(1,1)\, f_{yy}(1,1) - [f_{xy}(1,1)]^2 = (4)(2) - (1)^2 = 7$. Since $D(1,1) > 0$ and $f_{xx}(1,1) > 0$, f has a local minimum at $(1,1)$ by the Second Derivatives Test.

(b) $D(1,1) = f_{xx}(1,1)\, f_{yy}(1,1) - [f_{xy}(1,1)]^2 = (4)(2) - (3)^2 = -1$. Since $D(1,1) < 0$, f has a saddle point at $(1,1)$ by the Second Derivatives Test.

3. In the figure, a point at approximately $(1,1)$ is enclosed by level curves which are oval in shape and indicate that as we move away from the point in any direction the values of f are increasing. Hence we would expect a local minimum at or near $(1,1)$. The level curves near $(0,0)$ resemble hyperbolas, and as we move away from the origin, the values of f increase in some directions and decrease in others, so we would expect to find a saddle point there.

To verify our predictions, we have $f(x,y) = 4 + x^3 + y^3 - 3xy \Rightarrow f_x(x,y) = 3x^2 - 3y$, $f_y(x,y) = 3y^2 - 3x$. We have critical points where these partial derivatives are equal to 0: $3x^2 - 3y = 0$, $3y^2 - 3x = 0$. Substituting $y = x^2$ from the first equation into the second equation gives $3(x^2)^2 - 3x = 0 \Rightarrow 3x(x^3 - 1) = 0 \Rightarrow x = 0$ or $x = 1$. Then we have two critical points, $(0,0)$ and $(1,1)$. The second partial derivatives are $f_{xx}(x,y) = 6x$, $f_{xy}(x,y) = -3$, and $f_{yy}(x,y) = 6y$, so $D(x,y) = f_{xx}(x,y)\, f_{yy}(x,y) - [f_{xy}(x,y)]^2 = (6x)(6y) - (-3)^2 = 36xy - 9$. Then $D(0,0) = 36(0)(0) - 9 = -9$, and $D(1,1) = 36(1)(1) - 9 = 27$. Since $D(0,0) < 0$, f has a saddle point at $(0,0)$ by the Second Derivatives Test. Since $D(1,1) > 0$ and $f_{xx}(1,1) > 0$, f has a local minimum at $(1,1)$.

5. $f(x,y) = 9 - 2x + 4y - x^2 - 4y^2 \Rightarrow f_x = -2 - 2x$, $f_y = 4 - 8y$, $f_{xx} = -2$, $f_{xy} = 0$, $f_{yy} = -8$. Then $f_x = 0$ and $f_y = 0$ imply $x = -1$ and $y = \frac{1}{2}$, and the only critical point is $\left(-1, \frac{1}{2}\right)$. $D(x,y) = f_{xx} f_{yy} - (f_{xy})^2 = (-2)(-8) - 0^2 = 16$, and since $D\left(-1, \frac{1}{2}\right) = 16 > 0$ and $f_{xx}\left(-1, \frac{1}{2}\right) = -2 < 0$, $f\left(-1, \frac{1}{2}\right) = 11$ is a local maximum by the Second Derivatives Test.