15 □ PARTIAL DERIVATIVES □ ET 14

15.1 Functions of Several Variables ET 14.1

1. (a) From Table 1, $f(-15, 40) = -27$, which means that if the temperature is $-15°\text{C}$ and the wind speed is 40 km/h, then the air would feel equivalent to approximately $-27°\text{C}$ without wind.

(b) The question is asking: when the temperature is $-20°\text{C}$, what wind speed gives a wind-chill index of $-30°\text{C}$? From Table 1, the speed is 20 km/h.

(c) The question is asking: when the wind speed is 20 km/h, what temperature gives a wind-chill index of $-49°\text{C}$? From Table 1, the temperature is $-35°\text{C}$.

(d) The function $W = f(-5, v)$ means that we fix T at -5 and allow v to vary, resulting in a function of one variable. In other words, the function gives wind-chill index values for different wind speeds when the temperature is $-5°\text{C}$. From Table 1 (look at the row corresponding to $T = -5$), the function decreases and appears to approach a constant value as v increases.

(e) The function $W = f(T, 50)$ means that we fix v at 50 and allow T to vary, again giving a function of one variable. In other words, the function gives wind-chill index values for different temperatures when the wind speed is 50 km/h . From Table 1 (look at the column corresponding to $v = 50$), the function increases almost linearly as T increases.

3. If the amounts of labor and capital are both doubled, we replace L, K in the function with $2L, 2K$, giving

$$P(2L, 2K) = 1.01(2L)^{0.75}(2K)^{0.25} = 1.01(2^{0.75})(2^{0.25})L^{0.75}K^{0.25} = (2^1)1.01L^{0.75}K^{0.25} = 2P(L, K)$$

Thus, the production is doubled. It is also true for the general case $P(L, K) = bL^{\alpha}K^{1-\alpha}$:
$P(2L, 2K) = b(2L)^{\alpha}(2K)^{1-\alpha} = b(2^{\alpha})(2^{1-\alpha})L^{\alpha}K^{1-\alpha} = (2^{\alpha+1-\alpha})bL^{\alpha}K^{1-\alpha} = 2P(L, K)$.

5. (a) According to Table 4, $f(40, 15) = 25$, which means that if a 40-knot wind has been blowing in the open sea for 15 hours, it will create waves with estimated heights of 25 feet.

(b) $h = f(30, t)$ means we fix v at 30 and allow t to vary, resulting in a function of one variable. Thus here, $h = f(30, t)$ gives the wave heights produced by 30-knot winds blowing for t hours. From the table (look at the row corresponding to $v = 30$), the function increases but at a declining rate as t increases. In fact, the function values appear to be approaching a limiting value of approximately 19, which suggests that 30-knot winds cannot produce waves higher than about 19 feet.

(c) $h = f(v, 30)$ means we fix t at 30, again giving a function of one variable. So, $h = f(v, 30)$ gives the wave heights produced by winds of speed v blowing for 30 hours. From the table (look at the column corresponding to $t = 30$), the function appears to increase at an increasing rate, with no apparent limiting value. This suggests that faster winds (lasting 30 hours) always create higher waves.

7. (a) $f(2, 0) = 2^2 e^{3(2)(0)} = 4(1) = 4$

(b) Since both x^2 and the exponential function are defined everywhere, $x^2 e^{3xy}$ is defined for all choices of values for x and y. Thus the domain of f is $\mathbb{R}^2$.

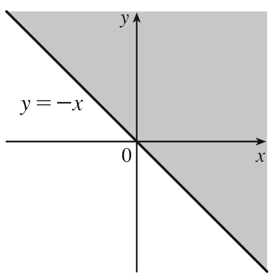
(c) Because the range of $g(x,y) = 3xy$ is $\mathbb{R}$, and the range of e^x is $(0,\infty)$, the range of $e^{g(x,y)} = e^{3xy}$ is $(0,\infty)$. The range of x^2 is $[0,\infty)$, so the range of the product $x^2 e^{3xy}$ is $[0,\infty)$.

9. (a) $f(2,-1,6) = e^{\sqrt{6-2^2-(-1)^2}} = e^{\sqrt{1}} = e$.

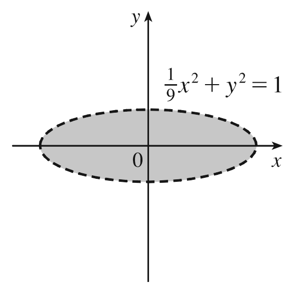
(b) $e^{\sqrt{z-x^2-y^2}}$ is defined when $z - x^2 - y^2 \geq 0 \;\;\Rightarrow\;\; z \geq x^2 + y^2$. Thus the domain of f is $\{(x,y,z) \mid z \geq x^2 + y^2\}$.

(c) Since $\sqrt{z - x^2 - y^2} \geq 0$, we have $e^{\sqrt{z-x^2-y^2}} \geq 1$. Thus the range of f is $[1,\infty)$.

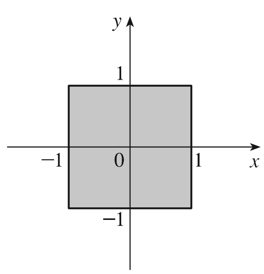
11. $\sqrt{x+y}$ is defined only when $x + y \geq 0$, or $y \geq -x$. So the domain of f is $\{(x,y) \mid y \geq -x\}$.



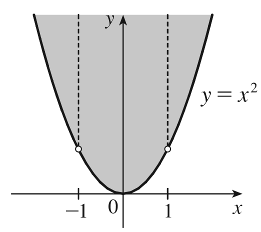
13. $\ln(9 - x^2 - 9y^2)$ is defined only when $9 - x^2 - 9y^2 > 0$, or $\frac{1}{9}x^2 + y^2 < 1$. So the domain of f is $\left\{(x,y) \mid \frac{1}{9}x^2 + y^2 < 1\right\}$, the interior of an ellipse.



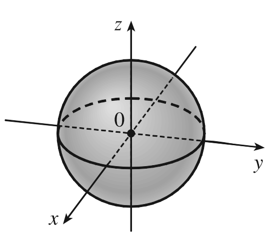
15. $\sqrt{1-x^2}$ is defined only when $1 - x^2 \geq 0$, or $x^2 \leq 1 \;\;\Leftrightarrow\;\; -1 \leq x \leq 1$, and $\sqrt{1-y^2}$ is defined only when $1 - y^2 \geq 0$, or $y^2 \leq 1 \;\;\Leftrightarrow\;\; -1 \leq y \leq 1$. Thus the domain of f is $\{(x,y) \mid -1 \leq x \leq 1,\; -1 \leq y \leq 1\}$.



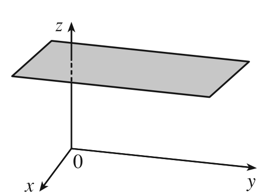
17. $\sqrt{y - x^2}$ is defined only when $y - x^2 \geq 0$, or $y \geq x^2$. In addition, f is not defined if $1 - x^2 = 0 \;\;\Rightarrow\;\; x = \pm 1$. Thus the domain of f is $\left\{(x,y) \mid y \geq x^2,\; x \neq \pm 1\right\}$.



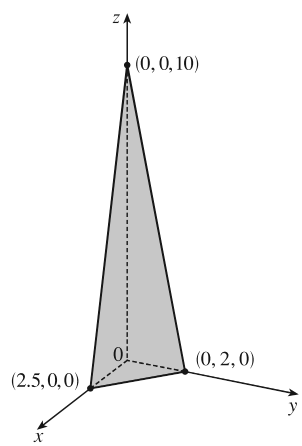
19. We need $1 - x^2 - y^2 - z^2 \geq 0$ or $x^2 + y^2 + z^2 \leq 1$, so $D = \left\{(x,y,z) \mid x^2 + y^2 + z^2 \leq 1\right\}$ (the points inside or on the sphere of radius 1, center the origin).



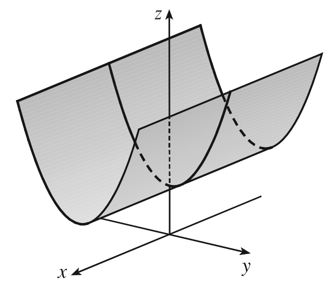
21. $z = 3$, a horizontal plane through the point $(0,0,3)$.



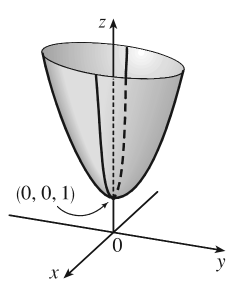
23. $z = 10 - 4x - 5y$ or $4x + 5y + z = 10$, a plane with intercepts 2.5, 2, and 10.



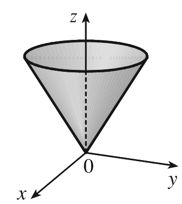
25. $z = y^2 + 1$, a parabolic cylinder



27. $z = 4x^2 + y^2 + 1$, an elliptic paraboloid with vertex at $(0, 0, 1)$.



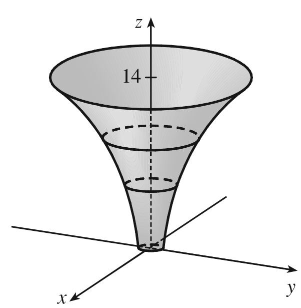
29. $z = \sqrt{x^2 + y^2}$ so $x^2 + y^2 = z^2$ and $z \geq 0$, the top half of a right circular cone.



31. The point $(-3, 3)$ lies between the level curves with z-values 50 and 60. Since the point is a little closer to the level curve with $z = 60$, we estimate that $f(-3, 3) \approx 56$. The point $(3, -2)$ appears to be just about halfway between the level curves with z-values 30 and 40, so we estimate $f(3, -2) \approx 35$. The graph rises as we approach the origin, gradually from above, steeply from below.

33. Near A, the level curves are very close together, indicating that the terrain is quite steep. At B, the level curves are much farther apart, so we would expect the terrain to be much less steep than near A, perhaps almost flat.

35.



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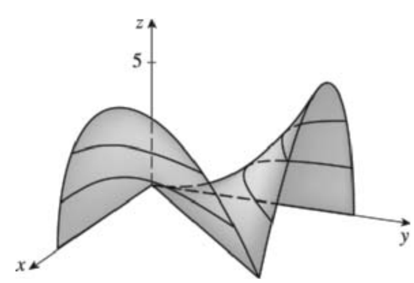
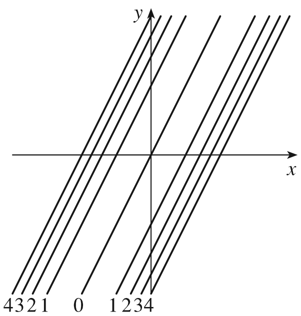
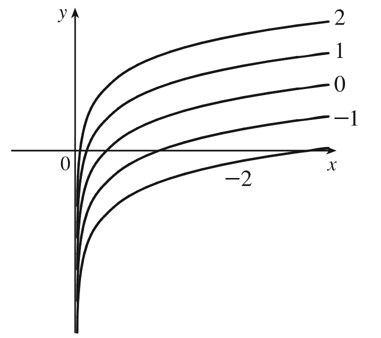


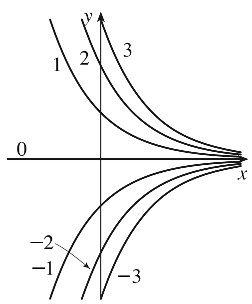
39. The level curves are $(y - 2x)^2 = k$ or $y = 2x \pm \sqrt{k}$, $k \geq 0$, a family of pairs of parallel lines.



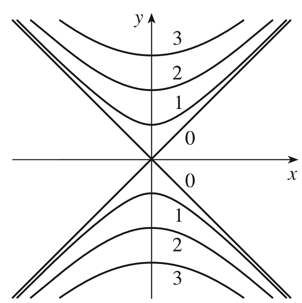
41. The level curves are $y - \ln x = k$ or $y = \ln x + k$.



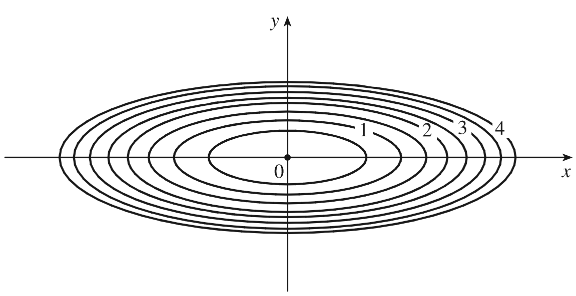
43. The level curves are $ye^x = k$ or $y = ke^{-x}$, a family of exponential curves.

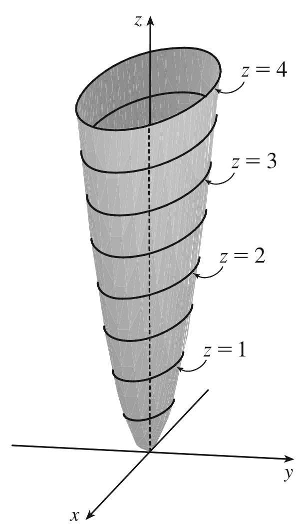


45. The level curves are $\sqrt{y^2 - x^2} = k$ or $y^2 - x^2 = k^2$, $k \geq 0$. When $k = 0$ the level curve is the pair of lines $y = \pm x$. For $k > 0$, the level curves are hyperbolas with axis the y-axis.



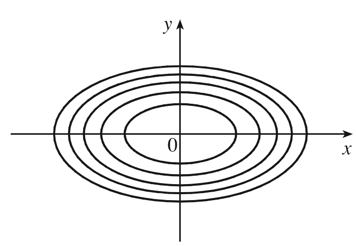
47. The contour map consists of the level curves $k = x^2 + 9y^2$, a family of ellipses with major axis the x-axis. (Or, if $k = 0$, the origin.)
The graph of $f(x, y)$ is the surface $z = x^2 + 9y^2$, an elliptic paraboloid.



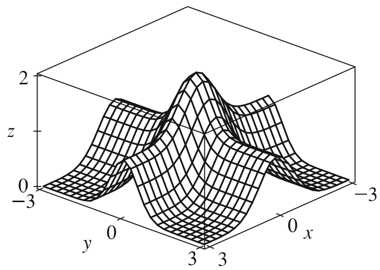


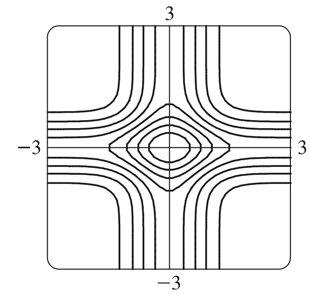
If we visualize lifting each ellipse $k = x^2 + 9y^2$ of the contour map to the plane $z = k$, we have horizontal traces that indicate the shape of the graph of f.

49. The isothermals are given by $k = 100/(1 + x^2 + 2y^2)$ or $x^2 + 2y^2 = (100 - k)/k$ $[0 < k \leq 100]$, a family of ellipses.

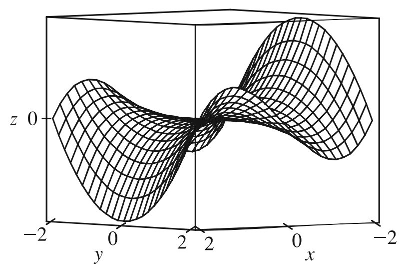


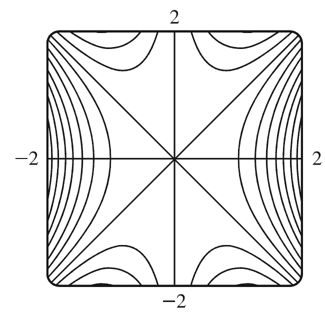
51. $f(x,y) = e^{-x^2} + e^{-2y^2}$





53. $f(x,y) = xy^2 - x^3$





The traces parallel to the yz-plane (such as the left-front trace in the graph above) are parabolas; those parallel to the xz-plane (such as the right-front trace) are cubic curves. The surface is called a monkey saddle because a monkey sitting on the surface near the origin has places for both legs and tail to rest.

55. (a) C (b) II
Reasons: This function is periodic in both x and y, and the function is the same when x is interchanged with y, so its graph is symmetric about the plane $y = x$. In addition, the function is 0 along the x- and y-axes. These conditions are satisfied only by C and II.

57. (a) F (b) I
Reasons: This function is periodic in both x and y but is constant along the lines $y = x + k$, a condition satisfied only by F and I.

59. (a) B (b) VI
Reasons: This function is 0 along the lines $x = \pm 1$ and $y = \pm 1$. The only contour map in which this could occur is VI. Also note that the trace in the xz-plane is the parabola $z = 1 - x^2$ and the trace in the yz-plane is the parabola $z = 1 - y^2$, so the graph is B.

61. $k = x + 3y + 5z$ is a family of parallel planes with normal vector $\langle 1, 3, 5 \rangle$.

63. $k = x^2 - y^2 + z^2$ are the equations of the level surfaces. For $k = 0$, the surface is a right circular cone with vertex the origin and axis the y-axis. For $k > 0$, we have a family of hyperboloids of one sheet with axis the y-axis. For $k < 0$, we have a family of hyperboloids of two sheets with axis the y-axis.

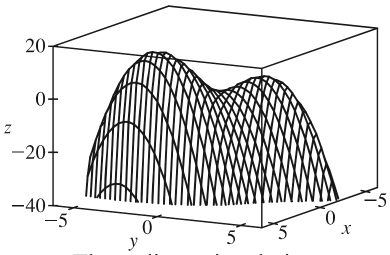
65. (a) The graph of g is the graph of f shifted upward 2 units.

(b) The graph of g is the graph of f stretched vertically by a factor of 2.

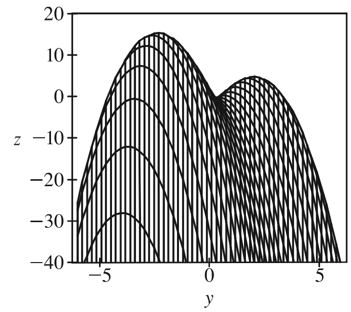
(c) The graph of g is the graph of f reflected about the xy-plane.

(d) The graph of $g(x,y) = -f(x,y) + 2$ is the graph of f reflected about the xy-plane and then shifted upward 2 units.

67. $f(x,y) = 3x - x^4 - 4y^2 - 10xy$



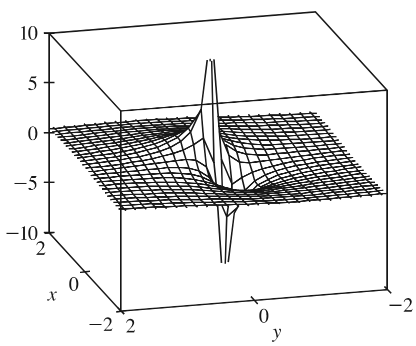
Three-dimensional view



Front view

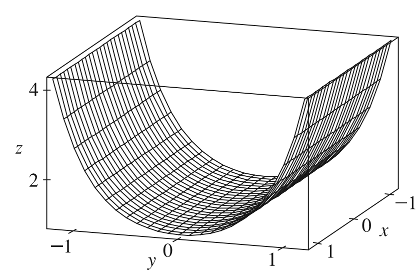
It does appear that the function has a maximum value, at the higher of the two "hilltops." From the front view graph, the maximum value appears to be approximately 15. Both hilltops could be considered local maximum points, as the values of f there are larger than at the neighboring points. There does not appear to be any local minimum point; although the valley shape between the two peaks looks like a minimum of some kind, some neighboring points have lower function values.

69.



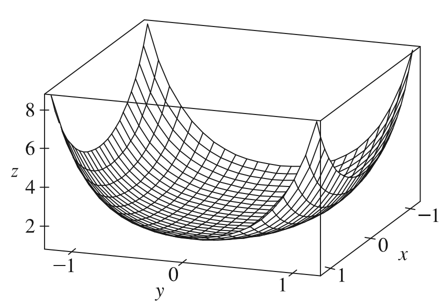
$f(x,y) = \dfrac{x+y}{x^2+y^2}$. As both x and y become large, the function values appear to approach 0, regardless of which direction is considered. As (x,y) approaches the origin, the graph exhibits asymptotic behavior. From some directions, $f(x,y) \to \infty$, while in others $f(x,y) \to -\infty$. (These are the vertical spikes visible in the graph.) If the graph is examined carefully, however, one can see that $f(x,y)$ approaches 0 along the line $y = -x$.

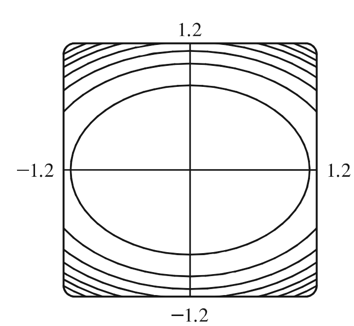
71. $f(x,y) = e^{cx^2+y^2}$. First, if $c = 0$, the graph is the cylindrical surface $z = e^{y^2}$ (whose level curves are parallel lines). When $c > 0$, the vertical trace above the y-axis remains fixed while the sides of the surface in the x-direction "curl" upward, giving the graph a shape resembling an elliptic paraboloid. The level curves of the surface are ellipses centered at the origin.



$c = 0$

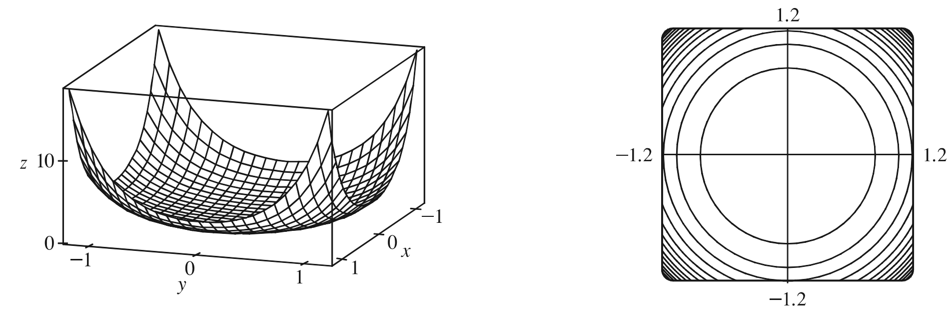
For $0 < c < 1$, the ellipses have major axis the x-axis and the eccentricity increases as $c \to 0$.





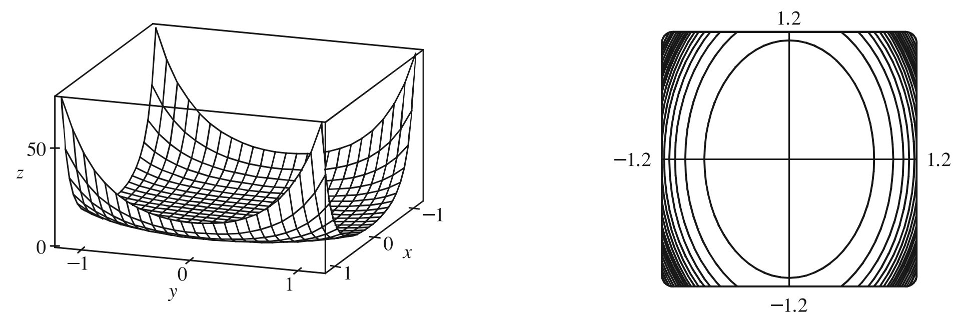
$c = 0.5$ (level curves in increments of 1)

For $c = 1$ the level curves are circles centered at the origin.



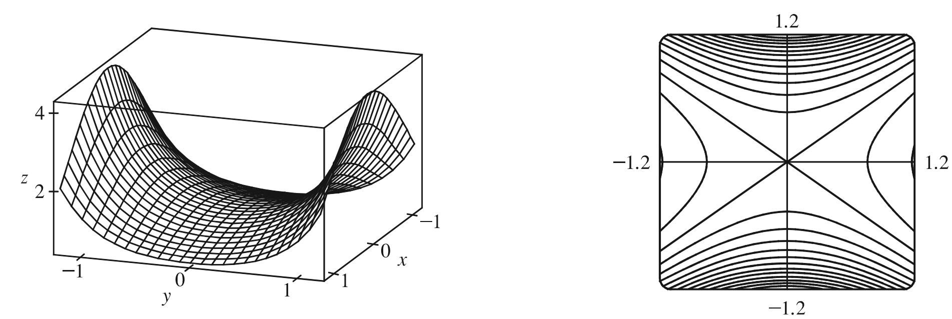
$c = 1$ (level curves in increments of 1)

When $c > 1$, the level curves are ellipses with major axis the y-axis, and the eccentricity increases as c increases.

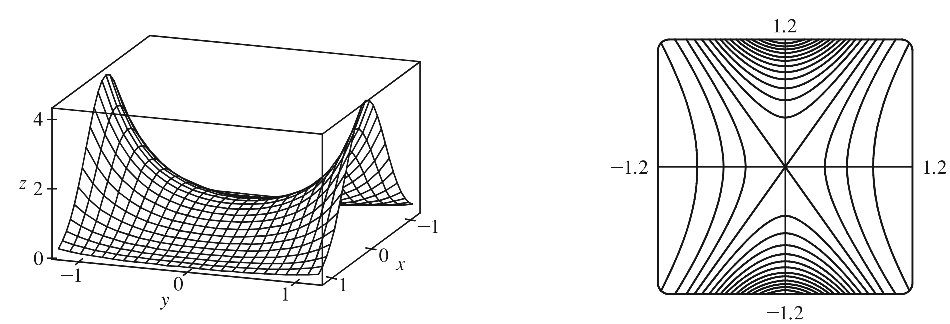


$c = 2$ (level curves in increments of 4)

For values of $c < 0$, the sides of the surface in the x-direction curl downward and approach the xy-plane (while the vertical trace $x = 0$ remains fixed), giving a saddle-shaped appearance to the graph near the point $(0, 0, 1)$. The level curves consist of a family of hyperbolas. As c decreases, the surface becomes flatter in the x-direction and the surface's approach to the curve in the trace $x = 0$ becomes steeper, as the graphs demonstrate.



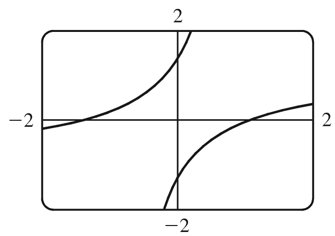
$c = -0.5$ (level curves in increments of 0.25)



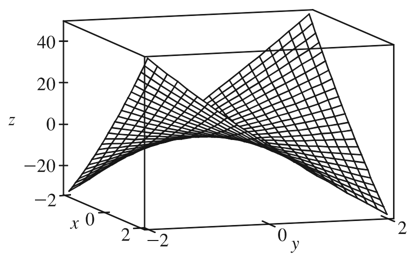
$c = -2$ (level curves in increments of 0.25)

73. $z = x^2 + y^2 + cxy$. When $c < -2$, the surface intersects the plane $z = k \neq 0$ in a hyperbola. (See graph below.) It intersects the plane $x = y$ in the parabola $z = (2 + c)x^2$, and the plane $x = -y$ in the parabola $z = (2 - c)x^2$. These parabolas open in opposite directions, so the surface is a hyperbolic paraboloid.

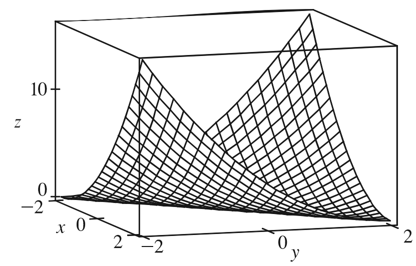
When $c = -2$ the surface is $z = x^2 + y^2 - 2xy = (x - y)^2$. So the surface is constant along each line $x - y = k$. That is, the surface is a cylinder with axis $x - y = 0$, $z = 0$. The shape of the cylinder is determined by its intersection with the plane $x + y = 0$, where $z = 4x^2$, and hence the cylinder is parabolic with minima of 0 on the line $y = x$.



$c = -5$, $z = 2$



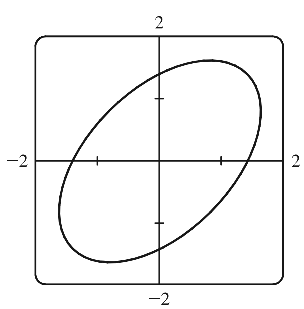
$c = -10$



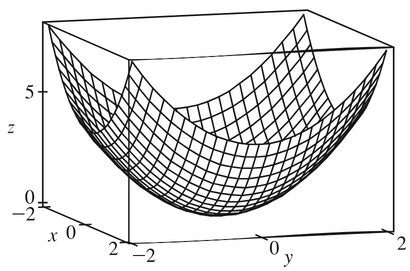
$c = -2$

When $-2 < c \leq 0$, $z \geq 0$ for all x and y. If x and y have the same sign, then $x^2 + y^2 + cxy \geq x^2 + y^2 - 2xy = (x - y)^2 \geq 0$. If they have opposite signs, then $cxy \geq 0$. The intersection with the surface and the plane $z = k > 0$ is an ellipse (see graph below). The intersection with the surface and the planes $x = 0$ and $y = 0$ are parabolas $z = y^2$ and $z = x^2$ respectively, so the surface is an elliptic paraboloid.

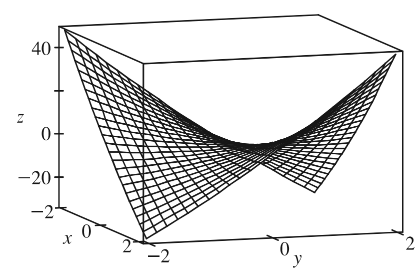
When $c > 0$ the graphs have the same shape, but are reflected in the plane $x = 0$, because $x^2 + y^2 + cxy = (-x)^2 + y^2 + (-c)(-x)y$. That is, the value of z is the same for c at (x, y) as it is for $-c$ at $(-x, y)$.



$c = -1$, $z = 2$



$c = 0$



$c = 10$

So the surface is an elliptic paraboloid for $0 < c < 2$, a parabolic cylinder for $c = 2$, and a hyperbolic paraboloid for $c > 2$.

75. (a) $P = bL^\alpha K^{1-\alpha} \quad \Rightarrow \quad \dfrac{P}{K} = bL^\alpha K^{-\alpha} \quad \Rightarrow \quad \dfrac{P}{K} = b\left(\dfrac{L}{K}\right)^\alpha \quad \Rightarrow \quad \ln \dfrac{P}{K} = \ln\left(b\left(\dfrac{L}{K}\right)^\alpha\right) \quad \Rightarrow$

$$\ln \frac{P}{K} = \ln b + \alpha \ln\left(\frac{L}{K}\right)$$

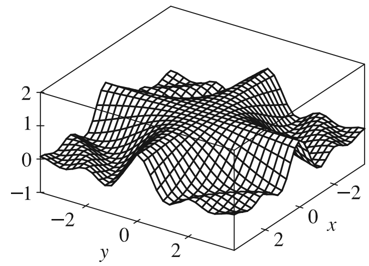
(b) We list the values for $\ln(L/K)$ and $\ln(P/K)$ for the years 1899–1922. (Historically, these values were rounded to 2 decimal places.)

39. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} = \lim_{r\to0^+} \frac{(r\cos\theta)^3+(r\sin\theta)^3}{r^2} = \lim_{r\to0^+}(r\cos^3\theta + r\sin^3\theta) = 0$

41. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{e^{-x^2-y^2}-1}{x^2+y^2} = \lim_{r\to0^+}\frac{e^{-r^2}-1}{r^2} = \lim_{r\to0^+}\frac{e^{-r^2}(-2r)}{2r}$ [using l'Hospital's Rule]

$$= \lim_{r\to0^+} -e^{-r^2} = -e^0 = -1$$

43. $f(x,y) = \begin{cases} \dfrac{\sin(xy)}{xy} & \text{if } (x,y)\neq(0,0) \\ 1 & \text{if } (x,y) = (0,0) \end{cases}$



From the graph, it appears that f is continuous everywhere. We know xy is continuous on $\mathbb{R}^2$ and $\sin t$ is continuous everywhere, so $\sin(xy)$ is continuous on $\mathbb{R}^2$ and $\dfrac{\sin(xy)}{xy}$ is continuous on $\mathbb{R}^2$ except possibly where $xy = 0$. To show that f is continuous at those points, consider any point (a,b) in $\mathbb{R}^2$ where $ab = 0$. Because xy is continuous, $xy \to ab = 0$ as $(x,y)\to(a,b)$. If we let $t = xy$, then $t\to 0$ as $(x,y)\to(a,b)$ and $\displaystyle\lim_{(x,y)\to(a,b)}\frac{\sin(xy)}{xy} = \lim_{t\to0}\frac{\sin(t)}{t} = 1$ by Equation 3.4.2 [ET 3.3.2]. Thus $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$ and f is continuous on $\mathbb{R}^2$.

45. Since $|\mathbf{x}-\mathbf{a}|^2 = |\mathbf{x}|^2 + |\mathbf{a}|^2 - 2\,|\mathbf{x}|\,|\mathbf{a}|\cos\theta \geq |\mathbf{x}|^2 + |\mathbf{a}|^2 - 2\,|\mathbf{x}|\,|\mathbf{a}| = (|\mathbf{x}| - |\mathbf{a}|)^2$, we have $\big||\mathbf{x}| - |\mathbf{a}|\big| \leq |\mathbf{x}-\mathbf{a}|$. Let $\epsilon > 0$ be given and set $\delta = \epsilon$. Then if $0 < |\mathbf{x}-\mathbf{a}| < \delta$, $\big||\mathbf{x}| - |\mathbf{a}|\big| \leq |\mathbf{x}-\mathbf{a}| < \delta = \epsilon$. Hence $\lim_{\mathbf{x}\to\mathbf{a}} |\mathbf{x}| = |\mathbf{a}|$ and $f(\mathbf{x}) = |\mathbf{x}|$ is continuous on $\mathbb{R}^n$.

15.3 Partial Derivatives

ET 14.3

1. (a) $\partial T/\partial x$ represents the rate of change of T when we fix y and t and consider T as a function of the single variable x, which describes how quickly the temperature changes when longitude changes but latitude and time are constant. $\partial T/\partial y$ represents the rate of change of T when we fix x and t and consider T as a function of y, which describes how quickly the temperature changes when latitude changes but longitude and time are constant. $\partial T/\partial t$ represents the rate of change of T when we fix x and y and consider T as a function of t, which describes how quickly the temperature changes over time for a constant longitude and latitude.

(b) $f_x(158, 21, 9)$ represents the rate of change of temperature at longitude 158°W, latitude 21°N at 9:00 AM when only longitude varies. Since the air is warmer to the west than to the east, increasing longitude results in an increased air temperature, so we would expect $f_x(158, 21, 9)$ to be positive. $f_y(158, 21, 9)$ represents the rate of change of temperature at the same time and location when only latitude varies. Since the air is warmer to the south and cooler to the north, increasing latitude results in a decreased air temperature, so we would expect $f_y(158, 21, 9)$ to be negative. $f_t(158, 21, 9)$ represents the rate of change of temperature at the same time and location when only time varies. Since typically air temperature increases from the morning to the afternoon as the sun warms it, we would expect $f_t(158, 21, 9)$ to be positive.

3. (a) By Definition 4, $f_T(-15, 30) = \lim\limits_{h \to 0} \dfrac{f(-15+h, 30) - f(-15, 30)}{h}$, which we can approximate by considering $h = 5$ and $h = -5$ and using the values given in the table:

$f_T(-15, 30) \approx \dfrac{f(-10, 30) - f(-15, 30)}{5} = \dfrac{-20 - (-26)}{5} = \dfrac{6}{5} = 1.2$,

$f_T(-15, 30) \approx \dfrac{f(-20, 30) - f(-15, 30)}{-5} = \dfrac{-33 - (-26)}{-5} = \dfrac{-7}{-5} = 1.4$. Averaging these values, we estimate $f_T(-15, 30)$ to be approximately 1.3. Thus, when the actual temperature is -15°C and the wind speed is 30 km/h, the apparent temperature rises by about 1.3°C for every degree that the actual temperature rises.

Similarly, $f_v(-15, 30) = \lim\limits_{h \to 0} \dfrac{f(-15, 30+h) - f(-15, 30)}{h}$ which we can approximate by considering $h = 10$ and $h = -10$: $f_v(-15, 30) \approx \dfrac{f(-15, 40) - f(-15, 30)}{10} = \dfrac{-27 - (-26)}{10} = \dfrac{-1}{10} = -0.1$,

$f_v(-15, 30) \approx \dfrac{f(-15, 20) - f(-15, 30)}{-10} = \dfrac{-24 - (-26)}{-10} = \dfrac{2}{-10} = -0.2$. Averaging these values, we estimate $f_v(-15, 30)$ to be approximately -0.15. Thus, when the actual temperature is -15°C and the wind speed is 30 km/h, the apparent temperature decreases by about 0.15°C for every km/h that the wind speed increases.

(b) For a fixed wind speed v, the values of the wind-chill index W increase as temperature T increases (look at a column of the table), so $\dfrac{\partial W}{\partial T}$ is positive. For a fixed temperature T, the values of W decrease (or remain constant) as v increases (look at a row of the table), so $\dfrac{\partial W}{\partial v}$ is negative (or perhaps 0).

(c) For fixed values of T, the function values $f(T, v)$ appear to become constant (or nearly constant) as v increases, so the corresponding rate of change is 0 or near 0 as v increases. This suggests that $\lim\limits_{v \to \infty} (\partial W / \partial v) = 0$.

5. (a) If we start at $(1, 2)$ and move in the positive x-direction, the graph of f increases. Thus $f_x(1, 2)$ is positive.

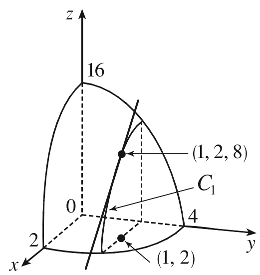
(b) If we start at $(1, 2)$ and move in the positive y-direction, the graph of f decreases. Thus $f_y(1, 2)$ is negative.

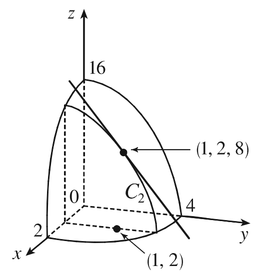
7. (a) $f_{xx} = \frac{\partial}{\partial x}(f_x)$, so f_{xx} is the rate of change of f_x in the x-direction. f_x is negative at $(-1, 2)$ and if we move in the positive x-direction, the surface becomes less steep. Thus the values of f_x are increasing and $f_{xx}(-1, 2)$ is positive.

(b) f_{yy} is the rate of change of f_y in the y-direction. f_y is negative at $(-1, 2)$ and if we move in the positive y-direction, the surface becomes steeper. Thus the values of f_y are decreasing, and $f_{yy}(-1, 2)$ is negative.

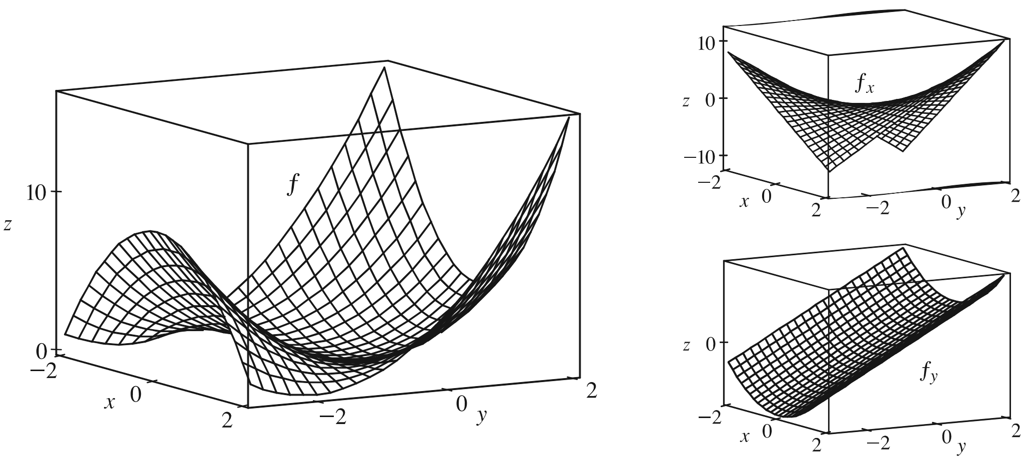
9. First of all, if we start at the point $(3, -3)$ and move in the positive y-direction, we see that both b and c decrease, while a increases. Both b and c have a low point at about $(3, -1.5)$, while a is 0 at this point. So a is definitely the graph of f_y, and one of b and c is the graph of f. To see which is which, we start at the point $(-3, -1.5)$ and move in the positive x-direction. b traces out a line with negative slope, while c traces out a parabola opening downward. This tells us that b is the x-derivative of c. So c is the graph of f, b is the graph of f_x, and a is the graph of f_y.

11. $f(x,y) = 16 - 4x^2 - y^2 \;\Rightarrow\; f_x(x,y) = -8x$ and $f_y(x,y) = -2y \;\Rightarrow\; f_x(1,2) = -8$ and $f_y(1,2) = -4$. The graph of f is the paraboloid $z = 16 - 4x^2 - y^2$ and the vertical plane $y = 2$ intersects it in the parabola $z = 12 - 4x^2$, $y = 2$ (the curve C_1 in the first figure). The slope of the tangent line to this parabola at $(1,2,8)$ is $f_x(1,2) = -8$. Similarly the plane $x = 1$ intersects the paraboloid in the parabola $z = 12 - y^2$, $x = 1$ (the curve C_2 in the second figure) and the slope of the tangent line at $(1,2,8)$ is $f_y(1,2) = -4$.





13. $f(x,y) = x^2 + y^2 + x^2y \;\Rightarrow\; f_x = 2x + 2xy, \quad f_y = 2y + x^2$



Note that the traces of f in planes parallel to the xz-plane are parabolas which open downward for $y < -1$ and upward for $y > -1$, and the traces of f_x in these planes are straight lines, which have negative slopes for $y < -1$ and positive slopes for $y > -1$. The traces of f in planes parallel to the yz-plane are parabolas which always open upward, and the traces of f_y in these planes are straight lines with positive slopes.

15. $f(x,y) = y^5 - 3xy \;\Rightarrow\; f_x(x,y) = 0 - 3y = -3y,\; f_y(x,y) = 5y^4 - 3x$

17. $f(x,t) = e^{-t}\cos\pi x \;\Rightarrow\; f_x(x,t) = e^{-t}(-\sin\pi x)(\pi) = -\pi e^{-t}\sin\pi x,\; f_t(x,t) = e^{-t}(-1)\cos\pi x = -e^{-t}\cos\pi x$

19. $z = (2x+3y)^{10} \;\Rightarrow\; \dfrac{\partial z}{\partial x} = 10(2x+3y)^9 \cdot 2 = 20(2x+3y)^9,\; \dfrac{\partial z}{\partial y} = 10(2x+3y)^9 \cdot 3 = 30(2x+3y)^9$

21. $f(x,y) = \dfrac{x-y}{x+y} \;\Rightarrow\; f_x(x,y) = \dfrac{(1)(x+y) - (x-y)(1)}{(x+y)^2} = \dfrac{2y}{(x+y)^2}$,

$f_y(x,y) = \dfrac{(-1)(x+y) - (x-y)(1)}{(x+y)^2} = -\dfrac{2x}{(x+y)^2}$

23. $w = \sin\alpha\,\cos\beta \;\Rightarrow\; \dfrac{\partial w}{\partial \alpha} = \cos\alpha\,\cos\beta,\; \dfrac{\partial w}{\partial \beta} = -\sin\alpha\,\sin\beta$

25. $f(r,s) = r\ln(r^2+s^2) \;\Rightarrow\; f_r(r,s) = r \cdot \dfrac{2r}{r^2+s^2} + \ln(r^2+s^2)\cdot 1 = \dfrac{2r^2}{r^2+s^2} + \ln(r^2+s^2)$,

$f_s(r,s) = r \cdot \dfrac{2s}{r^2+s^2} + 0 = \dfrac{2rs}{r^2+s^2}$

27. $u = te^{w/t} \quad \Rightarrow \quad \dfrac{\partial u}{\partial t} = t \cdot e^{w/t}(-wt^{-2}) + e^{w/t} \cdot 1 = e^{w/t} - \dfrac{w}{t}e^{w/t} = e^{w/t}\left(1 - \dfrac{w}{t}\right),\ \dfrac{\partial u}{\partial w} = te^{w/t} \cdot \dfrac{1}{t} = e^{w/t}$

29. $f(x, y, z) = xz - 5x^2y^3z^4 \quad \Rightarrow \quad f_x(x, y, z) = z - 10xy^3z^4,\ f_y(x, y, z) = -15x^2y^2z^4,\ f_z(x, y, z) = x - 20x^2y^3z^3$

31. $w = \ln(x + 2y + 3z) \quad \Rightarrow \quad \dfrac{\partial w}{\partial x} = \dfrac{1}{x + 2y + 3z},\ \dfrac{\partial w}{\partial y} = \dfrac{2}{x + 2y + 3z},\ \dfrac{\partial w}{\partial z} = \dfrac{3}{x + 2y + 3z}$

33. $u = xy\sin^{-1}(yz) \quad \Rightarrow \quad \dfrac{\partial u}{\partial x} = y\sin^{-1}(yz),\ \ \dfrac{\partial u}{\partial y} = xy \cdot \dfrac{1}{\sqrt{1 - (yz)^2}}\,(z) + \sin^{-1}(yz) \cdot x = \dfrac{xyz}{\sqrt{1 - y^2z^2}} + x\sin^{-1}(yz),$

$\dfrac{\partial u}{\partial z} = xy \cdot \dfrac{1}{\sqrt{1 - (yz)^2}}\,(y) = \dfrac{xy^2}{\sqrt{1 - y^2z^2}}$

35. $f(x, y, z, t) = xyz^2\tan(yt) \quad \Rightarrow \quad f_x(x, y, z, t) = yz^2\tan(yt),$

$f_y(x, y, z, t) = xyz^2 \cdot \sec^2(yt) \cdot t + xz^2\tan(yt) = xyz^2t\sec^2(yt) + xz^2\tan(yt),$

$f_z(x, y, z, t) = 2xyz\tan(yt),\ f_t(x, y, z, t) = xyz^2\sec^2(yt) \cdot y = xy^2z^2\sec^2(yt)$

37. $u = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$. For each $i = 1, \ldots, n$, $u_{x_i} = \frac{1}{2}\left(x_1^2 + x_2^2 + \cdots + x_n^2\right)^{-1/2}(2x_i) = \dfrac{x_i}{\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}}$.

39. $f(x, y) = \ln\left(x + \sqrt{x^2 + y^2}\right) \quad \Rightarrow$

$$f_x(x, y) = \frac{1}{x + \sqrt{x^2 + y^2}}\left[1 + \tfrac{1}{2}(x^2 + y^2)^{-1/2}(2x)\right] = \frac{1}{x + \sqrt{x^2 + y^2}}\left(1 + \frac{x}{\sqrt{x^2 + y^2}}\right),$$

so $f_x(3, 4) = \dfrac{1}{3 + \sqrt{3^2 + 4^2}}\left(1 + \dfrac{3}{\sqrt{3^2 + 4^2}}\right) = \frac{1}{8}\left(1 + \frac{3}{5}\right) = \frac{1}{5}$.

41. $f(x, y, z) = \dfrac{y}{x + y + z} \quad \Rightarrow \quad f_y(x, y, z) = \dfrac{1(x + y + z) - y(1)}{(x + y + z)^2} = \dfrac{x + z}{(x + y + z)^2},$

so $f_y(2, 1, -1) = \dfrac{2 + (-1)}{(2 + 1 + (-1))^2} = \dfrac{1}{4}$.

43. $f(x, y) = xy^2 - x^3y \quad \Rightarrow$

$$f_x\,(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h} = \lim_{h \to 0} \frac{(x + h)y^2 - (x + h)^3y - (xy^2 - x^3y)}{h}$$
$$= \lim_{h \to 0} \frac{h(y^2 - 3x^2y - 3xyh - yh^2)}{h} = \lim_{h \to 0}(y^2 - 3x^2y - 3xyh - yh^2) = y^2 - 3x^2y$$

$$f_y\,(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h} = \lim_{h \to 0} \frac{x(y + h)^2 - x^3(y + h) - (xy^2 - x^3y)}{h} = \lim_{h \to 0} \frac{h(2xy + xh - x^3)}{h}$$
$$= \lim_{h \to 0}(2xy + xh - x^3) = 2xy - x^3$$

45. $x^2 + y^2 + z^2 = 3xyz \quad \Rightarrow \quad \dfrac{\partial}{\partial x}\,(x^2 + y^2 + z^2) = \dfrac{\partial}{\partial x}\,(3xyz) \quad \Rightarrow \quad 2x + 0 + 2z\,\dfrac{\partial z}{\partial x} = 3y\left(x\,\dfrac{\partial z}{\partial x} + z \cdot 1\right) \quad \Leftrightarrow$

$2z\,\dfrac{\partial z}{\partial x} - 3xy\,\dfrac{\partial z}{\partial x} = 3yz - 2x \quad \Leftrightarrow \quad (2z - 3xy)\,\dfrac{\partial z}{\partial x} = 3yz - 2x$, so $\dfrac{\partial z}{\partial x} = \dfrac{3yz - 2x}{2z - 3xy}$.

$\dfrac{\partial}{\partial y}\,(x^2 + y^2 + z^2) = \dfrac{\partial}{\partial y}\,(3xyz) \quad \Rightarrow \quad 0 + 2y + 2z\,\dfrac{\partial z}{\partial y} = 3x\left(y\,\dfrac{\partial z}{\partial y} + z \cdot 1\right) \quad \Leftrightarrow \quad 2z\,\dfrac{\partial z}{\partial y} - 3xy\,\dfrac{\partial z}{\partial y} = 3xz - 2y \quad \Leftrightarrow$

$(2z - 3xy)\,\dfrac{\partial z}{\partial y} = 3xz - 2y$, so $\dfrac{\partial z}{\partial y} = \dfrac{3xz - 2y}{2z - 3xy}$.

47. $x - z = \arctan(yz) \quad\Rightarrow\quad \dfrac{\partial}{\partial x}(x - z) = \dfrac{\partial}{\partial x}(\arctan(yz)) \quad\Rightarrow\quad 1 - \dfrac{\partial z}{\partial x} = \dfrac{1}{1 + (yz)^2} \cdot y\,\dfrac{\partial z}{\partial x} \quad\Leftrightarrow$

$$1 = \left(\frac{y}{1 + y^2z^2} + 1\right)\frac{\partial z}{\partial x} \quad\Leftrightarrow\quad 1 = \left(\frac{y + 1 + y^2z^2}{1 + y^2z^2}\right)\frac{\partial z}{\partial x}, \text{ so } \frac{\partial z}{\partial x} = \frac{1 + y^2z^2}{1 + y + y^2z^2}.$$

$$\frac{\partial}{\partial y}(x - z) = \frac{\partial}{\partial y}(\arctan(yz)) \quad\Rightarrow\quad 0 - \frac{\partial z}{\partial y} = \frac{1}{1 + (yz)^2} \cdot \left(y\,\frac{\partial z}{\partial y} + z \cdot 1\right) \quad\Leftrightarrow$$

$$-\frac{z}{1 + y^2z^2} = \left(\frac{y}{1 + y^2z^2} + 1\right)\frac{\partial z}{\partial y} \quad\Leftrightarrow\quad -\frac{z}{1 + y^2z^2} = \left(\frac{y + 1 + y^2z^2}{1 + y^2z^2}\right)\frac{\partial z}{\partial y} \quad\Leftrightarrow\quad \frac{\partial z}{\partial y} = -\frac{z}{1 + y + y^2z^2}.$$

49. (a) $z = f(x) + g(y) \quad\Rightarrow\quad \dfrac{\partial z}{\partial x} = f'(x), \quad \dfrac{\partial z}{\partial y} = g'(y)$

(b) $z = f(x + y)$. Let $u = x + y$. Then $\dfrac{\partial z}{\partial x} = \dfrac{df}{du}\dfrac{\partial u}{\partial x} = \dfrac{df}{du}(1) = f'(u) = f'(x + y)$,

$$\frac{\partial z}{\partial y} = \frac{df}{du}\frac{\partial u}{\partial y} = \frac{df}{du}(1) = f'(u) = f'(x + y).$$

51. $f(x, y) = x^3y^5 + 2x^4y \quad\Rightarrow\quad f_x(x, y) = 3x^2y^5 + 8x^3y$, $f_y(x, y) = 5x^3y^4 + 2x^4$. Then $f_{xx}(x, y) = 6xy^5 + 24x^2y$, $f_{xy}(x, y) = 15x^2y^4 + 8x^3$, $f_{yx}(x, y) = 15x^2y^4 + 8x^3$, and $f_{yy}(x, y) = 20x^3y^3$.

53. $w = \sqrt{u^2 + v^2} \quad\Rightarrow\quad w_u = \frac{1}{2}(u^2 + v^2)^{-1/2} \cdot 2u = \dfrac{u}{\sqrt{u^2 + v^2}}$, $w_v = \frac{1}{2}(u^2 + v^2)^{-1/2} \cdot 2v = \dfrac{v}{\sqrt{u^2 + v^2}}$. Then

$$w_{uu} = \frac{1 \cdot \sqrt{u^2 + v^2} - u \cdot \frac{1}{2}(u^2 + v^2)^{-1/2}(2u)}{\left(\sqrt{u^2 + v^2}\right)^2} = \frac{\sqrt{u^2 + v^2} - u^2/\sqrt{u^2 + v^2}}{u^2 + v^2} = \frac{u^2 + v^2 - u^2}{(u^2 + v^2)^{3/2}} = \frac{v^2}{(u^2 + v^2)^{3/2}},$$

$$w_{uv} = u\left(-\tfrac{1}{2}\right)\left(u^2 + v^2\right)^{-3/2}(2v) = -\frac{uv}{(u^2 + v^2)^{3/2}},\ w_{vu} = v\left(-\tfrac{1}{2}\right)\left(u^2 + v^2\right)^{-3/2}(2u) = -\frac{uv}{(u^2 + v^2)^{3/2}},$$

$$w_{vv} = \frac{1 \cdot \sqrt{u^2 + v^2} - v \cdot \frac{1}{2}(u^2 + v^2)^{-1/2}(2v)}{\left(\sqrt{u^2 + v^2}\right)^2} = \frac{\sqrt{u^2 + v^2} - v^2/\sqrt{u^2 + v^2}}{u^2 + v^2} = \frac{u^2 + v^2 - v^2}{(u^2 + v^2)^{3/2}} = \frac{u^2}{(u^2 + v^2)^{3/2}}.$$

55. $z = \arctan \dfrac{x + y}{1 - xy} \quad\Rightarrow$

$$z_x = \frac{1}{1 + \left(\frac{x+y}{1-xy}\right)^2} \cdot \frac{(1)(1 - xy) - (x + y)(-y)}{(1 - xy)^2} = \frac{1 + y^2}{(1 - xy)^2 + (x + y)^2} = \frac{1 + y^2}{1 + x^2 + y^2 + x^2y^2}$$

$$= \frac{1 + y^2}{(1 + x^2)(1 + y^2)} = \frac{1}{1 + x^2},$$

$$z_y = \frac{1}{1 + \left(\frac{x+y}{1-xy}\right)^2} \cdot \frac{(1)(1 - xy) - (x + y)(-x)}{(1 - xy)^2} = \frac{1 + x^2}{(1 - xy)^2 + (x + y)^2} = \frac{1 + x^2}{(1 + x^2)(1 + y^2)} = \frac{1}{1 + y^2}.$$

Then $z_{xx} = -(1 + x^2)^{-2} \cdot 2x = -\dfrac{2x}{(1 + x^2)^2}$, $z_{xy} = 0$, $z_{yx} = 0$, $z_{yy} = -(1 + y^2)^{-2} \cdot 2y = -\dfrac{2y}{(1 + y^2)^2}$.

57. $u = x\sin(x + 2y) \quad\Rightarrow\quad u_x = x \cdot \cos(x + 2y)(1) + \sin(x + 2y) \cdot 1 = x\cos(x + 2y) + \sin(x + 2y)$,

$u_{xy} = x(-\sin(x + 2y)(2)) + \cos(x + 2y)(2) = 2\cos(x + 2y) - 2x\sin(x + 2y)$,

$u_y = x\cos(x + 2y)(2) = 2x\cos(x + 2y)$,

$u_{yx} = 2x \cdot (-\sin(x + 2y)(1)) + \cos(x + 2y) \cdot 2 = 2\cos(x + 2y) - 2x\sin(x + 2y)$. Thus $u_{xy} = u_{yx}$.

59. $u = \ln\sqrt{x^2+y^2} = \ln(x^2+y^2)^{1/2} = \frac{1}{2}\ln(x^2+y^2) \quad\Rightarrow\quad u_x = \dfrac{1}{2}\dfrac{1}{x^2+y^2}\cdot 2x = \dfrac{x}{x^2+y^2}$,

$u_{xy} = x(-1)(x^2+y^2)^{-2}(2y) = -\dfrac{2xy}{(x^2+y^2)^2}$ and $u_y = \dfrac{1}{2}\dfrac{1}{x^2+y^2}\cdot 2y = \dfrac{y}{x^2+y^2}$,

$u_{yx} = y(-1)(x^2+y^2)^{-2}(2x) = -\dfrac{2xy}{(x^2+y^2)^2}$. Thus $u_{xy} = u_{yx}$.

61. $f(x,y) = 3xy^4 + x^3y^2 \quad\Rightarrow\quad f_x = 3y^4 + 3x^2y^2,\ f_{xx} = 6xy^2,\ f_{xxy} = 12xy$ and

$f_y = 12xy^3 + 2x^3y,\ f_{yy} = 36xy^2 + 2x^3,\ f_{yyy} = 72xy.$

63. $f(x,y,z) = \cos(4x+3y+2z) \quad\Rightarrow$

$f_x = -\sin(4x+3y+2z)(4) = -4\sin(4x+3y+2z),\ f_{xy} = -4\cos(4x+3y+2z)(3) = -12\cos(4x+3y+2z),$

$f_{xyz} = -12(-\sin(4x+3y+2z))(2) = 24\sin(4x+3y+2z)$ and

$f_y = -\sin(4x+3y+2z)(3) = -3\sin(4x+3y+2z),$

$f_{yz} = -3\cos(4x+3y+2z)(2) = -6\cos(4x+3y+2z),\quad f_{yzz} = -6(-\sin(4x+3y+2z))(2) = 12\sin(4x+3y+2z).$

65. $u = e^{r\theta}\sin\theta \quad\Rightarrow\quad \dfrac{\partial u}{\partial\theta} = e^{r\theta}\cos\theta + \sin\theta\cdot e^{r\theta}\,(r) = e^{r\theta}\,(\cos\theta + r\sin\theta),$

$\dfrac{\partial^2 u}{\partial r\,\partial\theta} = e^{r\theta}\,(\sin\theta) + (\cos\theta + r\sin\theta)\,e^{r\theta}\,(\theta) = e^{r\theta}\,(\sin\theta + \theta\cos\theta + r\theta\sin\theta),$

$\dfrac{\partial^3 u}{\partial r^2\,\partial\theta} = e^{r\theta}\,(\theta\sin\theta) + (\sin\theta + \theta\cos\theta + r\theta\sin\theta)\cdot e^{r\theta}\,(\theta) = \theta e^{r\theta}\,(2\sin\theta + \theta\cos\theta + r\theta\sin\theta).$

67. $w = \dfrac{x}{y+2z} = x(y+2z)^{-1} \quad\Rightarrow\quad \dfrac{\partial w}{\partial x} = (y+2z)^{-1},\quad \dfrac{\partial^2 w}{\partial y\,\partial x} = -(y+2z)^{-2}(1) = -(y+2z)^{-2},$

$\dfrac{\partial^3 w}{\partial z\,\partial y\,\partial x} = -(-2)(y+2z)^{-3}(2) = 4(y+2z)^{-3} = \dfrac{4}{(y+2z)^3}$ and $\dfrac{\partial w}{\partial y} = x(-1)(y+2z)^{-2}(1) = -x(y+2z)^{-2},$

$\dfrac{\partial^2 w}{\partial x\,\partial y} = -(y+2z)^{-2},\quad \dfrac{\partial^3 w}{\partial x^2\,\partial y} = 0.$

69. By Definition 4, $f_x(3,2) = \lim\limits_{h\to 0}\dfrac{f(3+h,2) - f(3,2)}{h}$ which we can approximate by considering $h = 0.5$ and $h = -0.5$:

$f_x(3,2) \approx \dfrac{f(3.5,2) - f(3,2)}{0.5} = \dfrac{22.4 - 17.5}{0.5} = 9.8$, $f_x(3,2) \approx \dfrac{f(2.5,2) - f(3,2)}{-0.5} = \dfrac{10.2 - 17.5}{-0.5} = 14.6$. Averaging these values, we estimate $f_x(3,2)$ to be approximately 12.2. Similarly, $f_x(3,2.2) = \lim\limits_{h\to 0}\dfrac{f(3+h,2.2) - f(3,2.2)}{h}$ which we can approximate by considering $h = 0.5$ and $h = -0.5$: $f_x(3,2.2) \approx \dfrac{f(3.5,2.2) - f(3,2.2)}{0.5} = \dfrac{26.1 - 15.9}{0.5} = 20.4$,

$f_x(3,2.2) \approx \dfrac{f(2.5,2.2) - f(3,2.2)}{-0.5} = \dfrac{9.3 - 15.9}{-0.5} = 13.2$. Averaging these values, we have $f_x(3,2.2) \approx 16.8$.

To estimate $f_{xy}(3,2)$, we first need an estimate for $f_x(3,1.8)$:

$f_x(3,1.8) \approx \dfrac{f(3.5,1.8) - f(3,1.8)}{0.5} = \dfrac{20.0 - 18.1}{0.5} = 3.8$, $f_x(3,1.8) \approx \dfrac{f(2.5,1.8) - f(3,1.8)}{-0.5} = \dfrac{12.5 - 18.1}{-0.5} = 11.2$.

Averaging these values, we get $f_x(3,1.8) \approx 7.5$. Now $f_{xy}(x,y) = \dfrac{\partial}{\partial y}\,[f_x(x,y)]$ and $f_x(x,y)$ is itself a function of two variables, so Definition 4 says that $f_{xy}(x,y) = \dfrac{\partial}{\partial y}\left[f_x(x,y)\right] = \lim\limits_{h\to 0} \dfrac{f_x(x,y+h) - f_x(x,y)}{h} \quad\Rightarrow$

$f_{xy}(3,2) = \lim\limits_{h\to 0} \dfrac{f_x(3,2+h) - f_x(3,2)}{h}$.

We can estimate this value using our previous work with $h = 0.2$ and $h = -0.2$:

$$f_{xy}(3,2) \approx \frac{f_x(3,2.2) - f_x(3,2)}{0.2} = \frac{16.8 - 12.2}{0.2} = 23,\ f_{xy}(3,2) \approx \frac{f_x(3,1.8) - f_x(3,2)}{-0.2} = \frac{7.5 - 12.2}{-0.2} = 23.5.$$

Averaging these values, we estimate $f_{xy}(3,2)$ to be approximately 23.25.

71. $u = e^{-\alpha^2k^2t}\sin kx \quad\Rightarrow\quad u_x = ke^{-\alpha^2k^2t}\cos kx$, $u_{xx} = -k^2e^{-\alpha^2k^2t}\sin kx$, and $u_t = -\alpha^2k^2e^{-\alpha^2k^2t}\sin kx$.

Thus $\alpha^2 u_{xx} = u_t$.

73. $u = \dfrac{1}{\sqrt{x^2+y^2+z^2}} \quad\Rightarrow\quad u_x = \left(-\tfrac{1}{2}\right)(x^2+y^2+z^2)^{-3/2}(2x) = -x(x^2+y^2+z^2)^{-3/2}$ and

$$u_{xx} = -(x^2+y^2+z^2)^{-3/2} - x\left(-\tfrac{3}{2}\right)(x^2+y^2+z^2)^{-5/2}(2x) = \frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}}.$$

By symmetry, $u_{yy} = \dfrac{2y^2-x^2-z^2}{(x^2+y^2+z^2)^{5/2}}$ and $u_{zz} = \dfrac{2z^2-x^2-y^2}{(x^2+y^2+z^2)^{5/2}}$.

Thus $u_{xx} + u_{yy} + u_{zz} = \dfrac{2x^2-y^2-z^2+2y^2-x^2-z^2+2z^2-x^2-y^2}{(x^2+y^2+z^2)^{5/2}} = 0$.

75. Let $v = x + at$, $\ w = x - at$. Then $u_t = \dfrac{\partial[f(v)+g(w)]}{\partial t} = \dfrac{df(v)}{dv}\dfrac{\partial v}{\partial t} + \dfrac{dg(w)}{dw}\dfrac{\partial w}{\partial t} = af'(v) - ag'(w)$ and

$u_{tt} = \dfrac{\partial[af'(v) - ag'(w)]}{\partial t} = a[af''(v) + ag''(w)] = a^2[f''(v) + g''(w)]$. Similarly, by using the Chain Rule we have

$u_x = f'(v) + g'(w)$ and $u_{xx} = f''(v) + g''(w)$. Thus $u_{tt} = a^2 u_{xx}$.

77. $z = \ln(e^x + e^y) \quad\Rightarrow\quad \dfrac{\partial z}{\partial x} = \dfrac{e^x}{e^x+e^y}$ and $\dfrac{\partial z}{\partial y} = \dfrac{e^y}{e^x+e^y}$, so $\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = \dfrac{e^x}{e^x+e^y} + \dfrac{e^y}{e^x+e^y} = \dfrac{e^x+e^y}{e^x+e^y} = 1$.

$\dfrac{\partial^2 z}{\partial x^2} = \dfrac{e^x(e^x+e^y) - e^x(e^x)}{(e^x+e^y)^2} = \dfrac{e^{x+y}}{(e^x+e^y)^2}$, $\quad \dfrac{\partial^2 z}{\partial x\,\partial y} = \dfrac{0 - e^y(e^x)}{(e^x+e^y)^2} = -\dfrac{e^{x+y}}{(e^x+e^y)^2}$, and

$\dfrac{\partial^2 z}{\partial y^2} = \dfrac{e^y(e^x+e^y) - e^y(e^y)}{(e^x+e^y)^2} = \dfrac{e^{x+y}}{(e^x+e^y)^2}$. Thus

$$\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x\,\partial y}\right)^2 = \frac{e^{x+y}}{(e^x+e^y)^2}\cdot\frac{e^{x+y}}{(e^x+e^y)^2} - \left(-\frac{e^{x+y}}{(e^x+e^y)^2}\right)^2 = \frac{(e^{x+y})^2}{(e^x+e^y)^4} - \frac{(e^{x+y})^2}{(e^x+e^y)^4} = 0$$

79. If we fix $K = K_0$, $P(L, K_0)$ is a function of a single variable L, and $\dfrac{dP}{dL} = \alpha\dfrac{P}{L}$ is a separable differential equation. Then

$\dfrac{dP}{P} = \alpha\dfrac{dL}{L} \quad\Rightarrow\quad \displaystyle\int\frac{dP}{P} = \int\alpha\,\frac{dL}{L} \quad\Rightarrow\quad \ln|P| = \alpha\ln|L| + C(K_0)$, where $C(K_0)$ can depend on K_0. Then

$|P| = e^{\alpha\ln|L| + C(K_0)}$, and since $P > 0$ and $L > 0$, we have $P = e^{\alpha\ln L}e^{C(K_0)} = e^{C(K_0)}e^{\ln L^\alpha} = C_1(K_0)L^\alpha$ where

$C_1(K_0) = e^{C(K_0)}$.

81. By the Chain Rule, taking the partial derivative of both sides with respect to R_1 gives

$\dfrac{\partial R^{-1}}{\partial R}\dfrac{\partial R}{\partial R_1} = \dfrac{\partial\left[(1/R_1) + (1/R_2) + (1/R_3)\right]}{\partial R_1}$ or $-R^{-2}\dfrac{\partial R}{\partial R_1} = -R_1^{-2}$. Thus $\dfrac{\partial R}{\partial R_1} = \dfrac{R^2}{R_1^2}$.

83. By Exercise 82, $PV = mRT \quad\Rightarrow\quad P = \dfrac{mRT}{V}$, so $\dfrac{\partial P}{\partial T} = \dfrac{mR}{V}$. Also, $PV = mRT \quad\Rightarrow\quad V = \dfrac{mRT}{P}$ and $\dfrac{\partial V}{\partial T} = \dfrac{mR}{P}$.
Since $T = \dfrac{PV}{mR}$, we have $T\,\dfrac{\partial P}{\partial T}\,\dfrac{\partial V}{\partial T} = \dfrac{PV}{mR}\cdot\dfrac{mR}{V}\cdot\dfrac{mR}{P} = mR$.

85. $\dfrac{\partial K}{\partial m} = \frac{1}{2}v^2$, $\dfrac{\partial K}{\partial v} = mv$, $\dfrac{\partial^2 K}{\partial v^2} = m$. Thus $\dfrac{\partial K}{\partial m}\cdot\dfrac{\partial^2 K}{\partial v^2} = \frac{1}{2}v^2 m = K$.

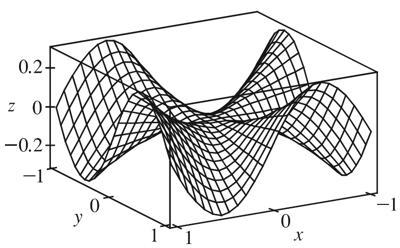
87. $f_x(x,y) = x + 4y \quad\Rightarrow\quad f_{xy}(x,y) = 4$ and $f_y(x,y) = 3x - y \quad\Rightarrow\quad f_{yx}(x,y) = 3$. Since f_{xy} and f_{yx} are continuous everywhere but $f_{xy}(x,y) \neq f_{yx}(x,y)$, Clairaut's Theorem implies that such a function $f(x,y)$ does not exist.

89. By the geometry of partial derivatives, the slope of the tangent line is $f_x(1,2)$. By implicit differentiation of $4x^2 + 2y^2 + z^2 = 16$, we get $8x + 2z\,(\partial z/\partial x) = 0 \quad\Rightarrow\quad \partial z/\partial x = -4x/z$, so when $x = 1$ and $z = 2$ we have $\partial z/\partial x = -2$. So the slope is $f_x(1,2) = -2$. Thus the tangent line is given by $z - 2 = -2(x - 1)$, $y = 2$. Taking the parameter to be $t = x - 1$, we can write parametric equations for this line: $x = 1 + t$, $y = 2$, $z = 2 - 2t$.

91. By Clairaut's Theorem, $f_{xyy} = (f_{xy})_y = (f_{yx})_y = f_{yxy} = (f_y)_{xy} = (f_y)_{yx} = f_{yyx}$.

93. Let $g(x) = f(x,0) = x(x^2)^{-3/2}e^0 = x\,|x|^{-3}$. But we are using the point $(1,0)$, so near $(1,0)$, $g(x) = x^{-2}$. Then $g'(x) = -2x^{-3}$ and $g'(1) = -2$, so using (1) we have $f_x(1,0) = g'(1) = -2$.

95. (a)



(b) For $(x,y) \neq (0,0)$,

$$f_x(x,y) = \frac{(3x^2y - y^3)(x^2+y^2) - (x^3y - xy^3)(2x)}{(x^2+y^2)^2} = \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2}$$

and by symmetry $f_y(x,y) = \dfrac{x^5 - 4x^3y^2 - xy^4}{(x^2+y^2)^2}$.

(c) $f_x(0,0) = \lim\limits_{h\to 0}\dfrac{f(h,0) - f(0,0)}{h} = \lim\limits_{h\to 0}\dfrac{(0/h^2) - 0}{h} = 0$ and $f_y(0,0) = \lim\limits_{h\to 0}\dfrac{f(0,h) - f(0,0)}{h} = 0$.

(d) By (3), $f_{xy}(0,0) = \dfrac{\partial f_x}{\partial y} = \lim\limits_{h\to 0}\dfrac{f_x(0,h) - f_x(0,0)}{h} = \lim\limits_{h\to 0}\dfrac{(-h^5 - 0)/h^4}{h} = -1$ while by (2),

$$f_{yx}(0,0) = \frac{\partial f_y}{\partial x} = \lim_{h\to 0}\frac{f_y(h,0) - f_y(0,0)}{h} = \lim_{h\to 0}\frac{h^5/h^4}{h} = 1.$$

(e) For $(x,y) \neq (0,0)$, we use a CAS to compute

$$f_{xy}(x,y) = \frac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{(x^2+y^2)^3}$$

Now as $(x,y) \to (0,0)$ along the x-axis, $f_{xy}(x,y) \to 1$ while as $(x,y) \to (0,0)$ along the y-axis, $f_{xy}(x,y) \to -1$. Thus f_{xy} isn't continuous at $(0,0)$ and Clairaut's Theorem doesn't apply, so there is no contradiction. The graphs of f_{xy} and f_{yx} are identical except at the origin, where we observe the discontinuity.