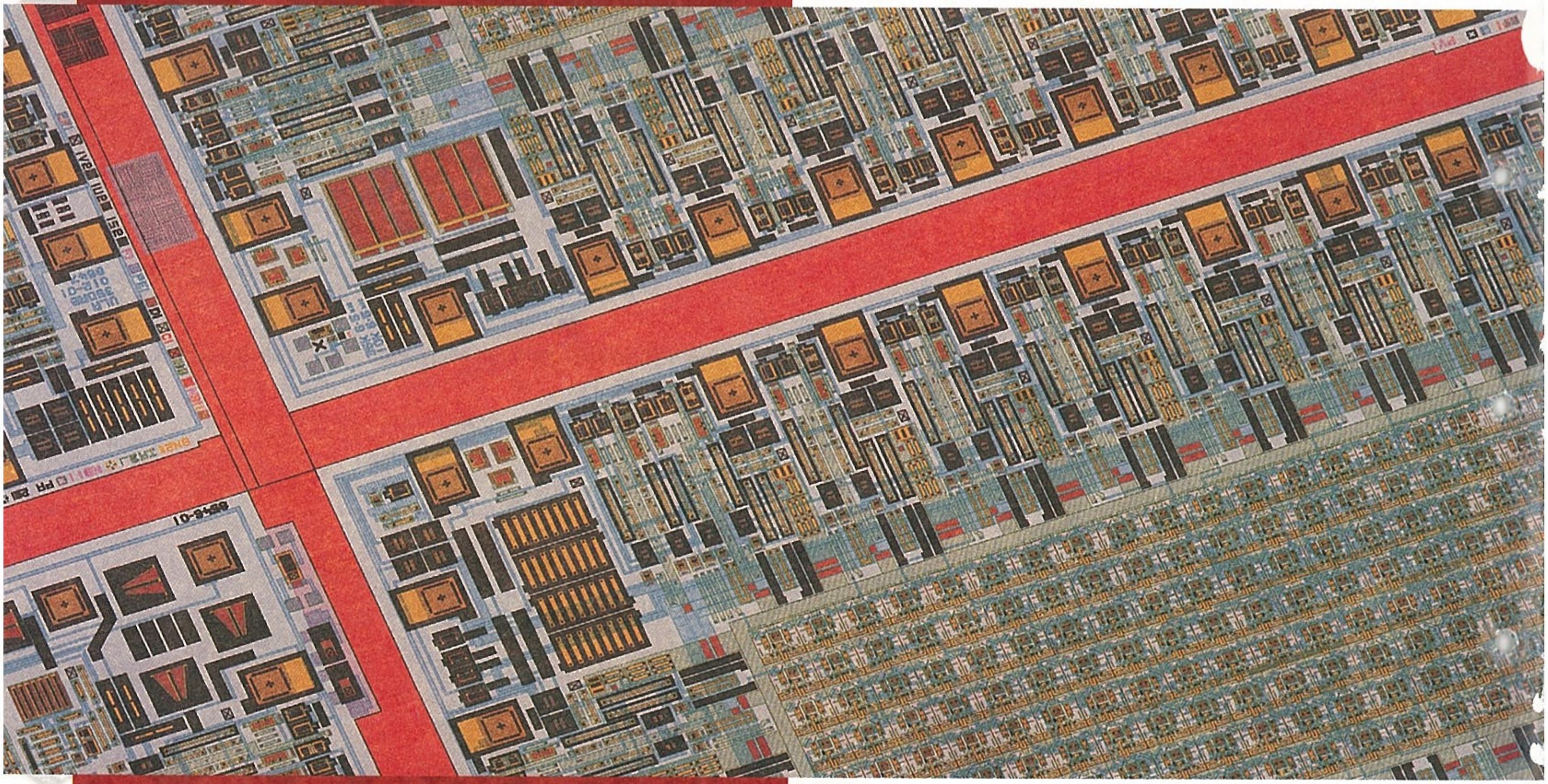
# Chapter 9 Infinite Series



One mathematical constant crucial to the analysis of the world is $\pi$. The $p$-series

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots$$

approximates the value of $\pi$. The error, or remainder, of such an approximation is the difference between the actual sum and the $n$th partial sum. For this $p$-series, the remainder is estimated by $R_n \leq 1/n$.

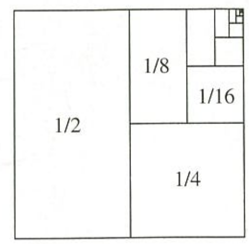
Shown here is a close-up of a high speed microprocessor chip. If a computer adds 1,000,000 terms of the $p$-series in one second, how many places of accuracy will it achieve in 24 hours? Section 9.5 provides a discussion of $p$-series.

## Chapter 9 Overview

One consequence of the early and dramatic successes that scientists enjoyed when using calculus to explain natural phenomena was that there suddenly seemed to be no limits, so to speak, on how infinite processes might be exploited. There was still considerable mystery about "infinite sums" and "division by infinitely small quantities" in the years after Newton and Leibniz, but even mathematicians normally insistent on rigorous proof were inclined to throw caution to the wind while things were working. The result was a century of unprecedented progress in understanding the physical universe. (Moreover, we can note happily in retrospect, the proofs eventually followed.)

One infinite process that had puzzled mathematicians for centuries was the summing of infinite series. Sometimes an infinite series of terms added to a number, as in

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1.$$




(You can see this by adding the areas in the "infinitely halved" unit square at the right.) But sometimes the infinite sum was infinite, as in

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots = \infty$$

(although this is far from obvious), and sometimes the infinite sum was impossible to pin down, as in

$$1 - 1 + 1 - 1 + 1 - 1 + \cdots$$

(Is it 0? Is it 1? Is it neither?).

Nonetheless, mathematicians like Gauss and Euler successfully used infinite series to derive previously inaccessible results. Laplace used infinite series to prove the stability of the solar system (although that does not stop some people from worrying about it today when they feel that "too many" planets have swung to the same side of the sun). It was years later that careful analysts like Cauchy developed the theoretical foundation for series computations, sending many mathematicians (including Laplace) back to their desks to verify their results.

Our approach in this chapter will be to discover the calculus of infinite series as the pioneers of calculus did: proceeding intuitively, accepting what works and rejecting what does not. Toward the end of the chapter we will return to the crucial question of convergence and take a careful look at it.

## 9.1 Power Series

**What you'll learn about**

- Geometric Series
- Representing Functions by Series
- Differentiation and Integration
- Identifying a Series

**. . . and why**

Power series are important in understanding the physical universe and can be used to represent functions.

### Geometric Series

The first thing to get straight about an infinite series is that it is not simply an example of addition. Addition of real numbers is a *binary* operation, meaning that we really add numbers two at a time. The only reason that $1 + 2 + 3$ makes sense as "addition" is that we can *group* the numbers and then add them two at a time. The associative property of addition guarantees that we get the same sum no matter how we group them:

$$1 + (2 + 3) = 1 + 5 = 6 \quad \text{and} \quad (1 + 2) + 3 = 3 + 3 = 6.$$

In short, a *finite sum* of real numbers always produces a real number (the result of a finite number of binary additions), but an *infinite sum* of real numbers is something else entirely. That is why we need the following definition.

**DEFINITION Infinite Series**

An **infinite series** is an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots, \quad \text{or} \quad \sum_{k=1}^{\infty} a_k.$$

The numbers $a_1, a_2, \ldots$ are the **terms** of the series; $a_n$ is the ***n*th term.**

The **partial sums** of the series form a sequence

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ s_n &= \sum_{k=1}^{n} a_k \\ &\vdots \end{aligned}$$

of real numbers, each defined as a finite sum. If the sequence of partial sums has a limit $S$ as $n \to \infty$, we say the series **converges** to the sum $S$, and we write

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots = \sum_{k=1}^{\infty} a_k = S.$$

Otherwise, we say the series **diverges.**

### EXAMPLE 1 Identifying a Divergent Series

Does the series $1 - 1 + 1 - 1 + 1 - 1 + \cdots$ converge?

**SOLUTION**

You might be tempted to pair the terms as

$$(1 - 1) + (1 - 1) + (1 - 1) + \cdots.$$

That strategy, however, requires an *infinite* number of pairings, so it cannot be justified by the associative property of addition. This is an infinite series, not a finite sum, so if it has a sum it *has to be* the limit of its sequence of partial sums,

$$1, 0, 1, 0, 1, 0, 1, \ldots.$$

Since this sequence has no limit, the series has no sum. It diverges.

*Now try Exercise 7.*

### EXAMPLE 2 Identifying a Convergent Series

Does the series

$$\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \cdots + \frac{3}{10^n} + \cdots$$

converge?

**SOLUTION**

Here is the sequence of partial sums, written in decimal form.

$$0.3,\ 0.33,\ 0.333,\ 0.3333,\ \ldots$$

This sequence has a limit $0.\overline{3}$, which we recognize as the fraction $1/3$. The series converges to the sum $1/3$.

*Now try Exercise 9.*

There is an easy way to identify some divergent series. In Exercise 62 you are asked to show that whenever an infinite series $\sum_{k=1}^{\infty} a_k$ converges, the limit of the $n$th term as $n \to \infty$ must be zero.

> If the **infinite series**
> $$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + \cdots + a_k + \cdots$$
> converges, then $\lim_{k\to\infty} a_k = 0$.

This means that if $\lim_{k\to\infty} a_k \neq 0$ the series must diverge.

The series in Example 2 is a **geometric series** because each term is obtained from its preceding term by multiplying by the same number $r$—in this case, $r = 1/10$. (The series of areas for the infinitely-halved square at the beginning of this chapter is also geometric.) The convergence of geometric series is one of the few infinite processes with which mathematicians were reasonably comfortable prior to calculus. You may have already seen the following result in a previous course.

> The **geometric series**
> $$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$$
> converges to the sum $a/(1 - r)$ if $|r| < 1$, and diverges if $|r| \geq 1$.

This completely settles the issue for geometric series. We know which ones converge and which ones diverge, and for the convergent ones we know what the sums must be. The interval $-1 < r < 1$ is the **interval of convergence.**

## EXAMPLE 3 Analyzing Geometric Series

Tell whether each series converges or diverges. If it converges, give its sum.

**(a)** $\displaystyle\sum_{n=1}^{\infty} 3\left(\frac{1}{2}\right)^{n-1}$

**(b)** $1 - \dfrac{1}{2} + \dfrac{1}{4} - \dfrac{1}{8} + \cdots + \left(-\dfrac{1}{2}\right)^{n-1} + \cdots$

**(c)** $\displaystyle\sum_{k=0}^{\infty} \left(\frac{3}{5}\right)^{k}$

**(d)** $\dfrac{\pi}{2} + \dfrac{\pi^2}{4} + \dfrac{\pi^3}{8} + \cdots$

**SOLUTION**

**(a)** First term is $a = 3$ and $r = 1/2$. The series converges to
$$\frac{3}{1 - (1/2)} = 6.$$

**(b)** First term is $a = 1$ and $r = -1/2$. The series converges to
$$\frac{1}{1 - (-1/2)} = \frac{2}{3}.$$

*continued*

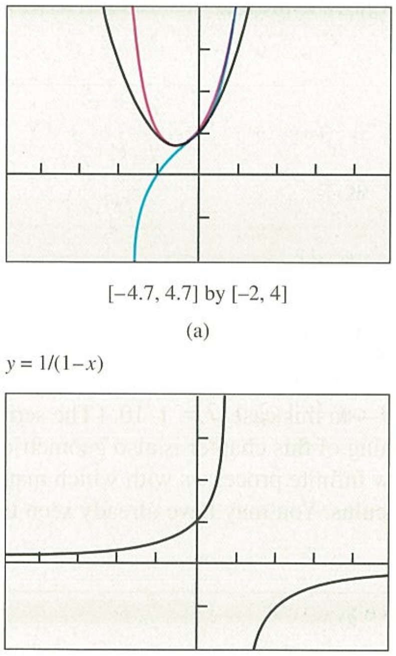



**Figure 9.1** (a) Partial sums converging to $1/(1-x)$ on the interval (0, 1). The partial sums graphed here are $1 + x + x^2$, $1 + x + x^2 + x^3$, and $1 + x + x^2 + x^3 + x^4$. (b) Notice how the graphs in (a) resemble the graph of $1/(1-x)$ on the interval $(-1, 1)$ but are not even close when $|x| \geq 1$.

**(c)** First term is $a = (3/5)^0 = 1$ and $r = 3/5$. The series converges to

$$\frac{1}{1-(3/5)} = \frac{5}{2}.$$

**(d)** In this series, $r = \pi/2 > 1$. The series diverges. **Now try Exercises 11 and 19.**

We have hardly begun our study of infinite series, but knowing everything there is to know about the convergence and divergence of an *entire class* of series (geometric) is an impressive start. Like the Renaissance mathematicians, we are ready to explore where this might lead. We are ready to bring in $x$.

## Representing Functions by Series

If $|x| < 1$, then the geometric series formula assures us that

$$1 + x + x^2 + x^3 + \cdots + x^n + \cdots = \frac{1}{1-x}.$$

Consider this statement for a moment. The expression on the right defines a function whose domain is the set of all numbers $x \neq 1$. The expression on the left defines a function whose domain is the interval of convergence, $|x| < 1$. The equality is understood to hold only on this latter domain, where both sides of the equation are defined. On this domain, the series *represents* the function $1/(1-x)$.

The partial sums of the infinite series on the left are all polynomials, so we can graph them (Figure 9.1). As expected, we see that the convergence is strong in the interval $(-1, 1)$ but breaks down when $|x| \geq 1$.

The expression $\sum_{n=0}^{\infty} x^n$ is like a polynomial in that it is a sum of coefficients times powers of $x$, but polynomials have *finite* degrees and do not suffer from divergence for the wrong values of $x$. Just as an infinite series of numbers is not a mere sum, this series of powers of $x$ is not a mere polynomial.

When we set $x = 0$ in the expression

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots,$$

we get $c_0$ on the right but $c_0 \cdot 0^0$ on the left. Since $0^0$ is not a number, this is a slight flaw in the notation, which we agree to overlook. The same situation arises when we set

$$x = a \quad \text{in} \quad \sum_{n=0}^{\infty} c_n (x-a)^n.$$

In either case, we agree that the expression will equal $c_0$. (It really *should* equal $c_0$, so we are not compromising the mathematics; we are clarifying the notation we use to convey the mathematics.)

**DEFINITION Power Series**

An expression of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$$

is a **power series centered at $x = 0$.** An expression of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots$$

is a **power series centered at $x = a$.** The term $c_n(x-a)^n$ is the **$n$th term;** the number $a$ is the **center.**

The geometric series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots$$

is a power series centered at $x = 0$. It converges on the interval $-1 < x < 1$, also centered at $x = 0$. This is typical behavior, as we will see in Section 9.4. A power series either converges for all $x$, converges on a finite interval with the same center as the series, or converges only at the center itself.

We have seen that the power series $\sum_{n=0}^{\infty} x^n$ represents the function $1/(1-x)$ on the domain $(-1, 1)$. Can we find power series to represent other functions?

**EXPLORATION 1** **Finding Power Series for Other Functions**

Given that $1/(1-x)$ is represented by the power series

$$1 + x + x^2 + \cdots + x^n + \cdots$$

on the interval $(-1, 1)$,

1. find a power series that represents $1/(1+x)$ on $(-1, 1)$.
2. find a power series that represents $x/(1+x)$ on $(-1, 1)$.
3. find a power series that represents $1/(1-2x)$ on $(-1/2, 1/2)$.
4. find a power series that represents

$$\frac{1}{x} = \frac{1}{1 + (x-1)}$$

on $(0, 2)$.

Could you have found the intervals of convergence yourself?

5. Find a power series that represents

$$\frac{1}{3x} = \frac{1}{3} \cdot \left(\frac{1}{1 + (x-1)}\right)$$

and give its interval of convergence.

## Differentiation and Integration

So far we have only represented functions by power series that happen to be geometric. The partial sums that converge to those power series, however, are *polynomials,* and we can apply calculus to polynomials. It would seem logical that the calculus of polynomials (the first rules we encountered in Chapter 3) would also apply to power series.

### EXAMPLE 4 Finding a Power Series by Differentiation

Given that $1/(1-x)$ is represented by the power series

$$1 + x + x^2 + \cdots + x^n + \cdots$$

on the interval $(-1, 1)$, find a power series to represent $1/(1-x)^2$.

**SOLUTION**

Notice that $1/(1-x)^2$ is the derivative of $1/(1-x)$. To find the power series, we differentiate both sides of the equation

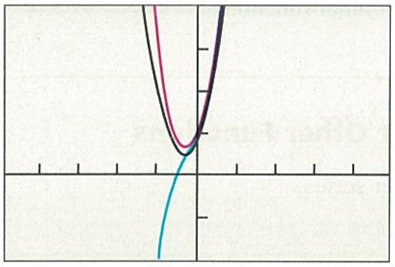
$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots.$$

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}(1 + x + x^2 + x^3 + \cdots + x^n + \cdots)$$

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1} + \cdots$$

*continued*

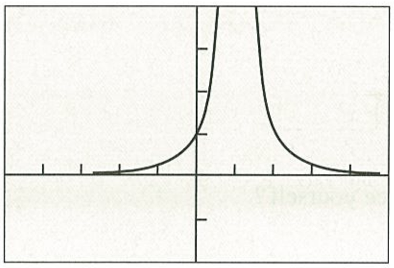
Partial Sums



[−4.7, 4.7] by [−2, 4]

(a)

$y = 1/(1-x)^2$



[−4.7, 4.7] by [−2, 4]

(b)

**Figure 9.2** (a) The polynomial partial sums of the power series we derived for (b) $1/(1-x)^2$ seem to converge on the open interval $(-1, 1)$. (Example 4)

What about the interval of convergence? Since the original series converges for $-1 < x < 1$, it would seem that the differentiated series ought to converge on the same open interval. Graphs (Figure 9.2) of the partial sums $1 + 2x + 3x^2$, $1 + 2x + 3x^2 + 4x^3$, and $1 + 2x + 3x^2 + 4x^3 + 5x^4$ suggest that this is the case (although such empirical evidence does not constitute a proof). ***Now try Exercise 27.***

The basic theorem about differentiating power series is the following.

**THEOREM 1 Term-by-Term Differentiation**

If $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots$

converges for $|x - a| < R$, then the series

$$\sum_{n=1}^{\infty} nc_n(x-a)^{n-1} = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots + nc_n(x-a)^{n-1} + \cdots,$$

obtained by differentiating the series for $f$ term by term, converges for $|x - a| < R$ and represents $f'(x)$ on that interval. If the series for $f$ converges for all $x$, then so does the series for $f'$.

Theorem 1 says that if a power series is differentiated term by term, the new series will converge on the same interval to the derivative of the function represented by the original series. This gives a way to generate new connections between functions and series.

Another way to reveal new connections between functions and series is by integration.

## EXAMPLE 5 Finding a Power Series by Integration

Given that

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots + (-x)^n + \cdots, \qquad -1 < x < 1$$

(Exploration 1, part 1), find a power series to represent $\ln(1 + x)$.

**SOLUTION**

Recall that $1/(1 + x)$ is the derivative of $\ln(1 + x)$. We can therefore integrate the series for $1/(1 + x)$ to obtain a series for $\ln(1 + x)$ (no absolute value bars are necessary because $(1 + x)$ is positive for $-1 < x < 1$).

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots + (-x)^n + \cdots$$
$$= 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + \cdots$$

$$\int_0^x \frac{1}{1+t}\,dt = \int_0^x (1 - t + t^2 - t^3 + \cdots + (-1)^n t^n + \cdots)\,dt$$ *t is a dummy variable.*

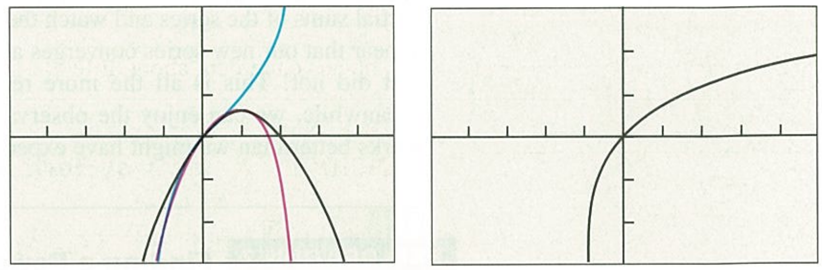
$$\ln(1+t)\Big]_0^x = \left[t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots + (-1)^n\frac{t^{n+1}}{n+1} + \cdots\right]_0^x$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^n\frac{x^{n+1}}{n+1} + \cdots$$

*continued*

It would seem logical for the new series to converge where the original series converges, on the open interval $(-1, 1)$. The graphs of the partial sums in Figure 9.3 support this idea. **Now try Exercise 33.**

Partial Sums $\qquad$ $y = \ln(1 + x)$



[–5, 5] by [–3, 3] $\qquad$ [–5, 5] by [–3, 3]

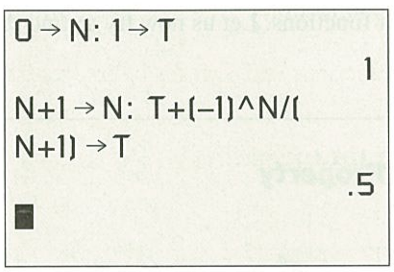
(a) $\qquad$ (b)

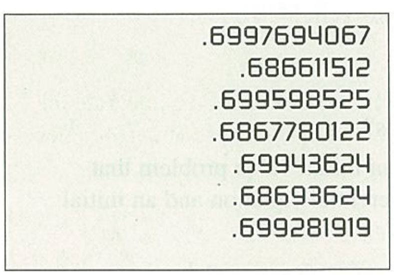
**Figure 9.3** (a) The graphs of the partial sums

$$x - \frac{x^2}{2}, \quad x - \frac{x^2}{2} + \frac{x^3}{3}, \quad \text{and} \quad x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$$

closing in on (b) the graph of $\ln(1 + x)$ over the interval $(-1, 1)$. (Example 5)

Some calculators have a *sequence mode* that enables you to generate a sequence of partial sums, but you can also do it with simple commands on the home screen. Try entering the two multiple-step commands shown on the first screen below.




If you are successful, then every time you hit ENTER, the calculator will display the next partial sum of the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^n}{n+1} + \cdots.$$

The second screen shows the result of about 80 ENTERs. The sequence certainly seems to be converging to $\ln 2 = 0.6931471806\ldots$.

The idea that the integrated series in Example 5 converges to $\ln(1 + x)$ for all $x$ between $-1$ and 1 is confirmed by the following theorem.

**THEOREM 2 Term-by-Term Integration**

If $f(x) = \sum_{n=0}^{\infty} c_n(x - a)^n = c_0 + c_1(x - a) + c_2(x - a)^2 + \cdots + c_n(x - a)^n + \cdots$

converges for $|x - a| < R$, then the series

$$\sum_{n=0}^{\infty} c_n \frac{(x - a)^{n+1}}{n + 1} = c_0(x - a) + c_1 \frac{(x - a)^2}{2} + c_2 \frac{(x - a)^3}{3} + \cdots + c_n \frac{(x - a)^{n+1}}{n + 1} + \cdots,$$

obtained by integrating the series for $f$ term by term, converges for $|x - a| < R$ and represents $\int_a^x f(t)\,dt$ on that interval. If the series for $f$ converges for all $x$, then so does the series for the integral.

Theorem 2 says that if a power series is integrated term by term, the new series will converge on the same interval to the integral of the function represented by the original series.

There is still more to be learned from Example 5. The original equation

$$\frac{1}{1 + x} = 1 - x + x^2 - x^3 + \cdots + (-x)^n + \cdots$$

clearly diverges at $x = 1$ (see Example 1). The behavior is not so apparent, however, for the new equation

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^n \frac{x^{n+1}}{n + 1} + \cdots.$$

If we let $x = 1$ on both sides of the previous equation, we get

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^n}{n+1} + \cdots,$$

which looks like a reasonable statement. It looks even more reasonable if you look at the partial sums of the series and watch them converge toward ln 2 (see margin note). It would appear that our new series converges at 1 despite the fact that we obtained it from a series that did not! This is all the more reason to take a careful look at convergence later. Meanwhile, we can enjoy the observation that we have created a series that apparently works better than we might have expected and better than Theorem 2 could guarantee.

**EXPLORATION 2** **Finding a Power Series for $\tan^{-1} x$**

1. Find a power series that represents $1/(1 + x^2)$ on $(-1, 1)$.
2. Use the technique of Example 5 to find a power series that represents $\tan^{-1} x$ on $(-1, 1)$.
3. Graph the first four partial sums. Do the graphs suggest convergence on the open interval $(-1, 1)$?
4. Do you think that the series for $\tan^{-1} x$ converges at $x = 1$? Can you support your answer with evidence?

## Identifying a Series

So far we have been finding power series to represent functions. Let us now try to find the function that a given power series represents.

**EXPLORATION 3** **A Series with a Curious Property**

Define a function $f$ by a power series as follows:

$$f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + \cdots.$$

1. Find $f'(x)$.
2. Find $f(0)$.
3. What well-known function do you suppose $f$ is?
4. Use your responses to parts 1 and 2 to set up an initial value problem that the function $f$ must solve. You will need a differential equation and an initial condition.
5. Solve the initial value problem to prove your conjecture in part 3.
6. Graph the first three partial sums. What appears to be the interval of convergence?
7. Graph the next three partial sums. Did you underestimate the interval of convergence?

The correct answer to part 7 in Exploration 3 above is "yes," unless you had the keen insight (or reckless bravado) to answer "all real numbers" in part 6. We will prove the remarkable fact that this series converges *for all x* when we revisit the question of convergence of this series in Section 9.3, Example 4.

## Quick Review 9.1 *(For help, go to Section 8.1.)*

In Exercises 1 and 2, find the first four terms and the 30th term of the sequence

$$\{u_n\}_{n=1}^{\infty} = \{u_1, u_2, \dots, u_n, \dots\}.$$

**1.** $u_n = \dfrac{4}{n+2}$ **2.** $u_n = \dfrac{(-1)^n}{n}$

In Exercises 3 and 4, the sequences are geometric ($a_{n+1}/a_n = r$, a constant). Find

**(a)** the common ratio $r$. **(b)** the tenth term.

**(c)** a rule for the $n$th term.

**3.** $\{2, 6, 18, 54, \dots\}$ **4.** $\{8, -4, 2, -1, \dots\}$

In Exercises 5–10,

**(a)** graph the sequence $\{a_n\}$.

**(b)** determine $\lim_{n\to\infty} a_n$.

**5.** $a_n = \dfrac{1-n}{n^2}$ **6.** $a_n = \left(1+\dfrac{1}{n}\right)^n$

**7.** $a_n = (-1)^n$ **8.** $a_n = \dfrac{1-2n}{1+2n}$

**9.** $a_n = 2 - \dfrac{1}{n}$ **10.** $a_n = \dfrac{\ln(n+1)}{n}$

## Section 9.1 Exercises

**1.** Replace the $*$ with an expression that will generate the series

$$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots.$$

**(a)** $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1}\left(\frac{1}{*}\right)$ **(b)** $\displaystyle\sum_{n=0}^{\infty} (-1)^{n}\left(\frac{1}{*}\right)$

**(c)** $\displaystyle\sum_{n=*}^{\infty} (-1)^{n}\left(\frac{-1}{(n-2)^2}\right)$

**2.** Write an expression for the $n$th term, $a_n$.

**(a)** $\displaystyle\sum_{n=0}^{\infty} a_n = 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots$

**(b)** $\displaystyle\sum_{n=1}^{\infty} a_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$

**(c)** $\displaystyle\sum_{n=0}^{\infty} a_n = 5 + 0.5 + 0.05 + 0.005 + 0.0005 + \cdots$

In Exercises 3–6, tell whether the series is the same as

$$\sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)^{n-1}.$$

**3.** $\displaystyle\sum_{n=1}^{\infty} -\left(\frac{1}{2}\right)^{n-1}$ **4.** $\displaystyle\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^{n}$

**5.** $\displaystyle\sum_{n=0}^{\infty} (-1)^n\left(\frac{1}{2}\right)^{n}$ **6.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{2^{n-1}}$

In Exercises 7–10, compute the limit of the partial sums to determine whether the series converges or diverges.

**7.** $1 + 1.1 + 1.11 + 1.111 + 1.1111 + \cdots$

**8.** $2 - 1 + 1 - 1 + 1 - 1 + \cdots$

**9.** $\dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} + \cdots + \dfrac{1}{2^k} + \cdots$

**10.** $3 + 0.5 + 0.05 + 0.005 + 0.0005 + \cdots$

In Exercises 11–20, tell whether the series converges or diverges. If it converges, give its sum.

**11.** $1 + \dfrac{2}{3} + \left(\dfrac{2}{3}\right)^2 + \left(\dfrac{2}{3}\right)^3 + \cdots + \left(\dfrac{2}{3}\right)^n + \cdots$

**12.** $1 - 2 + 3 - 4 + 5 - \cdots + (-1)^n(n+1) + \cdots$

**13.** $\displaystyle\sum_{n=0}^{\infty} \left(\frac{5}{4}\right)\left(\frac{2}{3}\right)^n$ **14.** $\displaystyle\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)\left(\frac{5}{4}\right)^n$

**15.** $\displaystyle\sum_{n=0}^{\infty} \cos(n\pi)$

**16.** $3 - 0.3 + 0.03 - 0.003 + 0.0003 - \cdots + 3(-0.1)^n + \cdots$

**17.** $\displaystyle\sum_{n=0}^{\infty} \sin^n\left(\frac{\pi}{4} + n\pi\right)$

**18.** $\dfrac{1}{2} + \dfrac{2}{3} + \dfrac{3}{4} + \dfrac{4}{5} + \cdots + \dfrac{n}{n+1} + \cdots$

**19.** $\displaystyle\sum_{n=1}^{\infty} \left(\frac{e}{\pi}\right)^n$ **20.** $\displaystyle\sum_{n=0}^{\infty} \frac{5^n}{6^{n+1}}$

In Exercises 21–24, find the interval of convergence and the function of $x$ represented by the geometric series.

**21.** $\displaystyle\sum_{n=0}^{\infty} 2^n x^n$ **22.** $\displaystyle\sum_{n=0}^{\infty} (-1)^n (x+1)^n$

**23.** $\displaystyle\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n (x-3)^n$ **24.** $\displaystyle\sum_{n=0}^{\infty} 3\left(\frac{x-1}{2}\right)^n$

In Exercises 25 and 26, find the values of $x$ for which the geometric series converges and find the function of $x$ it represents.

**25.** $\displaystyle\sum_{n=0}^{\infty} \sin^n x$ **26.** $\displaystyle\sum_{n=0}^{\infty} \tan^n x$

In Exercises 27–30, use the series and the function $f(x)$ that it represents from the indicated exercise to find a power series for $f'(x)$.

**27.** Exercise 21 **28.** Exercise 22

**29.** Exercise 23 **30.** Exercise 24

In Exercises 31–34, use the series and the function $f(x)$ that it represents from the indicated exercise to find a power series for $\int_0^x f(t)\,dt$.

**31.** Exercise 21 **32.** Exercise 22

**33.** Exercise 23 **34.** Exercise 24

**35. Writing to Learn** Each of the following series diverges in a slightly different way. Explain what is happening to the sequence of partial sums in each case.

**(a)** $\sum_{n=1}^{\infty} 2n$ **(b)** $\sum_{n=0}^{\infty} (-1)^n$ **(c)** $\sum_{n=1}^{\infty} (-1)^n(2n)$

**36.** Prove that $\sum_{n=0}^{\infty} \frac{e^{n\pi}}{\pi^{ne}}$ diverges.

**37.** Solve for $x$: $\sum_{n=0}^{\infty} x^n = 20.$

**38. Writing to Learn** Explain how it is possible, given any real number at all, to construct an infinite series of non-zero terms that converges to it.

**39.** Make up a geometric series $\sum ar^{n-1}$ that converges to the number 5 if

**(a)** $a = 2$ **(b)** $a = 13/2$

In Exercises 40 and 41, express the repeating decimal as a geometric series and find its sum.

**40.** $0.\overline{21}$ **41.** $0.\overline{234}$

In Exercises 42–47, express the number as the ratio of two integers.

**42.** $0.\overline{7} = 0.7777\ldots$

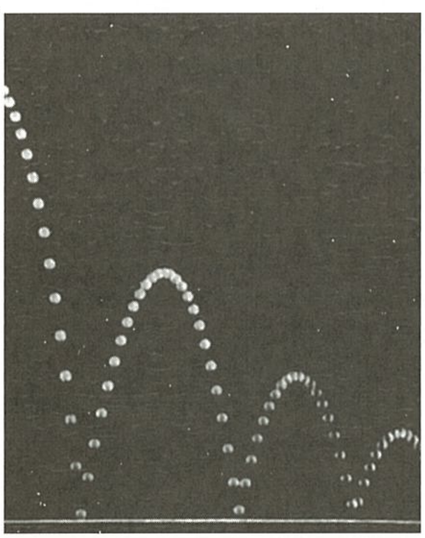
**43.** $0.\overline{d} = 0.dddd\ldots,$ where $d$ is a digit

**44.** $0.0\overline{6} = 0.06666\ldots$ **45.** $1.\overline{414} = 1.414\ 414\ 414\ldots$

**46.** $1.24\overline{123} = 1.24\ 123\ 123\ 123\ldots$

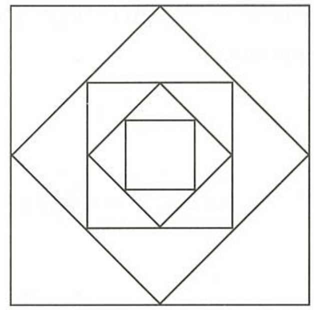
**47.** $3.\overline{142857} = 3.142857\ 142857\ldots$

**48. *Bouncing Ball*** A ball is dropped from a height of 4 m. Each time it strikes the pavement after falling from a height of $h$ m, it rebounds to a height of $0.6h$ m. Find the total distance the ball travels up and down.

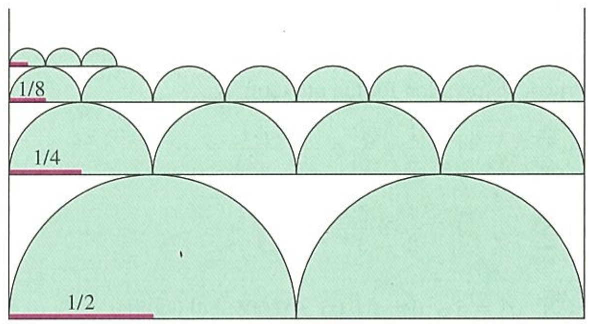


**49. *(Continuation of Exercise 48)*** Find the total number of seconds that the ball in Exercise 48 travels. (*Hint:* A freely falling ball travels $4.9t^2$ meters in $t$ seconds, so it will fall $h$ meters in $\sqrt{h/4.9}$ seconds. Bouncing from ground to apex takes the same time as falling from apex to ground.)

**50. *Summing Areas*** The figure below shows the first five of an infinite sequence of squares. The outermost square has an area of 4 $m^2$. Each of the other squares is obtained by joining the midpoints of the sides of the preceding square. Find the sum of the areas of all the squares.



**51. *Summing Areas*** The accompanying figure shows the first three rows and part of the fourth row of a sequence of rows of semicircles. There are $2^n$ semicircles in the $n$th row, each of radius $1/(2^n)$. Find the sum of the areas of all the semicircles.




**52. *Sum of a Finite Geometric Progression*** Let $a$ and $r$ be real numbers with $r \neq 1$, and let

$$S = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}.$$

**(a)** Find $S - rS$.

**(b)** Use the result in part (a) to show that $S = \frac{a - ar^n}{1 - r}$.

**53. *Sum of a Convergent Geometric Series*** Exercise 52 gives a formula for the $n$th partial sum of an infinite geometric series. Use this formula to show that $\sum_{n=1}^{\infty} ar^{n-1}$ diverges when $|r| \geq 1$ and converges to $a/(1 - r)$ when $|r| < 1$.

In Exercises 54–59, find a power series to represent the given function and identify its interval of convergence. When writing the power series, include a formula for the $n$th term.

**54.** $\frac{1}{1 + 3x}$ **55.** $\frac{x}{1 - 2x}$

**56.** $\frac{3}{1 - x^3}$ **57.** $\frac{1}{1 + (x - 4)}$

**58.** $\frac{1}{4x} = \frac{1}{4}\left(\frac{1}{1 + (x - 1)}\right)$ **59.** $\frac{1}{2 - x}$ (*Hint:* Rewrite $2 - x$.)

**60.** Find the value of $b$ for which $1 + e^b + e^{2b} + e^{3b} + \cdots = 9$.

**61.** Let $S$ be the series $\sum_{n=0}^{\infty} \left(\frac{t}{1+t}\right)^n, \quad t \neq 0$.

**(a)** Find the value to which $S$ converges when $t = 1$.

**(b)** Determine all values of $t$ for which $S$ converges.

**(c)** Find all values of $t$ that make the sum of $S$ greater than 10.

**62.** ***nth Term Test*** Assume that the series $\sum_{k=1}^{\infty} a_k$ converges to $S$.

**(a)** **Writing to Learn** Explain why $\lim_{n\to\infty} \sum_{k=1}^{k=n} a_k = S$.

**(b)** Show that $S_n = S_{n-1} + a_n$, where $S_n$ denotes the $n$th partial sum of the series.

**(c)** Show that $\lim_{n\to\infty} a_n = 0$.

**63.** ***A Series for ln x*** Starting with the power series found for $1/x$ in Exploration 1, Part 4, find a power series for $\ln x$ centered at $x = 1$.

**64.** ***Differentiation*** Use differentiation to find a series for $f(x) = 2/(1-x)^3$. What is the interval of convergence of your series?

**65.** **Group Activity** ***Intervals of Convergence*** How much can the interval of convergence of a power series be changed by integration or differentiation? To be specific, suppose that the power series

$$f(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$$

converges for $-1 < x < 1$ and diverges for all other values of $x$.

**(a)** **Writing to Learn** Could the series obtained by integrating the series for $f$ term by term possibly converge for $-2 < x < 2$? Explain. (*Hint:* Apply Theorem 1, not Theorem 2.)

**(b)** **Writing to Learn** Could the series obtained by differentiating the series for $f$ term by term possibly converge for $-2 < x < 2$? Explain.

## Standardized Test Questions

You should solve the following problems without using a graphing calculator.

**66.** **True or False** The series

$$\frac{1}{2} + \frac{1.01}{2} + \frac{(1.01)^2}{2} + \cdots + \frac{(1.01)^n}{2} + \cdots$$

converges. Justify your answer.

**67.** **True or False** The series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$ diverges. Justify your answer.

**68.** **Multiple Choice** To which of the following numbers does the series $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots$ converge?

(A) 2/3 (B) 9/8 (C) 3/2 (D) 2 (E) It diverges

In Exercises 69–71, use the geometric series $\sum_{n=0}^{\infty} (x-1)^n$, which represents the function $f(x)$.

**69.** **Multiple Choice** Find the values of $x$ for which the series converges.

(A) $0 < x < 2$ (B) $0 < x < 1$ (C) $-1 < x < 0$
(D) $-1 < x < 1$ (E) $-2 < x < 0$

**70.** **Multiple Choice** Which of the following is the function that the power series represents?

(A) $\frac{1}{x-1}$ (B) $\frac{1}{1-2x}$ (C) $-\frac{1}{x}$ (D) $\frac{1}{x-2}$ (E) $\frac{1}{2-x}$

**71.** **Multiple Choice** Which of the following is a function that $\int_0^x f(t)\,dt$ represents?

(A) $-\ln\left(\frac{x-2}{2}\right)$ (B) $\ln\left(\frac{x-2}{2}\right)$ (C) $\frac{1}{(x-2)^2}$
(D) $-\ln\left(\frac{|x-2|}{2}\right)$ (E) $\ln\left(\frac{|x-2|}{2}\right)$

## Exploration

**72.** Let $f(t) = \frac{4}{1+t^2}$ and $G(x) = \int_0^x f(t)\,dt$.

**(a)** Find the first four nonzero terms and the general term for a power series for $f(t)$ centered at $t = 0$.

**(b)** Find the first four nonzero terms and the general term for a power series for $G(x)$ centered at $x = 0$.

**(c)** Find the interval of convergence of the power series in part (a).

**(d)** The interval of convergence of the power series in part (b) is almost the same as the interval in part (c), but includes two more numbers. What are the numbers?

## Extending the Ideas

The sequence $\{a_n\}$ **converges** to the number $L$ if to every positive number $\varepsilon$ there corresponds an integer $N$ such that for all $n$,

$$n > N \quad\Rightarrow\quad |a_n - L| < \varepsilon.$$

$L$ is the **limit** of the sequence and we write $\lim_{n\to\infty} a_n = L$. If no such number $L$ exists, we say that $\{a_n\}$ **diverges.**

**73.** ***Tail of a Sequence*** Prove that if $\{a_n\}$ is a convergent sequence, then to every positive number $\varepsilon$ there corresponds an integer $N$ such that for all $m$ and $n$,

$$m > N \quad\text{and}\quad n > N \quad\Rightarrow\quad |a_m - a_n| < \varepsilon.$$

(*Hint:* Let $\lim_{n\to\infty} a_n = L$. As the terms approach $L$, how far apart can they be?)

**74.** ***Uniqueness of Limits*** Prove that limits of sequences are unique. That is, show that if $L_1$ and $L_2$ are numbers such that $\lim_{n\to\infty} a_n = L_1$ and $\lim_{n\to\infty} a_n = L_2$, then $L_1 = L_2$.

**75.** ***Limits and Subsequences*** Prove that if two subsequences of a sequence $\{a_n\}$ have different limits $L_1 \neq L_2$, then $\{a_n\}$ diverges.

**76.** ***Limits and Asymptotes***

**(a)** Show that the sequence with $n$th term $a_n = (3n+1)/(n+1)$ converges.

**(b)** If $\lim_{n\to\infty} a_n = L$, explain why $y = L$ is a horizontal asymptote of the graph of the function

$$f(x) = \frac{3x+1}{x+1}$$

obtained by replacing $n$ by $x$ in the $n$th term.

# 9.2 Taylor Series

**What you'll learn about**

- Constructing a Series
- Series for sin $x$ and cos $x$
- Beauty Bare
- Maclaurin and Taylor Series
- Combining Taylor Series
- Table of Maclaurin Series

**. . . and why**

The partial sums of a Taylor series are polynomials that can be used to approximate the function represented by the series.

## Constructing a Series

A comprehensive understanding of geometric series served us well in Section 9.1, enabling us to find power series to represent certain functions, and functions that are equivalent to certain power series (all of these equivalencies being subject to the condition of convergence). In this section we learn a more general technique for constructing power series, one that makes good use of the tools of calculus.

Let us start by constructing a polynomial.

**EXPLORATION 1 Designing a Polynomial to Specifications**

Construct a polynomial $P(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$ with the following behavior at $x = 0$:

$$P(0) = 1,$$
$$P'(0) = 2,$$
$$P''(0) = 3,$$
$$P'''(0) = 4, \text{ and}$$
$$P^{(4)}(0) = 5.$$

This task might look difficult at first, but when you try it you will find that the predictability of differentiation when applied to polynomials makes it straightforward. (Be sure to check this out before you move on.)

There is nothing special about the number of derivatives in Exploration 1. We could have prescribed the value of the polynomial and its first $n$ derivatives at $x = 0$ for any $n$ and found a polynomial of degree at most $n$ to match. Our plan now is to use the technique of Exploration 1 to construct polynomials that approximate functions by emulating their behavior at 0.

**EXAMPLE 1 Approximating ln (1 + $x$) by a Polynomial**

Construct a polynomial $P(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$ that matches the behavior of $\ln(1 + x)$ at $x = 0$ through its first four derivatives. That is,

$$P(0) = \ln(1+x) \quad \text{at } x = 0,$$
$$P'(0) = (\ln(1+x))' \quad \text{at } x = 0,$$
$$P''(0) = (\ln(1+x))'' \quad \text{at } x = 0,$$
$$P'''(0) = (\ln(1+x))''' \quad \text{at } x = 0, \text{ and}$$
$$P^{(4)}(0) = (\ln(1+x))^{(4)} \quad \text{at } x = 0.$$

**SOLUTION**

This is just like Exploration 1, except that first we need to find out what the numbers are.

$$P(0) = \ln(1+x)\Big|_{x=0} = 0$$

$$P'(0) = \frac{1}{1+x}\Big|_{x=0} = 1$$

*continued*

$$P''(0) = -\frac{1}{(1+x)^2}\bigg|_{x=0} = -1$$

$$P'''(0) = \frac{2}{(1+x)^3}\bigg|_{x=0} = 2$$

$$P^{(4)}(0) = -\frac{6}{(1+x)^4}\bigg|_{x=0} = -6$$

In working through Exploration 1, you probably noticed that the coefficient of the term $x^n$ in the polynomial we seek is $P^{(n)}(0)$ divided by $n!$. The polynomial is

$$P(x) = 0 + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}.$$

*Now try Exercise 1.*

We have just constructed the fourth order **Taylor polynomial** for the function $\ln(1+x)$ at $x = 0$. You might recognize it as the beginning of the power series we discovered for $\ln(1+x)$ in Example 5 of Section 9.1, when we came upon it by integrating a geometric series. If we keep going, of course, we will gradually reconstruct that entire series one term at a time, improving the approximation near $x = 0$ with every term we add. The series is called the **Taylor series** generated by the function $\ln(1+x)$ at $x = 0$.

You might also recall Figure 9.3, which shows how the polynomial approximations converge nicely to $\ln(1+x)$ near $x = 0$, but then gradually peel away from the curve as $x$ gets farther away from 0 in either direction. Given that the coefficients are totally determined by specifying behavior at $x = 0$, that is exactly what we ought to expect.

## Series for sin *x* and cos *x*

We can use the technique of Example 1 to construct Taylor series about $x = 0$ for any function, as long as we can keep taking derivatives there. Two functions that are particularly well-suited for this treatment are the sine and cosine.

### EXAMPLE 2 Constructing a Power Series for sin *x*

Construct the seventh order Taylor polynomial and the Taylor series for $\sin x$ at $x = 0$.

**SOLUTION**

We need to evaluate $\sin x$ and its first seven derivatives at $x = 0$. Fortunately, this is not hard to do.

$$\sin(0) = 0$$
$$\sin'(0) = \cos(0) = 1$$
$$\sin''(0) = -\sin(0) = 0$$
$$\sin'''(0) = -\cos(0) = -1$$
$$\sin^{(4)}(0) = \sin(0) = 0$$
$$\sin^{(5)}(0) = \cos(0) = 1$$
$$\vdots$$

The pattern 0, 1, 0, −1 will keep repeating forever.

*continued*

The unique seventh order Taylor polynomial that matches all these derivatives at $x = 0$ is

$$P_7(x) = 0 + 1x - 0x^2 - \frac{1}{3!}x^3 + 0x^4 + \frac{1}{5!}x^5 - 0x^6 - \frac{1}{7!}x^7$$
$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}.$$

$P_7$ is the seventh order Taylor polynomial for $\sin x$ at $x = 0$. (It also happens to be of seventh *degree,* but that does not always happen. For example, you can see that $P_8$ for $\sin x$ will be the same polynomial as $P_7$.)

To form the Taylor series, we just keep on going:

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

*Now try Exercise 3.*

**EXPLORATION 2** **A Power Series for the Cosine**

Group Activity

1. Construct the sixth order Taylor polynomial and the Taylor series at $x = 0$ for $\cos x$.
2. Compare your method for attacking part 1 with the methods of other groups. Did anyone find a shortcut?

## Beauty Bare

Edna St. Vincent Millay, an early twentieth-century American poet, referring to the experience of simultaneously seeing and understanding the geometric underpinnings of nature, wrote "Euclid alone has looked on Beauty bare." In case you have never experienced that sort of reverie when gazing upon something geometric, we intend to give you that opportunity now.

In Example 2 we constructed a power series for $\sin x$ by matching the behavior of $\sin x$ at $x = 0$. Let us graph the first nine partial sums together with $y = \sin x$ to see how well we did (Figure 9.4).

Behold what is occurring here! These polynomials were constructed to mimic the behavior of $\sin x$ near $x = 0$. The *only* information we used to construct the coefficients of these polynomials was information about the sine function and its derivatives at 0. Yet, somehow, the information at $x = 0$ is producing a series whose graph not only looks like sine near the origin, but appears to be a clone of the *entire* sine curve. This is no deception, either; we will show in Section 9.3, Example 3 that the Taylor series for $\sin x$ does, in fact, converge to $\sin x$ over the entire real line. We have managed to construct an entire function by knowing its behavior at a single point! (The same is true about the series for $\cos x$ found in Exploration 2.)

We still must remember that convergence is an infinite process. Even the one-billionth order Taylor polynomial begins to peel away from $\sin x$ as we move away from 0, although imperceptibly at first, and eventually becomes unbounded, as any polynomial must. Nonetheless, we can approximate the sine of *any* number to whatever accuracy we want if we just have the patience to work out enough terms of this series!

This kind of dramatic convergence does not occur for all Taylor series. The Taylor polynomials for $\ln(1 + x)$ do not converge outside the interval from $-1$ to 1, no matter how many terms we add.

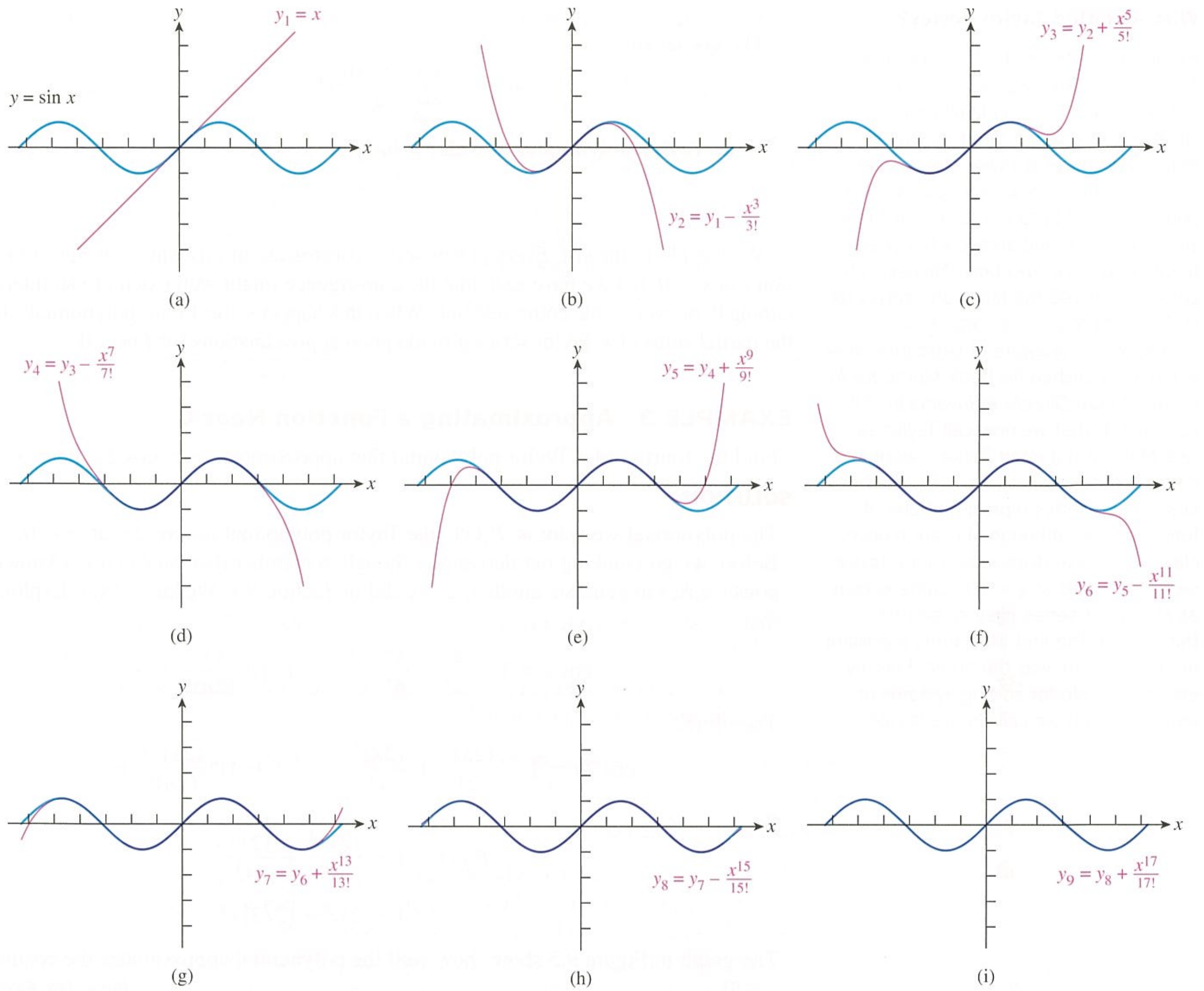



**Figure 9.4** $y = \sin x$ and its nine Taylor polynomials $P_1, P_3, \ldots, P_{17}$ for $-2\pi \le x \le 2\pi$. Try graphing these functions in the window $[-2\pi, 2\pi]$ by $[-5, 5]$.

## Maclaurin and Taylor Series

If we generalize the steps we followed in constructing the coefficients of the power series in this section so far, we arrive at the following definition.

**DEFINITION Taylor Series Generated by *f* at $x = 0$ (Maclaurin Series)**

Let $f$ be a function with derivatives of all orders throughout some open interval containing 0. Then the **Taylor series generated by $f$ at $x = 0$** is

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k.$$

This series is also called the **Maclaurin series generated by $f$.**

*continued*

### Who invented Taylor series?

Brook Taylor (1685–1731) did not invent Taylor series, and Maclaurin series were not developed by Colin Maclaurin (1698–1746). James Gregory was already working with Taylor series when Taylor was only a few years old, and he published the Maclaurin series for tan $x$, sec $x$, arctan $x$, and arcsec $x$ ten years before Maclaurin was born. Nicolaus Mercator discovered the Maclaurin series for ln $(1 + x)$ at about the same time.

Taylor was unaware of Gregory's work when he published his book *Methodus Incrementorum Directa et Inversa* in 1715, containing what we now call Taylor series. Maclaurin quoted Taylor's work in a calculus book he wrote in 1742. The book popularized series representations of functions and although Maclaurin never claimed to have discovered them, Taylor series centered at $x = 0$ became known as Maclaurin series. History evened things up in the end. Maclaurin, a brilliant mathematician, was the original discoverer of the rule for solving systems of equations that we call Cramer's rule.

The partial sum

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$$

is the **Taylor polynomial of order $n$ for $f$ at $x = 0$.**

We use $f^{(0)}$ to mean $f$. *Every* power series constructed in this way converges to the function $f$ at $x = 0$, but we have seen that the convergence might well extend to an interval containing 0, or even to the entire real line. When this happens, the Taylor polynomials that form the partial sums of a Taylor series provide good approximations for $f$ near 0.

## EXAMPLE 3 Approximating a Function Near 0

Find the fourth order Taylor polynomial that approximates $y = \cos 2x$ near $x = 0$.

**SOLUTION**

The polynomial we want is $P_4(x)$, the Taylor polynomial for $\cos 2x$ at $x = 0$. Before we go cranking out derivatives though, remember that we can use a known power series to generate another, as we did in Section 9.1. We know from Exploration 2 that

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots.$$

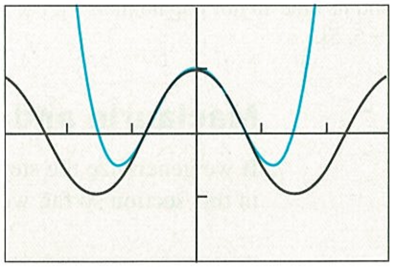
Therefore,

$$\cos 2x = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \cdots + (-1)^n \frac{(2x)^{2n}}{(2n)!} + \cdots.$$

So,

$$P_4(x) = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!}$$
$$= 1 - 2x^2 + (2/3)x^4.$$

The graph in Figure 9.5 shows how well the polynomial approximates the cosine near $x = 0$. *Now try Exercise 5.*



[–3, 3] by [–2, 2]

**Figure 9.5** The graphs of $y = 1 - 2x^2 + (2/3)x^4$ and $y = \cos 2x$ near $x = 0$. (Example 3)

These polynomial approximations can be useful in a variety of ways. For one thing, it is easy to do calculus with polynomials. For another thing, polynomials are built using only the two basic operations of addition and multiplication, so computers can handle them easily.

## EXPLORATION 3 Approximating sin 13

How many terms of the series

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

are required to approximate sin 13 accurate to the third decimal place?

1. Find sin 13 on your calculator (radians, of course).
2. Enter these two multiple-step commands on your home screen. They will give you the first order and second order Taylor polynomial approximations for sin 13. Notice that the second order approximation, in particular, is not very good.

```
0→N: 13→T
                          13
N+1→N: T+(-1)^N*1
3^(2N+1)/(2N+1)!
→T
                 -353.1666667
```

3. Continue to hit ENTER. Each time you will add one more term to the Taylor polynomial approximation. Be patient; things will get worse before they get better.
4. How many terms are required before the polynomial approximations stabilize in the thousandths place for $x = 13$?

This strategy for approximation would be of limited practical value if we were restricted to power series at $x = 0$—but we are not. We can match a power series with $f$ in the same way at *any* value $x = a$, provided we can take the derivatives. In fact, we can get a formula for doing that by simply "shifting horizontally" the formula we already have.

### DEFINITION Taylor Series Generated by $f$ at $x = a$

Let $f$ be a function with derivatives of all orders throughout some open interval containing $a$. Then the **Taylor series generated by $f$ at $x = a$** is

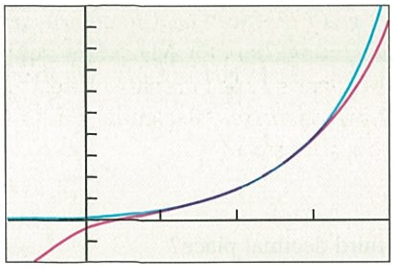
$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

The partial sum

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

is the **Taylor polynomial of order $n$ for $f$ at $x = a$.**



[–1, 4] by [–10, 50]

**Figure 9.6** The graphs of $y = e^x$ and $y = P_3(x)$ (the third order Taylor polynomial for $e^x$ at $x = 2$). (Example 4)

### EXAMPLE 4 A Taylor Series at $x = 2$

Find the Taylor series generated by $f(x) = e^x$ at $x = 2$.

**SOLUTION**

We first observe that $f(2) = f'(2) = f''(2) = \cdots = f^{(n)}(2) = e^2$. The series, therefore, is

$$e^x = e^2 + e^2(x-2) + \frac{e^2}{2!}(x-2)^2 + \cdots + \frac{e^2}{n!}(x-2)^n + \cdots$$

$$= \sum_{k=0}^{\infty} \left(\frac{e^2}{k!}\right)(x-2)^k.$$

We illustrate the convergence near $x = 2$ by sketching the graphs of $y = e^x$ and $y = P_3(x)$ in Figure 9.6. *Now try Exercise 13.*

### EXAMPLE 5 A Taylor Polynomial for a Polynomial

Find the third order Taylor polynomial for $f(x) = 2x^3 - 3x^2 + 4x - 5$

**(a)** at $x = 0$. **(b)** at $x = 1$.

**SOLUTION**

**(a)** This is easy. This polynomial is already written in powers of $x$ and is of degree three, so it is its own third order (and fourth order, etc.) Taylor polynomial at $x = 0$.

**(b)** This would also be easy if we could quickly rewrite the formula for $f$ as a polynomial in powers of $x - 1$, but that would require some messy tinkering. Instead, we apply the Taylor series formula.

$$f(1) = 2x^3 - 3x^2 + 4x - 5\Big|_{x=1} = -2$$

$$f'(1) = 6x^2 - 6x + 4\Big|_{x=1} = 4$$

$$f''(1) = 12x - 6\Big|_{x=1} = 6$$

$$f'''(1) = 12$$

So,

$$P_3(x) = -2 + 4(x-1) + \frac{6}{2!}(x-1)^2 + \frac{12}{3!}(x-1)^3$$

$$= 2(x-1)^3 + 3(x-1)^2 + 4(x-1) - 2.$$

This polynomial function agrees with $f$ at every value of $x$ (as you can verify by multiplying it out) but it is written in powers of $(x - 1)$ instead of $x$. *Now try Exercise 15.*

## Combining Taylor Series

On the intersection of their intervals of convergence, Taylor series can be added, subtracted, and multiplied by constants and powers of $x$, and the results are once again Taylor series. The Taylor series for $f(x) + g(x)$ is the sum of the Taylor series for $f(x)$ and the

Taylor series for $g(x)$ because the $n$th derivative of $f + g$ is $f^{(n)} + g^{(n)}$, and so on. We can obtain the Maclaurin series for $(1 + \cos 2x)/2$ by substituting $2x$ in the Maclaurin series for $\cos x$, adding 1, and dividing the result by 2. The Maclaurin series for $\sin x + \cos x$ is the term-by-term sum of the series for $\sin x$ and $\cos x$. We obtain the Maclaurin series for $x \sin x$ by multiplying all the terms of the Maclaurin series for $\sin x$ by $x$.

## Table of Maclaurin Series

We conclude the section by listing some of the most useful Maclaurin series, all of which have been derived in one way or another in the first two sections of this chapter. The exercises will ask you to use these series as basic building blocks for constructing other series (e.g., $\tan^{-1} x^2$ or $7xe^x$). We also list the intervals of convergence, although rigorous proofs of convergence are deferred until we develop convergence tests in Sections 9.4 and 9.5.

**Maclaurin Series**

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n \quad (|x| < 1)$$

$$\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-x)^n + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n \quad (|x| < 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad (\text{all real } x)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad (\text{all real } x)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \quad (\text{all real } x)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1} \frac{x^n}{n} + \cdots$$
$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} \quad (-1 < x \le 1)$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \quad (|x| \le 1)$$

## Quick Review 9.2 *(For help, go to Sections 3.3 and 3.6.)*

In Exercises 1–5, find a formula for the $n$th derivative of the function.

**1.** $e^{2x}$

**2.** $\dfrac{1}{x-1}$

**3.** $3^x$

**4.** $\ln x$

**5.** $x^n$

In Exercises 6–10, find $dy/dx$. (Assume that letters other than $x$ represent constants.)

**6.** $y = \dfrac{x^n}{n!}$

**7.** $y = \dfrac{2^n(x-a)^n}{n!}$

**8.** $y = (-1)^n \dfrac{x^{2n+1}}{(2n+1)!}$

**9.** $y = \dfrac{(x+a)^{2n}}{(2n)!}$

**10.** $y = \dfrac{(1-x)^n}{n!}$

## Section 9.2 Exercises

In Exercises 1 and 2, construct the fourth order Taylor polynomial at $x = 0$ for the function.

**1.** $f(x) = \sqrt{1+x^2}$

**2.** $f(x) = e^{2x}$

In Exercises 3 and 4, construct the fifth order Taylor polynomial and the Taylor series for the function at $x = 0$.

**3.** $f(x) = \dfrac{1}{x+2}$

**4.** $f(x) = e^{1-x}$

In Exercises 5–12, use the table of Maclaurin series on the preceding page. Construct the first three nonzero terms and the general term of the Maclaurin series generated by the function and give the interval of convergence.

**5.** $\sin 2x$

**6.** $\ln(1-x)$

**7.** $\tan^{-1} x^2$

**8.** $7x\,e^x$

**9.** $\cos(x+2)$ (*Hint:* $\cos(x+2) = (\cos 2)(\cos x) - (\sin 2)(\sin x)$)

**10.** $x^2 \cos x$

**11.** $\dfrac{x}{1-x^3}$

**12.** $e^{-2x}$

In Exercises 13 and 14, find the Taylor series generated by the function at the given point.

**13.** $f(x) = \dfrac{1}{x+1}$, $x = 2$

**14.** $f(x) = e^{x/2}$, $x = 1$

In Exercises 15–17, find the Taylor polynomial of order 3 generated by $f$

**(a)** at $x = 0$; **(b)** at $x = 1$.

**15.** $f(x) = x^3 - 2x + 4$

**16.** $f(x) = 2x^3 + x^2 + 3x - 8$

**17.** $f(x) = x^4$

In Exercises 18–21, find the Taylor polynomials of orders 0, 1, 2, and 3 generated by $f$ at $x = a$.

**18.** $f(x) = \dfrac{1}{x}$, $a = 2$

**19.** $f(x) = \sin x$, $a = \pi/4$

**20.** $f(x) = \cos x$, $a = \pi/4$

**21.** $f(x) = \sqrt{x}$, $a = 4$

**22.** Let $f$ be a function that has derivatives of all orders for all real numbers. Assume $f(0) = 4$, $f'(0) = 5$, $f''(0) = -8$, and $f'''(0) = 6$.

**(a)** Write the third order Taylor polynomial for $f$ at $x = 0$ and use it to approximate $f(0.2)$.

**(b)** Write the second order Taylor polynomial for $f'$, the derivative of $f$, at $x = 0$ and use it to approximate $f'(0.2)$.

**23.** Let $f$ be a function that has derivatives of all orders for all real numbers. Assume $f(1) = 4$, $f'(1) = -1$, $f''(1) = 3$, and $f'''(1) = 2$.

**(a)** Write the third order Taylor polynomial for $f$ at $x = 1$ and use it to approximate $f(1.2)$.

**(b)** Write the second order Taylor polynomial for $f'$, the derivative of $f$, at $x = 1$ and use it to approximate $f'(1.2)$.

**24.** The Maclaurin series for $f(x)$ is

$$f(x) = 1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \cdots + \frac{x^n}{(n+1)!} + \cdots.$$

**(a)** Find $f'(0)$ and $f^{(10)}(0)$.

**(b)** Let $g(x) = xf(x)$. Write the Maclaurin series for $g(x)$, showing the first three nonzero terms and the general term.

**(c)** Write $g(x)$ in terms of a familiar function without using series.

**25.** **(a)** Write the first three nonzero terms and the general term of the Taylor series generated by $e^{x/2}$ at $x = 0$.

**(b)** Write the first three nonzero terms and the general term of a power series to represent

$$g(x) = \frac{e^x - 1}{x}.$$

**(c)** For the function $g$ in part (b), find $g'(1)$ and use it to show that

$$\sum_{n=1}^{\infty} \frac{n}{(n+1)!} = 1.$$

**26.** Let

$$f(t) = \frac{2}{1-t^2} \quad \text{and} \quad G(x) = \int_0^x f(t)\,dt.$$

**(a)** Find the first four terms and the general term for the Maclaurin series generated by $f$.

**(b)** Find the first four nonzero terms and the Maclaurin series for $G$.

**27.** **(a)** Find the first four nonzero terms in the Taylor series generated by $f(x) = \sqrt{1+x}$ at $x = 0$.

**(b)** Use the results found in part (a) to find the first four nonzero terms in the Taylor series for $g(x) = \sqrt{1+x^2}$ at $x = 0$.

**(c)** Find the first four nonzero terms in the Taylor series at $x = 0$ for the function $h$ such that $h'(x) = \sqrt{1+x^2}$ and $h(0) = 5$.

**28.** Consider the power series

$$\sum_{n=0}^{\infty} a_n x^n, \text{ where } a_0 = 1 \text{ and } a_n = \left(\frac{3}{n}\right) a_{n-1} \text{ for } n \geq 1.$$

(This defines the coefficients *recursively*.)

**(a)** Find the first four terms and the general term of the series.

**(b)** What function $f$ is represented by this power series?

**(c)** Find the exact value of $f'(1)$.

**29.** Use the technique of Exploration 3 to determine the number of terms of the Maclaurin series for $\cos x$ that are needed to approximate the value of cos 18 accurate to within 0.001 of the true value.

**30.** **Writing to Learn** Based on what you know about polynomial functions, explain why no Taylor polynomial of any order could actually equal $\sin x$.

**31.** **Writing to Learn** Your friend has memorized the Maclaurin series for both $\sin x$ and $\cos x$ but is having a hard time remembering which is which. Assuming that your friend knows the trigonometric functions well, what are some tips you could give that would help match $\sin x$ and $\cos x$ with their correct series?

**32.** What is the coefficient of $x^5$ in the Maclaurin series generated by $\sin 3x$?

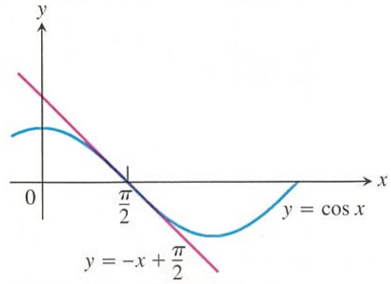
**33.** What is the coefficient of $(x-2)^3$ in the Taylor series generated by $\ln x$ at $x = 2$?

**34.** **Writing to Learn** Review the definition of the *linearization* of a differentiable function $f$ at $a$ in Chapter 4. What is the connection between the linearization of $f$ and Taylor polynomials?

**35.** ***Linearizations at Inflection Points***

**(a)** As the figure below suggests, linearizations fit particularly well at inflection points. As another example, graph *Newton's serpentine* $f(x) = 4x/(x^2+1)$ together with its linearizations at $x = 0$ and $x = \sqrt{3}$.

**(b)** Show that if the graph of a twice-differentiable function $f(x)$ has an inflection point at $x = a$, then the linearization of $f$ at $x = a$ is also the second order Taylor polynomial of $f$ at $x = a$. This explains why tangent lines fit so well at inflection points.




The graph of $f(x) = \cos x$ and its linearization at $\pi/2$. (Exercise 35)

**36.** According to the table of Maclaurin series, the power series

$$x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots$$

converges at $x = \pm 1$. To what number does it converge when $x = 1$? To what number does it converge when $x = -1$?

## Standardized Test Questions

You should solve the following problems without using a graphing calculator.

In Exercises 37 and 38, the Taylor series generated by $f(x)$ at $x = 0$ is

$$x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots.$$

**37.** **True or False** $f(0) = 0$. Justify your answer.

**38.** **True or False** $f'''(0) = -1/3$. Justify your answer.

**39.** **Multiple Choice** If $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, and $f'''(0) = 2$, then which of the following is the third order Taylor polynomial generated by $f(x)$ at $x = 0$?

**(A)** $2x^3 + x$ **(B)** $\frac{1}{3}x^3 + \frac{1}{2}x$ **(C)** $\frac{2}{3}x^3 + x$

**(D)** $2x^3 - x$ **(E)** $\frac{1}{3}x^3 + x$

**40.** **Multiple Choice** Which of the following is the coefficient of $x^4$ in the Maclaurin series generated by $\cos(3x)$?

**(A)** 27/8 **(B)** 9 **(C)** 1/24 **(D)** 0 **(E)** −27/8

In Exercises 41 and 42, let $f(x) = \sin x$.

**41.** **Multiple Choice** Which of the following is the fourth order Taylor polynomial generated by $f(x)$ at $x = \pi/2$?

**(A)** $(x - \pi/2) - \frac{(x-\pi/2)^2}{2!} + \frac{(x-\pi/2)^4}{4!}$

**(B)** $1 + \frac{(x-\pi/2)^2}{2!} + \frac{(x-\pi/2)^4}{4!}$

**(C)** $1 - \frac{(x-\pi/2)^2}{2!} + \frac{(x-\pi/2)^4}{4!}$

**(D)** $1 - (x-\pi/2)^2 + (x-\pi/2)^4$

**(E)** $1 + (x-\pi/2)^2 + (x-\pi/2)^4$

**42.** **Multiple Choice** Which of the following is the Taylor series generated by $f(x)$ at $x = \pi/2$?

**(A)** $\sum_{n=0}^{\infty} (-1)^n \frac{(x-\pi/2)^{2n}}{(2n)!}$

**(B)** $\sum_{n=0}^{\infty} (-1)^n \frac{(x-\pi/2)^{2n+1}}{(2n)!}$

**(C)** $\sum_{n=0}^{\infty} \frac{(x-\pi/2)^{2n}}{(2n)!}$

**(D)** $\sum_{n=0}^{\infty} (-1)^n (x-\pi/2)^{2n}$

**(E)** $\sum_{n=0}^{\infty} (x-\pi/2)^{2n}$

## Explorations

**43.** **(a)** Using the table of Maclaurin series, find a power series to represent $f(x) = (\sin x)/x$.

**(b)** The power series you found in part (a) is not quite a Maclaurin series for $f$, because $f$ is technically not eligible to have a Maclaurin series. Why not?

**(c)** If we redefine $f$ as follows, then the power series in part (a) *will* be a Maclaurin series for $f$. What is the value of $k$?

$$f(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0, \\ k, & x = 0 \end{cases}$$

**44. Group Activity** Find a function $f$ whose Maclaurin series is

$$1x^1 + 2x^2 + 3x^3 + \cdots + nx^n + \cdots.$$

## Extending the Ideas

**45.** ***The Binomial Series*** Let $f(x) = (1 + x)^m$ for some nonzero constant $m$.

**(a)** Show that $f'''(x) = m(m-1)(m-2)(1+x)^{m-3}$.

**(b)** Extend the result of part (a) to show that

$$f^{(k)}(0) = m(m-1)(m-2)\cdots(m-k+1).$$

**(c)** Find the coefficient of $x^k$ in the Maclaurin series generated by $f$.

**(d)** We define the symbol $\binom{m}{k}$ as follows:

$$\binom{m}{k} = \frac{m(m-1)(m-2)\cdots(m-k+1)}{k!},$$

with the understanding that

$$\binom{m}{0} = 1 \quad \text{and} \quad \binom{m}{1} = m.$$

With this notation, show that the Maclaurin series generated by $f(x) = (1 + x)^m$ is

$$\sum_{k=0}^{\infty} \binom{m}{k} x^k.$$

This is called the **binomial series.**

**46.** ***(Continuation of Exercise 45)*** If $m$ is a positive integer, explain why the Maclaurin series generated by $f$ is a polynomial of degree $m$. (This means that

$$(1 + x)^m = \sum_{k=0}^{m} \binom{m}{k} x^k.$$

You may recognize this result as the **Binomial Theorem** from algebra.)

# 9.3 Taylor's Theorem

**What you'll learn about**

- Taylor Polynomials
- The Remainder
- Remainder Estimation Theorem
- Euler's Formula

**. . . and why**

If we approximate a function represented by a power series by its Taylor polynomials, it is important to know how to determine the error in the approximation.

## Taylor Polynomials

While there is a certain unspoiled beauty in the exactness of a convergent Taylor series, it is the inexact Taylor polynomials that essentially do all the work. It is satisfying to know, for example, that $\sin x$ can be found *exactly* by summing an infinite Taylor series, but if we want to use that information to find sin 3, we will have to evaluate Taylor polynomials until we arrive at an *approximation* with which we are satisfied. Even a computer must deal with finite sums.

### EXAMPLE 1 Approximating a Function to Specifications

Find a Taylor polynomial that will serve as an adequate substitute for $\sin x$ on the interval $[-\pi, \pi]$.

**SOLUTION**

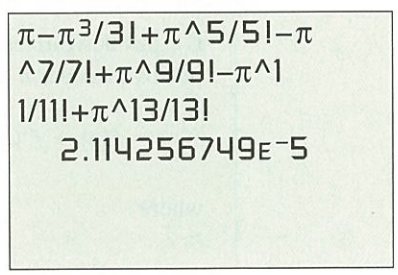
You do not have to be a professional mathematician to appreciate the imprecision of this problem as written. We are simply unable to proceed until someone decides what an "adequate" substitute is! We will revisit this issue shortly, but for now let us accept the following clarification of "adequate."

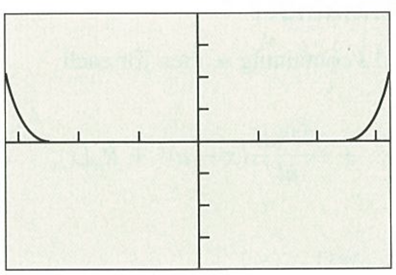
By "adequate," we mean that the polynomial should differ from $\sin x$ by less than 0.0001 anywhere on the interval.

Now we have a clear mission: Choose $P_n(x)$ so that $|P_n(x) - \sin x| < 0.0001$ for every $x$ in the interval $[-\pi, \pi]$. How do we do this?

Recall the nine graphs of the partial sums of the Maclaurin series for $\sin x$ in Section 9.2. They show that the approximations get worse as $x$ moves away from 0, suggesting that if we can make $|P_n(\pi) - \sin \pi| < 0.0001$, then $P_n$ will be adequate throughout the interval. Since $\sin \pi = 0$, this means that we need to make $|P_n(\pi)| < 0.0001$.

We evaluate the partial sums at $x = \pi$, adding a term at a time, eventually arriving at the following:



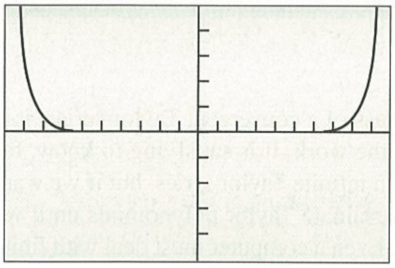
$[-\pi, \pi]$ by $[-0.00004, 0.00004]$

**Figure 9.7** The graph shows that $|P_{13}(x) - \sin x| < 0.00010$ throughout the interval $[-\pi, \pi]$. (Example 1)

As graphical support that the polynomial $P_{13}(x)$ is adequate throughout the interval, we graph the *absolute error* of the approximation, namely $|P_{13}(x) - \sin x|$, in the window $[-\pi, \pi]$ by $[-0.00004, 0.00004]$ (Figure 9.7). *Now try Exercise 11.*

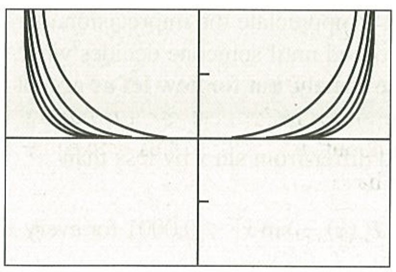
In practical terms, then, we would like to be able to use Taylor polynomials to approximate functions over the intervals of convergence of the Taylor series, and we would like to keep the error of the approximation within specified bounds. Since the error results from *truncating* the series down to a polynomial (that is, cutting it off after some number of terms), we call it the **truncation error.**

$y = x^8/(1 - x^2)$



$[-1, 1]$ by $[-5, 5]$

**Figure 9.8** A graph of the truncation error on $(-1, 1)$ if $P_6(x)$ is used to approximate $1/(1 - x^2)$. (Example 2)



$[-1, 1]$ by $[-2, 2]$

**Figure 9.9** The truncation errors for $n = 2, 4, 6, 8, 10$, when we approximate $1/(1 - x^2)$ by its Taylor polynomials of higher and higher order. (The errors for the higher order polynomials are on the bottom.)

### EXAMPLE 2 Truncation Error for a Geometric Series

Find a formula for the truncation error if we use $1 + x^2 + x^4 + x^6$ to approximate $1/(1 - x^2)$ over the interval $(-1, 1)$.

**SOLUTION**

We recognize this polynomial as the fourth partial sum of the geometric series for $1/(1 - x^2)$. Since this series converges to $1/(1 - x^2)$ on $(-1, 1)$, the truncation error is the absolute value of the part that we threw away, namely

$$|x^8 + x^{10} + \cdots + x^{2n} + \cdots|.$$

This is the absolute value of a geometric series with first term $x^8$ and $r = x^2$. Therefore,

$$|x^8 + x^{10} + \cdots + x^{2n} + \cdots| = \left|\frac{x^8}{1 - x^2}\right| = \frac{x^8}{1 - x^2}.$$

Figure 9.8 shows that the error is small near 0, but increases as $x$ gets closer to 1 or $-1$.

*Now try Exercise 13.*

You can probably infer from our solution in Example 2 that the truncation error after 5 terms would be $x^{10}/(1 - x^2)$, and after $n$ terms would be $x^{2n}/(1 - x^2)$. Figure 9.9 shows how these errors get closer to 0 on the interval $(-1, 1)$ as $n$ gets larger, and that they still get worse as we approach $-1$ and 1.

It was fortunate for our error analysis that this series was geometric, since the error was consequently a geometric series itself. This enabled us to write it as a (non-series) function and study it exactly. But how could we handle the error if we were to truncate a *nongeometric* series? That practical question sets the stage for Taylor's Theorem.

## The Remainder

Every truncation splits a Taylor series into two equally significant pieces: the Taylor polynomial $P_n(x)$ that gives us the approximation, and the *remainder* $R_n(x)$ that tells us whether the approximation is any good. Taylor's Theorem is about both pieces.

**THEOREM 3 Taylor's Theorem with Remainder**

If $f$ has derivatives of all orders in an open interval $I$ containing $a$, then for each positive integer $n$ and for each $x$ in $I$,

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n + R_n(x),$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n + 1)!}(x - a)^{n+1}$$

for some $c$ between $a$ and $x$.

Pause for a moment to consider how remarkable this theorem is. If we wish to approximate $f$ by a polynomial of degree $n$ over an interval $I$, the theorem gives us both a formula for the *polynomial* and a formula for the *error* involved in using that approximation over the interval $I$.

The first equation in Taylor's Theorem is **Taylor's formula.** The function $R_n(x)$ is the **remainder of order $n$** or the **error term** for the approximation of $f$ by $P_n(x)$ over $I$. It is also called the **Lagrange form** of the remainder, and bounds on $R_n(x)$ found using this form are **Lagrange error bounds.**

The introduction of $R_n(x)$ finally gives us a mathematically precise way to define what we mean when we say that a Taylor series converges to a function on an interval. If $R_n(x) \to 0$ as $n \to \infty$ for all $x$ in $I$, we say that the Taylor series generated by $f$ at $x = a$ **converges to $f$** on $I$, and we write

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

### EXAMPLE 3 Proving Convergence of a Maclaurin Series

Prove that the series

$$\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

converges to $\sin x$ for all real $x$.

**SOLUTION**

We need to consider what happens to $R_n(x)$ as $n \to \infty$.

By Taylor's Theorem,

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-0)^{n+1},$$

where $f^{(n+1)}(c)$ is the $(n+1)$st derivative of $\sin x$ evaluated at some $c$ between $x$ and 0. This does not seem at first glance to give us much information, but *for this particular function* we can say something very significant about $f^{(n+1)}(c)$: it lies between $-1$ and 1 inclusive. Therefore, no matter what $x$ is, we have

$$\begin{aligned} |R_n(x)| &= \left| \frac{f^{(n+1)}(c)}{(n+1)!}(x-0)^{n+1} \right| \\ &= \frac{|f^{(n+1)}(c)|}{(n+1)!}|x^{n+1}| \\ &\le \frac{1}{(n+1)!}|x^{n+1}| = \frac{|x|^{n+1}}{(n+1)!}. \end{aligned}$$

What happens to $|x|^{n+1}/(n+1)!$ as $n \to \infty$? The numerator is a product of $n+1$ factors, all of them $|x|$. The denominator is a product of $n+1$ factors, the largest of which eventually exceed $|x|$ and keep on growing as $n \to \infty$. The factorial growth in the denominator, therefore, eventually outstrips the power growth in the numerator, and we have $|x|^{n+1}/(n+1)! \to 0$ for all $x$. This means that $R_n(x) \to 0$ for all $x$, which completes the proof. *Now try Exercise 15.*

### EXPLORATION 1 Your Turn

Modify the steps of the proof in Example 3 to prove that

$$\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

converges to $\cos x$ for all real $x$.

## Remainder Estimation Theorem

Notice that we were able to use the remainder formula in Taylor's Theorem to verify the convergence of two Taylor series to their generating functions ($\sin x$ and $\cos x$), and yet in neither case did we have to find an actual value for $f^{(n+1)}(c)$. Instead, we were able to put an *upper bound* on $|f^{(n+1)}(c)|$, which was enough to ensure that $R_n(x) \to 0$ for all $x$. This strategy is so convenient that we state it as a theorem for future reference.

### THEOREM 4 Remainder Estimation Theorem

If there are positive constants $M$ and $r$ such that $|f^{(n+1)}(t)| \le Mr^{n+1}$ for all $t$ between $a$ and $x$, then the remainder $R_n(x)$ in Taylor's Theorem satisfies the inequality

$$|R_n(x)| \le M \frac{r^{n+1}|x-a|^{n+1}}{(n+1)!}.$$

If these conditions hold for every $n$ and all the other conditions of Taylor's Theorem are satisfied by $f$, then the series converges to $f(x)$.

It does not matter if $M$ and $r$ are huge; the important thing is that they do not get any *more huge* as $n \to \infty$. This allows the factorial growth to outstrip the power growth and thereby sweep $R_n(x)$ to zero.

### EXAMPLE 4 Proving Convergence

Prove that

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

for all real $x$ by showing that the Lagrange form of the remainder goes to zero as $n$ goes to infinity.

**SOLUTION**

We have already seen that this is the Taylor series generated by $e^x$ at $x = 0$, so all that remains is to verify that $R_n(x) \to 0$ for all $x$. By the Remainder Estimation Theorem, it suffices to find $M$ and $r$ such that $|f^{(n+1)}(t)| = e^t$ is bounded by $Mr^{n+1}$ for $t$ between 0 and an arbitrary $x$.

We know that $e^t$ is an increasing function on any interval, so it reaches its maximum value at the right-hand endpoint. We can pick $M$ to be that maximum value and simply let $r = 1$. If the interval is $[0, x]$, we let $M = e^x$; if the interval is $[x, 0]$, we let $M = e^0 = 1$. In either case, we have $e^t \le M$ throughout the interval, and the Remainder Estimation Theorem guarantees convergence. *Now try Exercise 17.*

### EXAMPLE 5 Estimating a Remainder

The approximation $\ln(1 + x) \approx x - (x^2/2)$ is used when $x$ is small. Use the Lagrange form of the remainder to get a bound for the maximum error when $|x| \le 0.1$. Support the answer graphically.

**SOLUTION**

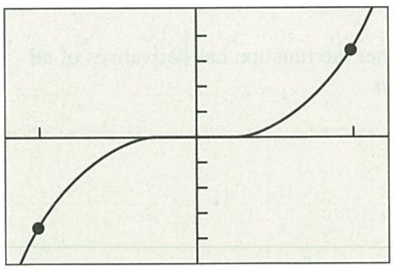
In the notation of the Remainder Estimation Theorem, $f(x) = \ln(1 + x)$, the polynomial is $P_2(x)$, and we need a bound for $|R_2(x)|$. On the interval $[-0.1, 0.1]$, the function $|f^{(3)}(t)| = 2/(1 + t)^3$ is strictly decreasing, achieving its maximum value at the left-hand endpoint, $-0.1$. We can therefore bound $|f^{(3)}(t)|$ by

$$M = \left|\frac{2}{(1 + (-0.1))^3}\right| = \frac{2000}{729}.$$

We can let $r = 1$.

*continued*

$y = \ln(1 + x) - (x - x^2/2)$



[−0.12, 0.12] by [−0.0005, 0.0005]

**Figure 9.10** The graph of the error term $R_2(x)$ in Example 5. Maximum error for $|x| \leq 0.1$ occurs at the left-hand endpoint of the interval.

By the Remainder Estimation Theorem, we may conclude that

$$|R_2(x)| \leq \frac{2000}{729} \cdot \frac{|x|^3}{3!} \leq \frac{2000}{729} \cdot \frac{|\pm 0.1|^3}{3!} < 4.6 \times 10^{-4}.$$

Rounded up, to be safe

Since $R_2(x) = \ln(1 + x) - (x - x^2/2)$, it is an easy matter to produce a graph to *observe* the behavior of the error on the interval $[-0.1, 0.1]$ (Figure 9.10).

The graph almost appears to have odd-function symmetry, but evaluation shows that $R_2(-0.1) \approx -3.605 \times 10^{-4}$ and $R_2(0.1) \approx 3.102 \times 10^{-4}$. The maximum *absolute* error on the interval is $3.605 \times 10^{-4}$, which is indeed less than the bound, $4.6 \times 10^{-4}$.

***Now try Exercise 23.***

## Euler's Formula

We have seen that $\sin x$, $\cos x$, and $e^x$ equal their respective Maclaurin series for all real numbers $x$. It can also be shown that this is true for all *complex* numbers, although we would need to extend our concept of limit to know what convergence would mean in that context. Accept for the moment that we can substitute complex numbers into these power series, and let us see where that might lead.

We mentioned at the beginning of the chapter that Leonhard Euler had derived some powerful results using infinite series. One of the most impressive was the surprisingly simple relationship he discovered that connects the exponential function $e^x$ to the trigonometric functions $\sin x$ and $\cos x$. You do not need a deep understanding of complex numbers to understand what Euler did, but you do need to recall the powers of $i = \sqrt{-1}$.

$$i^1 = i$$
$$i^2 = -1$$
$$i^3 = -i$$
$$i^4 = 1$$
$$i^5 = i$$
$$i^6 = -1$$
$$i^7 = -i$$
$$i^8 = 1$$

etc.

Now try this exploration!

### Srinivasa Ramanujan (1887–1920)


Ramanujan, from southern India, wrote with amazing originality and depth on a wide range of topics in mathematics, including infinite series, prime and composite numbers, integers as the sum of squares, function theory, and combinatorics. His theorems have influenced medical research and statistical mechanics. One of his identities has been used by computer programmers to calculate the decimal expansion of pi to millions of digits. There are still areas of his work that have not been explored. Ramanujan was largely self taught and, although he worked with the British mathematician G. H. Hardy of Cambridge, he never graduated from college because he neglected his other studies for mathematics.

### EXPLORATION 2 Euler's Formula

Assume that $e^x$, $\cos x$, and $\sin x$ equal their Maclaurin series (as in the table in Section 9.2) for complex numbers as well as for real numbers.

1. Find the Maclaurin series for $e^{ix}$.
2. Use the result of part 1 and the Maclaurin series for $\cos x$ and $\sin x$ to prove that $e^{ix} = \cos x + i \sin x$. This equation is known as **Euler's formula**.
3. Use Euler's formula to prove that $e^{i\pi} + 1 = 0$. This beautiful equation, which brings together the five most celebrated numbers in mathematics in such a stunningly unexpected way, is also widely known as Euler's formula. (There are still others. The prolific Euler had more than his share.)

## Quick Review 9.3 *(For help, go to Sections 3.3 and 3.6.)*

In Exercises 1–5, find the smallest number $M$ that bounds $|f|$ from above on the interval $I$ (that is, find the smallest $M$ such that $|f(x)| \le M$ for all $x$ in $I$).

**1.** $f(x) = 2\cos(3x), \quad I = [-2\pi, 2\pi]$

**2.** $f(x) = x^2 + 3, \quad I = [1, 2]$

**3.** $f(x) = 2^x, \quad I = [-3, 0]$

**4.** $f(x) = \dfrac{x}{x^2+1}, \quad I = [-2, 2]$

**5.** $f(x) = \begin{cases} 2 - x^2, & x \le 1, \\ 2x - 1, & x > 1, \end{cases} \quad I = [-3, 3]$

In Exercises 6–10, tell whether the function has derivatives of all orders at the given value of $a$.

**6.** $\dfrac{x}{x+1}, \quad a = 0$

**7.** $|x^2 - 4|, \quad a = 2$

**8.** $\sin x + \cos x, \quad a = \pi$

**9.** $e^{-x}, \quad a = 0$

**10.** $x^{3/2}, \quad a = 0$

## Section 9.3 Exercises

In Exercises 1–5, find the Taylor polynomial of order four for the function at $x = 0$, and use it to approximate the value of the function at $x = 0.2$.

**1.** $e^{-2x}$

**2.** $\cos(\pi x/2)$

**3.** $5\sin(-x)$

**4.** $\ln(1 + x^2)$

**5.** $(1 - x)^{-2}$

In Exercises 6–10, find the Maclaurin series for the function.

**6.** $\sin x - x + \dfrac{x^3}{3!}$

**7.** $xe^x$

**8.** $\cos^2 x \quad \left(= \dfrac{1 + \cos 2x}{2}\right)$

**9.** $\sin^2 x$

**10.** $\dfrac{x^2}{1 - 2x}$

**11.** Use graphs to find a Taylor polynomial $P_n(x)$ for $\ln(1 + x)$ so that $|P_n(x) - \ln(1 + x)| < 0.001$ for every $x$ in $[-0.5, 0.5]$.

**12.** Use graphs to find a Taylor polynomial $P_n(x)$ for $\cos x$ so that $|P_n(x) - \cos x| < 0.001$ for every $x$ in $[-\pi, \pi]$.

**13.** Find a formula for the truncation error if we use $P_6(x)$ to approximate $\dfrac{1}{1 - 2x}$ on $(-1/2, 1/2)$.

**14.** Find a formula for the truncation error if we use $P_9(x)$ to approximate $\dfrac{1}{1 - x}$ on $(-1, 1)$.

In Exercises 15–18, use the Lagrange form of the remainder to prove that the Maclaurin series converges to the generating function from the given exercise.

**15.** Exercise 7

**16.** Exercise 6

**17.** Exercise 9

**18.** Exercise 8

**19.** For approximately what values of $x$ can you replace $\sin x$ by $x - (x^3/6)$ with an error magnitude no greater than $5 \times 10^{-4}$? Give reasons for your answer.

**20.** If $\cos x$ is replaced by $1 - (x^2/2)$ and $|x| < 0.5$, what estimate can be made of the error? Does $1 - (x^2/2)$ tend to be too large or too small? Support your answer graphically.

**21.** How close is the approximation $\sin x \approx x$ when $|x| < 10^{-3}$? For which of these values of $x$ is $x < \sin x$? Support your answer graphically.

**22.** The approximation $\sqrt{1 + x} \approx 1 + (x/2)$ is used when $x$ is small. Estimate the maximum error when $|x| < 0.01$.

**23.** The approximation $e^x \approx 1 + x + (x^2/2)$ is used when $x$ is small. Use the Remainder Estimation Theorem to estimate the error when $|x| < 0.1$.

**24.** ***Hyperbolic sine and cosine*** The hyperbolic sine and hyperbolic cosine functions, denoted sinh and cosh respectively, are defined as

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh x = \frac{e^x + e^{-x}}{2}.$$

(Appendix A6 gives more information about hyperbolic functions.)

Find the Maclaurin series generated by $\sinh x$ and $\cosh x$.

**25.** ***(Continuation of Exercise 24)*** Use the Remainder Estimation Theorem to prove that $\cosh x$ equals its Maclaurin series for all real numbers $x$.

**26.** **Writing to Learn** Review the statement of the Mean Value Theorem (Section 4.2) and explain its relationship to Taylor's Theorem.

***Quadratic Approximations*** Just as we call the Taylor polynomial of order 1 generated by $f$ at $x = a$ the *linearization* of $f$ at $a$, we call the Taylor polynomial of order 2 generated by $f$ at $x = a$ the *quadratic approximation* of $f$ at $a$.

In Exercises 27–31, find **(a)** the linearization and **(b)** the quadratic approximation of $f$ at $x = 0$. Then **(c)** graph the function and its linear and quadratic approximations together around $x = 0$ and comment on how the graphs are related.

**27.** $f(x) = \ln(\cos x)$

**28.** $f(x) = e^{\sin x}$

**29.** $f(x) = 1/\sqrt{1 - x^2}$

**30.** $f(x) = \sec x$

**31.** $f(x) = \tan x$

**32.** Use the Taylor polynomial of order 2 to find the quadratic approximation of $f(x) = (1 + x)^k$ at $x = 0$ ($k$ a constant). If $k = 3$, for approximately what values of $x$ in the interval $[0, 1]$ will the magnitude of the error in the quadratic approximation be less than $1/100$?

**33. *A Cubic Approximation of $e^x$*** The approximation

$$e^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

is used on small intervals about the origin. Estimate the magnitude of the approximation error for $|x| \le 0.1$.

**34. *A Cubic Approximation*** Use the Taylor polynomial of order 3 to find the *cubic approximation* of $f(x) = 1/(1 - x)$ at $x = 0$. Give an upper bound for the magnitude of the approximation error for $|x| \le 0.1$.

**35.** Consider the initial value problem,

$$\frac{dy}{dx} = e^{-x^2} \quad \text{and} \quad y = 2 \quad \text{when } x = 0.$$

**(a)** Can you find a formula for the function $y$ that does not involve any integrals?

**(b)** Can you represent $y$ by a power series?

**(c)** For what values of $x$ does this power series actually equal the function $y$? Give a reason for your answer.

**36. (a)** Construct the Maclaurin series for $\ln(1 - x)$.

**(b)** Use this series and the series for $\ln(1 + x)$ to construct a Maclaurin series for

$$\ln\frac{1 + x}{1 - x}.$$

**37. *Identifying Graphs*** Which well-known functions are approximated on the interval $(-\pi/2, \pi/2)$ by the following Taylor polynomials?

**(a)** $x + \dfrac{x^3}{3} + \dfrac{2x^5}{15} + \dfrac{17x^7}{315} + \dfrac{62x^9}{2835}$

**(b)** $1 + \dfrac{x^2}{2} + \dfrac{5x^4}{24} + \dfrac{61x^6}{720} + \dfrac{277x^8}{8064}$

## Standardized Test Questions

You may use a graphing calculator to solve the following problems.

**38. True or False** The degree of the linearization of a function $f$ at $x = a$ must be 1. Justify your answer.

**39. True or False** If $\sum_{n=0}^{\infty} \dfrac{x^{n+1}}{n!} = x + x^2 + \dfrac{x^3}{2!} + \cdots$ is the Maclaurin series for the function $f(x)$, then $f'(0) = 1$. Justify your answer.

**40. Multiple Choice** Which of the following gives the Taylor polynomial of order 5 approximation to $\sin(1.5)$?

**(A)** 0.965 **(B)** 0.985 **(C)** 0.997 **(D)** 1.001 **(E)** 1.005

**41. Multiple Choice** Let $\sum_{n=0}^{\infty} \dfrac{x^{n+1}}{n!} = x + x^2 + \dfrac{x^3}{2!} + \cdots$ be the Maclaurin series for $f(x)$. Which of the following is $f^{(12)}(0)$, the 12th derivative of $f$ at $x = 0$?

**(A)** $1/11!$ **(B)** $1/12!$ **(C)** 0 **(D)** 1 **(E)** 12

**42. Multiple Choice** Let $\sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!} = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \cdots$

be the Maclaurin series for $\cos x$. Which of the following gives the smallest value of $n$ for which $|P_n(x) - \cos x| < 0.01$ for all $x$ in the interval $[-\pi, \pi]$?

**(A)** 12 **(B)** 10 **(C)** 8 **(D)** 6 **(E)** 4

**43. Multiple Choice** Which of the following is the quadratic approximation for $f(x) = e^{-x}$ at $x = 0$?

**(A)** $1 - x + \dfrac{1}{2}x^2$ **(B)** $1 - x - \dfrac{1}{2}x^2$

**(C)** $1 + x + \dfrac{1}{2}x^2$ **(D)** $1 + x$ **(E)** $1 - x$

## Explorations

**44. Group Activity** Try to reinforce each other's ideas and verify your computations at each step.

**(a)** Use the identity

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

to obtain the Maclaurin series for $\sin^2 x$.

**(b)** Differentiate this series to obtain the Maclaurin series for $2 \sin x \cos x$.

**(c)** Verify that this is the series for $\sin 2x$.

**45. *Improving Approximations to $\pi$***

**(a)** Let $P$ be an approximation of $\pi$ accurate to $n$ decimal places. Check with a calculator to see that $P + \sin P$ gives an approximation correct to $3n$ decimal places!

**(b)** Use the Remainder Estimation Theorem and the Maclaurin series for $\sin x$ to explain what is happening in part (a). (*Hint:* Let $P = \pi + x$, where $x$ is the error of the estimate. Why should $(P + \sin P) - \pi$ be less than $x^3$?)

**46. *Euler's Identities*** Use Euler's formula to show that

**(a)** $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$, and

**(b)** $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$.

## Extending the Ideas

**47.** When $a$ and $b$ are real numbers, we define $e^{(a+ib)x}$ with the equation

$$e^{(a+ib)x} = e^{ax} \cdot e^{ibx} = e^{ax}(\cos bx + i \sin bx).$$

Differentiate the right-hand side of this equation to show that

$$\frac{d}{dx}e^{(a+ib)x} = (a+ib)\,e^{(a+ib)x}.$$

Thus, the familiar rule

$$\frac{d}{dx}e^{kx} = ke^{kx}$$

holds for complex values of $k$ as well as for real values.

**48.** ***(Continuation of Exercise 47)***

**(a)** Confirm the antiderivative formula

$$\int e^{(a+ib)x}\,dx = \frac{a-ib}{a^2+b^2}\,e^{(a+ib)x} + C$$

by differentiating both sides. (In this case, $C = C_1 + iC_2$ is a complex constant of integration.)

**(b)** Two complex numbers $a + ib$ and $c + id$ are equal if and only if $a = c$ and $b = d$. Use this fact and the formula in part (a) to evaluate $\int e^{ax}\cos bx\,dx$ and $\int e^{ax}\sin bx\,dx$.

# Quick Quiz for AP* Preparation: Sections 9.1–9.3

You should solve the following problems without using a graphing calculator.

**1. Multiple Choice** Which of the following is the sum of the series $\sum_{n=0}^{\infty} \frac{\pi^n}{e^{2n}}$?

**(A)** $\frac{e}{e-\pi}$ **(B)** $\frac{\pi}{\pi-e}$ **(C)** $\frac{\pi}{\pi-e^2}$

**(D)** $\frac{e^2}{e^2-\pi}$ **(E)** The series diverges.

**2. Multiple Choice** Assume that $f$ has derivatives of all orders for all real numbers $x$, $f(0) = 2, f'(0) = -1, f''(0) = 6$, and $f'''(0) = 12$. Which of the following is the third order Taylor polynomial for $f$ at $x = 0$?

**(A)** $2 - x + 3x^2 + 2x^3$ **(B)** $2 - x + 6x^2 + 12x^3$

**(C)** $2 - \frac{1}{2}x + 3x^2 + 2x^3$ **(D)** $-2 + x - 3x^2 - 2x^3$

**(E)** $2 - x + 6x^2$

**3. Multiple Choice** Which of the following is the Taylor series generated by $f(x) = 1/x$ at $x = 1$?

**(A)** $\sum_{n=0}^{\infty} (x-1)^n$ **(B)** $\sum_{n=0}^{\infty} (-1)^n x^n$

**(C)** $\sum_{n=0}^{\infty} (-1)^n (x+1)^n$ **(D)** $\sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^n}{n!}$

**(E)** $\sum_{n=0}^{\infty} (-1)^n (x-1)^n$

**4. Free Response** Let $f$ be the function defined by

$$f(x) = \sum_{n=0}^{\infty} 2\left(\frac{x+2}{3}\right)^n$$

for all values of $x$ for which the series converges.

**(a)** Find the interval of convergence for the series.

**(b)** Find the function that the series represents.

# 9.4 Radius of Convergence

**What you'll learn about**

- Convergence
- *n*th-Term Test
- Comparing Nonnegative Series
- Ratio Test
- Endpoint Convergence

**. . . and why**

It is important to develop a strategy for finding the interval of convergence of a power series and to obtain some tests that can be used to determine convergence of a series.

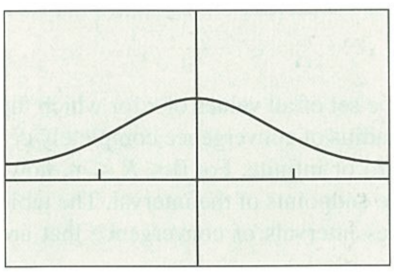
## Convergence

Throughout our explorations of infinite series we stressed the importance of convergence. In terms of numbers, the difference between a convergent series and a divergent series could hardly be more stark: a convergent series is a number and may be treated as such; a divergent series is not a number and must not be treated as one.

Recall that the symbol "=" means many different things in mathematics.

1. $1 + 1 = 2$ signifies *equality of real numbers.* It is a true sentence.
2. $2(x - 3) = 2x - 6$ signifies *equivalent expressions.* It is a true sentence.
3. $x^2 + 3 = 7$ is an *equation.* It is an *open sentence,* because it can be true or false, depending on whether $x$ is a solution to the equation.
4. $(x^2 - 1)/(x + 1) = x - 1$ is an *identity.* It is a true sentence (very much like the equation in (2)), but with the important qualification that *$x$ must be in the domain of both expressions.* If either side of the equality is undefined, the sentence is meaningless. Substituting $-1$ into both sides of the equation in (3) gives a sentence that is mathematically false (i.e., $4 = 7$); substituting $-1$ into both sides of this identity gives a sentence that is meaningless.

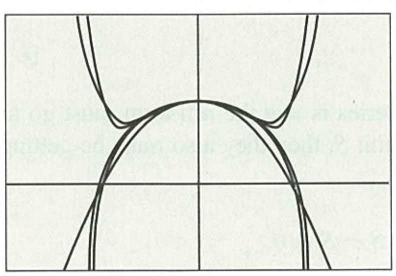
$y = \dfrac{1}{1 + x^2}$



[–2, 2] by [–1, 2]

(a)

Partial Sums



[–2, 2] by [–1, 2]

(b)

**Figure 9.11** (a) The graph of $y = 1/(1 + x^2)$ and (b) the graphs of the Taylor polynomials $P_2(x)$, $P_4(x)$, $P_6(x)$, $P_8(x)$, and $P_{10}(x)$. The approximations become better and better, but only over the interval of convergence $(-1, 1)$. (Example 1)

### EXAMPLE 1 The Importance of Convergence

Consider the sentence

$$\frac{1}{1 + x^2} = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \cdots.$$

For what values of $x$ is this an identity?

**SOLUTION**

The function on the left has domain all real numbers. The function on the right can be viewed as a limit of Taylor polynomials. Each Taylor polynomial has domain all real numbers, but the polynomial values *converge* only when $|x| < 1$, so the *series* has the domain $(-1, 1)$. If we graph the Taylor polynomials (Figure 9.11), we can see the dramatic convergence to $1/(1 + x^2)$ over the interval $(-1, 1)$. The divergence is just as dramatic for $|x| \geq 1$.

For values of $x$ outside the interval, the statement in this example is meaningless. The Taylor series on the right diverges so it is not a number. The sentence is an identity for $x$ in $(-1, 1)$.

*Now try Exercise 1.*

**Seki Kowa** (1642–1708)

Child prodigy, brilliant mathematician, and inspirational teacher, Seki Kowa was born into a samurai warrior family in Fujioka, Kozuke, Japan, and adopted by the family of an accountant. Among his contributions were an improved method of solving higher-degree equations, the use of determinants in solving simultaneous equations, and a form of calculus known in Japan as *yenri.* It is difficult to know the full extent of his work because the samurai code demanded great modesty. Seki Kowa is credited with awakening in Japan a scientific spirit that continues to this day.

As convincing as these graphs are, they do not *prove* convergence or divergence as $n \to \infty$. The series in Example 1 happens to be geometric, so we do have an analytic proof that it converges for $|x| < 1$ and diverges for $|x| \geq 1$, but for nongeometric series we do not have such undeniable assurance about convergence (yet).

In this section we develop a strategy for finding the interval of convergence of an arbitrary power series and backing it up with proof. We begin by noting that any power series of the form $\sum_{n=0}^{\infty} c_n(x-a)^n$ always converges at $x = a$, thus assuring us of at least one coordinate on the real number line where the series must converge. We have encountered power series that converge for all real numbers (the Maclaurin series for $\sin x$, $\cos x$, and $e^x$), and we have encountered power series like the series in Example 1 that converge only on a finite interval centered at $a$. A useful fact about power series is that those are the only possibilities, as the following theorem attests.

**THEOREM 5 The Convergence Theorem for Power Series**

There are three possibilities for $\sum_{n=0}^{\infty} c_n(x-a)^n$ with respect to convergence:

1. There is a positive number $R$ such that the series diverges for $|x-a| > R$ but converges for $|x-a| < R$. The series may or may not converge at either of the endpoints $x = a - R$ and $x = a + R$.
2. The series converges for every $x$ ($R = \infty$).
3. The series converges at $x = a$ and diverges elsewhere ($R = 0$).

The number $R$ is the **radius of convergence,** and the set of all values of $x$ for which the series converges is the **interval of convergence.** The radius of convergence completely determines the interval of convergence if $R$ is either zero or infinite. For $0 < R < \infty$, however, there remains the question of what happens at the endpoints of the interval. The table of Maclaurin series at the end of Section 9.2 includes intervals of convergence that are open, half-open, and closed.

We will learn how to find the radius of convergence first, and then we will settle the endpoint question in Section 9.5.

## *n*th-Term Test

The most obvious requirement for convergence of a series is that the $n$th term must go to zero as $n \to \infty$. If the partial sums are approaching a limit $S$, then they also must be getting close to one another, so that for a convergent series $\sum a_n$,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (S_n - S_{n-1}) = S - S = 0.$$

This gives a handy test for divergence:

**THEOREM 6 The *n*th-Term Test for Divergence**

$\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n$ fails to exist or is different from zero.

## Comparing Nonnegative Series

An effective way to show that a series $\sum a_n$ of nonnegative numbers *converges* is to compare it term by term with a known convergent series $\sum c_n$.

**THEOREM 7 The Direct Comparison Test**

Let $\sum a_n$ be a series with no negative terms.

**(a)** $\sum a_n$ converges if there is a convergent series $\sum c_n$ with $a_n \le c_n$ for all $n > N$, for some integer $N$.

**(b)** $\sum a_n$ diverges if there is a divergent series $\sum d_n$ of nonnegative terms with $a_n \ge d_n$ for all $n > N$, for some integer $N$.

If we can show that $\sum a_n$, $a_n \ge 0$ is eventually dominated by a convergent series, that will establish the convergence of $\sum a_n$. If we can show that $\sum a_n$ eventually dominates a divergent series of nonnegative terms, that will establish the divergence of $\sum a_n$.

We leave the proof to Exercises 61 and 62.

### EXAMPLE 2 Proving Convergence by Comparison

Prove that $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{(n!)^2}$ converges for all real $x$.

**SOLUTION**

Let $x$ be any real number. The series

$$\sum_{n=0}^{\infty} \frac{x^{2n}}{(n!)^2}$$

has no negative terms.

For any $n$, we have

$$\frac{x^{2n}}{(n!)^2} \le \frac{x^{2n}}{n!} = \frac{(x^2)^n}{n!}.$$

We recognize

$$\sum_{n=0}^{\infty} \frac{(x^2)^n}{n!}$$

as the Taylor series for $e^{x^2}$, which we know converges to $e^{x^2}$ for all real numbers. Since the $e^{x^2}$ series dominates

$$\sum_{n=0}^{\infty} \frac{x^{2n}}{(n!)^2}$$

term by term, the latter series must also converge for all real numbers by the Direct Comparison Test. *Now try Exercise 3.*

For the Direct Comparison Test to apply, the terms of the unknown series must be *nonnegative*. The fact that $\sum a_n$ is dominated by a convergent positive series means nothing if $\sum a_n$ diverges to $-\infty$. You might think that the requirement of nonnegativity would limit the usefulness of the Direct Comparison Test, but in practice this does not turn out to be the case.

We can apply our test to $\sum |a_n|$ (which certainly has no negative terms); if $\sum |a_n|$ converges, then $\sum a_n$ converges.

**DEFINITION Absolute Convergence**

If the series $\sum |a_n|$ of absolute values converges, then $\sum a_n$ **converges absolutely.**

**THEOREM 8 Absolute Convergence Implies Convergence**

If $\sum |a_n|$ converges, then $\sum a_n$ converges.

**Proof** For each $n$,

$$-|a_n| \le a_n \le |a_n|, \quad \text{so} \quad 0 \le a_n + |a_n| \le 2\,|a_n|.$$

If $\sum |a_n|$ converges, then $\sum 2|a_n|$ converges, and by the Direct Comparison Test, the nonnegative series $\sum (a_n + |a_n|)$ converges. The equality $a_n = (a_n + |a_n|) - |a_n|$ now allows us to express $\sum a_n$ as the difference of two convergent series:

$$\sum a_n = \sum (a_n + |a_n| - |a_n|) = \sum (a_n + |a_n|) - \sum |a_n|.$$

Therefore, $\sum a_n$ converges. ■

**EXAMPLE 3 Using Absolute Convergence**

Show that

$$\sum_{n=0}^{\infty} \frac{(\sin x)^n}{n!}$$

converges for all $x$.

**SOLUTION**

Let $x$ be any real number. The series

$$\sum_{n=0}^{\infty} \frac{|\sin x|^n}{n!}$$

has no negative terms, and it is term-by-term less than or equal to the series $\sum_{n=0}^{\infty} (1/n!)$, which we know converges to $e$. Therefore,

$$\sum_{n=0}^{\infty} \frac{|\sin x|^n}{n!}$$

converges by direct comparison. Since

$$\sum_{n=0}^{\infty} \frac{(\sin x)^n}{n!}$$

converges absolutely, it converges. *Now try Exercise 5.*

## Ratio Test

Our strategy for finding the radius of convergence for an arbitrary power series will be to check for absolute convergence using a powerful test called the *Ratio Test.*

**THEOREM 9 The Ratio Test**

Let $\sum a_n$ be a series with positive terms, and with

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L.$$

Then,

**(a)** the series *converges* if $L < 1$,
**(b)** the series *diverges* if $L > 1$,
**(c)** the test is *inconclusive* if $L = 1$.

L'Hôpital's rule is occasionally helpful in determining the limits that arise here.

## Proof

**(a)** $L < 1$:

Choose some number $r$ such that $L < r < 1$. Since

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L,$$

we know that there is some $N$ large enough so that $a_{n+1}/a_n$ is arbitrarily close to $L$ for all $n \ge N$. In particular, we can guarantee that for some $N$ large enough, $(a_{n+1}/a_n) < r$ for all $n \ge N$. (See Figure 9.12.)

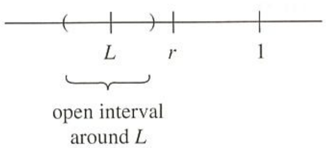



**Figure 9.12** Since

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L,$$

there is some $N$ large enough so that $a_{n+1}/a_n$ lies inside this open interval around $L$ for all $n \ge N$. This guarantees that $a_{n+1}/a_n < r < 1$ for all $n \ge N$.

Thus,

$$\frac{a_{N+1}}{a_N} < r \quad \text{so} \quad a_{N+1} < ra_N$$

$$\frac{a_{N+2}}{a_{N+1}} < r \quad \text{so} \quad a_{N+2} < ra_{N+1} < r^2 a_N$$

$$\frac{a_{N+3}}{a_{N+2}} < r \quad \text{so} \quad a_{N+3} < ra_{N+2} < r^3 a_N$$

$$\vdots$$

This shows that for $n \ge N$ we can dominate $\sum a_n$ by $a_N(1 + r + r^2 + \cdots)$. Since $0 < r < 1$, this latter series is a convergent geometric series, and so $\sum a_n$ converges by the Direct Comparison Test.

**(b)** $L > 1$:

From some index $M$,

$$\frac{a_{n+1}}{a_n} > 1$$

for all $n \ge M$. In particular,

$$a_M < a_{M+1} < a_{M+2} < \cdots.$$

The terms of the series do not approach 0, so $\sum a_n$ diverges by the $n$th-Term Test.

**(c)** $L = 1$:

In Exploration 1 you will finish the proof by showing that the Ratio Test is inconclusive when $L = 1$. ■

**A Note on Absolute Convergence:** The proof of the Ratio Test shows that the convergence of a power series inside its radius of convergence is *absolute* convergence, a stronger result than we first stated in Theorem 5. We will learn more about the distinction between convergence and absolute convergence in Section 9.5.

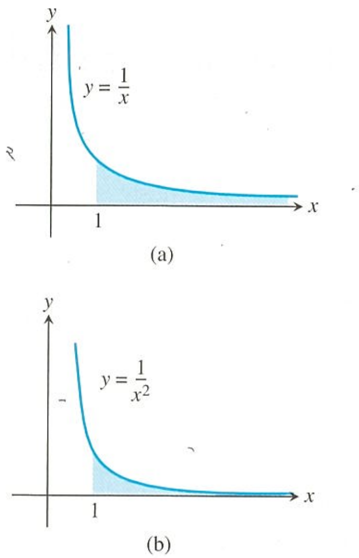



**Figure 9.13** Find these areas. (Exploration 1)

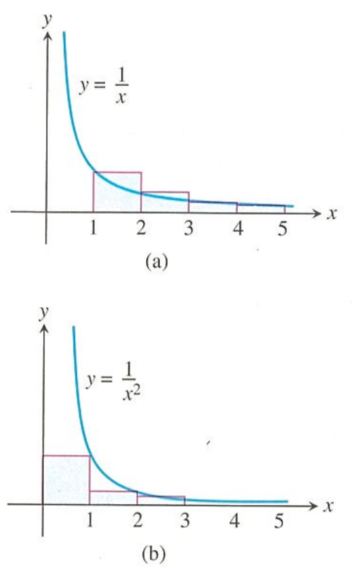



**Figure 9.14** The areas of the rectangles form a series in each case. (Exploration 1)

## EXPLORATION 1 Finishing the Proof of the Ratio Test

Consider

$$\sum_{n=1}^{\infty} \frac{1}{n} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

(We will refer to them hereafter in this exploration as $\sum 1/n$ and $\sum 1/n^2$.)

1. Show that the Ratio Test yields $L = 1$ for both series.
2. Use improper integrals to find the areas shaded in Figures 9.13a and 9.13b for $1 \le x < \infty$.
3. Explain how Figure 9.14a shows that $\sum 1/n$ diverges, while Figure 9.14b shows that $\sum 1/n^2$ converges.
4. Explain how this proves the last part of the Ratio Test.

## EXAMPLE 4 Finding the Radius of Convergence

Find the radius of convergence of

$$\sum_{n=0}^{\infty} \frac{nx^n}{10^n}.$$

**SOLUTION**

We check for absolute convergence using the Ratio Test.

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(n+1)\,|x^{n+1}|}{10^{n+1}} \cdot \frac{10^n}{n|x^n|}$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)\frac{|x|}{10} = \frac{|x|}{10}$$

Setting $|x|/10 < 1$, we see that the series converges absolutely (and hence converges) for $-10 < x < 10$. The series diverges for $|x| > 10$, which means (by Theorem 5, the Convergence Theorem for Power Series) that it diverges for $x > 10$ and for $x < -10$. The radius of convergence is 10. *Now try Exercise 9.*

## EXAMPLE 5 A Series with Radius of Convergence 0

Find the radius of convergence of the series $\sum_{n=0}^{\infty} n!x^n$.

**SOLUTION**

We check for absolute convergence using the Ratio Test.

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(n+1)!|x|^{n+1}}{n!|x|^n}$$

$$= \lim_{n\to\infty} (n+1)|x|$$

$$= \infty, \quad x \ne 0$$

The series converges only for $x = 0$. The radius of convergence is $R = 0$.

*Now try Exercise 17.*

## Endpoint Convergence

The Ratio Test, which is really a test for absolute convergence, establishes the radius of convergence for $\sum |c_n(x-a)^n|$. Theorem 5 guarantees that this is the same as the radius of convergence of $\sum c_n(x-a)^n$. Therefore, all that remains to be resolved about the convergence of an arbitrary power series is the question of convergence at the endpoints of the convergence interval when the radius of convergence is a finite, nonzero number.

**EXPLORATION 2** **Revisiting a Maclaurin Series**

For what values of $x$ does the series

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n} + \cdots$$

converge?

1. Apply the Ratio Test to determine the radius of convergence.
2. Substitute the left-hand endpoint of the interval into the power series. Use Figure 9.14a of Exploration 1 to help you decide whether the resulting series converges or diverges.
3. Substitute the right-hand endpoint of the interval into the power series. You should get

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n-1}}{n} + \cdots.$$

   Chart the progress of the partial sums of this series geometrically on a number line as follows: Start at 0. Go forward 1. Go back 1/2. Go forward 1/3. Go back 1/4. Go forward 1/5, and so on.
4. Does the series converge at the right-hand endpoint? Give a convincing argument based on your geometric journey in part 3.
5. Does the series converge *absolutely* at the right-hand endpoint?

**EXAMPLE 6** **Determining Convergence of a Series**

Determine the convergence or divergence of the series $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{5^n+1}$.

**SOLUTION** We use the Ratio Test.

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\dfrac{3^{n+1}}{5^{n+1}+1}}{\dfrac{3^n}{5^n+1}} = \lim_{n\to\infty} \left(\frac{3^{n+1}}{5^{n+1}+1}\right)\left(\frac{5^n+1}{3^n}\right)$$

$$= \lim_{n\to\infty} 3\left(\frac{5^n+1}{5^{n+1}+1}\right)$$

$$= \lim_{n\to\infty} 3\,\frac{1+\dfrac{1}{5^n}}{5+\dfrac{1}{5^n}} \qquad \text{Divide numerator and denominator by } 5^n.$$

$$= \frac{3}{5}$$

The series converges because the ratio $3/5 < 1$.

***Now try Exercise 31.***

The question of convergence of a power series at an endpoint is really a question about the convergence of a series of numbers. If the series is geometric with first term $a$ and common ratio $r$, then the series converges to $a/(1-r)$ if $|r|<1$ and diverges if $|r| \geq 1$. Another type of series whose sums are easily found are **telescoping series,** as illustrated in Example 7.

### EXAMPLE 7 Summing a Telescoping Series

Find the sum of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

**SOLUTION**

Use partial fractions to rewrite the $n$th term.

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

We compute a few partial sums to find a general formula.

$$s_1 = 1 - \frac{1}{2}$$

$$s_2 = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) = 1 - \frac{1}{3}$$

$$s_3 = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) = 1 - \frac{1}{4}$$

We can see that, in general,

$$s_n = 1 - \frac{1}{n},$$

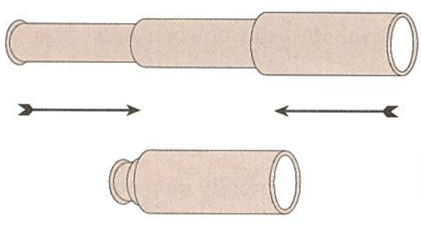
because all the terms between the first and last cancel when the parentheses are removed. Therefore, the sum of the series is

$$S = \lim_{n\to\infty} s_n = 1.$$

*Now try Exercise 48.*

**Telescoping Series**

We call the series in Example 7 a *telescoping series* because its partial sums collapse like an old handheld telescope.



The final section of this chapter will formalize some of the strategies used in Exploration 2 and Example 7 and will develop additional tests that can be used to determine series behavior at endpoints.

## Quick Review 9.4 *(For help, go to Sections 2.2 and 9.1.)*

In Exercises 1–5, find the limit of the expression as $n\to\infty$. Assume $x$ remains fixed as $n$ changes.

1. $\dfrac{n|x|}{n+1}$

2. $\dfrac{n^2|x-3|}{n(n-1)}$

3. $\dfrac{|x|^n}{n!}$

4. $\dfrac{(n+1)^4x^2}{(2n)^4}$

5. $\dfrac{|2x+1|^{n+1}2^n}{2^{n+1}|2x+1|^n}$

In Exercises 6–10, let $a_n$ be the $n$th term of the first and $b_n$ the $n$th term of the second series. Find the smallest positive integer $N$ for which $a_n > b_n$ for all $n \geq N$. Identify $a_n$ and $b_n$.

6. $\sum 5n, \quad \sum n^2$

7. $\sum n^5, \quad \sum 5^n$

8. $\sum \ln n, \quad \sum \sqrt{n}$

9. $\sum \dfrac{1}{10^n}, \quad \sum \dfrac{1}{n!}$

10. $\sum \dfrac{1}{n^2}, \quad \sum n^{-3}$

# Section 9.4 Exercises

In Exercises 1 and 2, find the values of $x$ for which the equation is an identity. Support your answer graphically.

**1.** $\dfrac{-1}{x+4} = 1 + (x+5) + (x+5)^2 + (x+5)^3 + \cdots$

**2.** $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$

In Exercises 3 and 4, use a comparison test to show that the series converges for all $x$.

**3.** $\displaystyle\sum_{n=0}^{\infty} \frac{x^{3n}}{2n!+1}$

**4.** $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{n!+2}$

In Exercises 5 and 6, show that the series converges absolutely.

**5.** $\displaystyle\sum_{n=0}^{\infty} \frac{(\cos x)^n}{n!+1}$

**6.** $\displaystyle\sum_{n=0}^{\infty} \frac{2(\sin x)^n}{n!+3}$

In Exercises 7–22, find the *radius* of convergence of the power series.

**7.** $\displaystyle\sum_{n=0}^{\infty} x^n$

**8.** $\displaystyle\sum_{n=0}^{\infty} (x+5)^n$

**9.** $\displaystyle\sum_{n=0}^{\infty} (-1)^n(4x+1)^n$

**10.** $\displaystyle\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n}$

**11.** $\displaystyle\sum_{n=0}^{\infty} \frac{(x-2)^n}{10^n}$

**12.** $\displaystyle\sum_{n=0}^{\infty} \frac{nx^n}{n+2}$

**13.** $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n\sqrt{n}\,3^n}$

**14.** $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!}$

**15.** $\displaystyle\sum_{n=0}^{\infty} \frac{n(x+3)^n}{5^n}$

**16.** $\displaystyle\sum_{n=0}^{\infty} \frac{nx^n}{4^n(n^2+1)}$

**17.** $\displaystyle\sum_{n=0}^{\infty} n!(x-4)^n$

**18.** $\displaystyle\sum_{n=0}^{\infty} \frac{\sqrt{n}\,x^n}{3^n}$

**19.** $\displaystyle\sum_{n=0}^{\infty} (-2)^n(n+1)(x-1)^n$

**20.** $\displaystyle\sum_{n=1}^{\infty} \frac{(4x-5)^{2n+1}}{n^{3/2}}$

**21.** $\displaystyle\sum_{n=1}^{\infty} \frac{(x+\pi)^n}{\sqrt{n}}$

**22.** $\displaystyle\sum_{n=0}^{\infty} \frac{(x-\sqrt{2})^{2n+1}}{2^n}$

In Exercises 23–28, find the *interval* of convergence of the series and, within this interval, the sum of the series as a function of $x$.

**23.** $\displaystyle\sum_{n=0}^{\infty} \frac{(x-1)^{2n}}{4^n}$

**24.** $\displaystyle\sum_{n=0}^{\infty} \frac{(x+1)^{2n}}{9^n}$

**25.** $\displaystyle\sum_{n=0}^{\infty} \left(\frac{\sqrt{x}}{2} - 1\right)^n$

**26.** $\displaystyle\sum_{n=0}^{\infty} (\ln x)^n$

**27.** $\displaystyle\sum_{n=0}^{\infty} \left(\frac{x^2-1}{3}\right)^n$

**28.** $\displaystyle\sum_{n=0}^{\infty} \left(\frac{\sin x}{2}\right)^n$

In Exercises 29–44, determine the convergence or divergence of the series. Identify the test (or tests) you use. There may be more than one correct way to determine convergence or divergence of a given series.

**29.** $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+1}$

**30.** $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n+1}$

**31.** $\displaystyle\sum_{n=1}^{\infty} \frac{n^2-1}{2^n}$

**32.** $\displaystyle\sum_{n=1}^{\infty} -\frac{1}{8^n}$

**33.** $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{3^n+1}$

**34.** $\displaystyle\sum_{n=1}^{\infty} n\sin\left(\frac{1}{n}\right)$

**35.** $\displaystyle\sum_{n=0}^{\infty} n^2e^{-n}$

**36.** $\displaystyle\sum_{n=0}^{\infty} \frac{n^{10}}{10^n}$

**37.** $\displaystyle\sum_{n=1}^{\infty} \frac{(n+3)!}{3!n!3^n}$

**38.** $\displaystyle\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n$

**39.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-2)^n}{3^n}$

**40.** $\displaystyle\sum_{n=1}^{\infty} n!e^{-n}$

**41.** $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n^3 2^n}$

**42.** $\displaystyle\sum_{n=1}^{\infty} \frac{n\ln n}{2^n}$

**43.** $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{(2n+1)!}$

**44.** $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$ (*Hint:* If you do not recognize $L$, try recognizing the reciprocal of $L$.)

**45.** Give an example to show that the converse of the $n$th-Term Test is false. That is, $\sum a_n$ might diverge even though $\lim_{n\to\infty} a_n = 0$.

**46.** Find two convergent series $\sum a_n$ and $\sum b_n$ such that $\sum (a_n/b_n)$ diverges.

**47. Writing to Learn** We reviewed in Section 9.1 how to find the interval of convergence for the geometric series $\sum_{n=0}^{\infty} x^n$. Can we find the interval of convergence of a geometric series by using the Ratio Test? Explain.

In Exercises 48–54, find the sum of the telescoping series.

**48.** $\displaystyle\sum_{n=1}^{\infty} \frac{4}{(4n-3)(4n+1)}$

**49.** $\displaystyle\sum_{n=1}^{\infty} \frac{6}{(2n-1)(2n+1)}$

**50.** $\displaystyle\sum_{n=1}^{\infty} \frac{40n}{(2n-1)^2(2n+1)^2}$

**51.** $\displaystyle\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}$

**52.** $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right)$

**53.** $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{\ln(n+2)} - \frac{1}{\ln(n+1)}\right)$

**54.** $\displaystyle\sum_{n=1}^{\infty} \left(\tan^{-1}(n) - \tan^{-1}(n+1)\right)$

## Standardized Test Questions

You may use a graphing calculator to solve the following problems.

**55. True or False** If a series converges absolutely, then it converges. Justify your answer.

**56. True or False** If the radius of convergence of a power series is 0, then the series diverges for all real numbers. Justify your answer.

**57. Multiple Choice** Which of the following gives $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ for the series $\sum_{n=0}^{\infty} \frac{2^n}{(-3)^n}$?

(A) $-3/2$ (B) $-2/3$ (C) 1 (D) 0 (E) $\infty$

**58. Multiple Choice** Which of the following gives the radius of convergence of the series $\sum_{n=1}^{\infty} \frac{(2x-3)^n}{n}$?

(A) 2 (B) 1 (C) 1/2 (D) 0 (E) $\infty$

**59. Multiple Choice** Which of the following describes the behavior of the series $\sum_{n=1}^{\infty} \frac{(\sin x)^n}{2^n n^2}$?

I. diverges

II. converges

III. converges absolutely

(A) I only (B) II only (C) III only

(D) I & II only (E) II & III only

**60. Multiple Choice** Which of the following gives the sum of the telescoping series $\sum_{n=1}^{\infty} \frac{3}{(3n-1)(3n+2)}$?

(A) 3/10 (B) 3/8 (C) 9/22 (D) 1/2 (E) The series diverges.

## Explorations

**Group Activity** ***Nondecreasing Sequences*** As you already know, a nondecreasing (or increasing) function $f(x)$ that is bounded from above on an interval $[a, \infty)$ has a limit as $x \to \infty$ that is less than or equal to the bound. The same is true of sequences of numbers. If $s_1 \le s_2 \le s_3 \le \dots \le s_n \dots$ and there is a number $M$ such that $|s_n| \le M$ for all $n$, then the sequence converges to a limit $S \le M$. You will need this fact as you work through Exercises 61 and 62.

**61. *Proof of the Direct Comparison Test, Part a*** Let $\sum a_n$ be a series with no negative terms, and let $\sum c_n$ be a convergent series such that $a_n \le c_n$ for all $n \ge N$, for some integer $N$.

**(a)** Show that the partial sums of $\sum a_n$ are bounded above by

$$a_1 + \cdots + a_N + \sum_{n=N+1}^{\infty} c_n.$$

**(b)** Explain why this shows that $\sum a_n$ must converge.

**62. *Proof of the Direct Comparison Test, Part b*** Let $\sum a_n$ be a series with no negative terms, and let $\sum d_n$ be a divergent series of nonnegative terms such that $a_n \ge d_n$ for all $n \ge N$, for some integer $N$.

**(a)** Show that the partial sums of $\sum d_n$ are bounded above by

$$d_1 + \cdots + d_N + \sum_{n=N+1}^{\infty} a_n.$$

**(b)** Explain why this leads to a contradiction if we assume that $\sum a_n$ converges.

**63. Group Activity** Within your group, have each student make up a power series with radius of convergence equal to one of the numbers $1, 2, \dots, n$. Then exchange series with another group and match the other group's series with the correct radii of convergence.

## Extending the Ideas

**64.** We can show that the series

$$\sum_{n=0}^{\infty} \frac{n^2}{2^n}$$

converges by the Ratio Test, but what is its sum?

To find out, express $1/(1-x)$ as a geometric series. Differentiate both sides of the resulting equation with respect to $x$, multiply both sides of the result by $x$, differentiate again, multiply by $x$ again, and set $x$ equal to 1/2. What do you get? (*Source:* David E. Dobbs's letter to the editor, *Illinois Mathematics Teacher,* Vol. 33, Issue 4, 1982, p. 27.)

# 9.5 Testing Convergence at Endpoints

**What you'll learn about**

- Integral Test
- Harmonic Series and *p*-series
- Comparison Tests
- Alternating Series
- Absolute and Conditional Convergence
- Intervals of Convergence
- A Word of Caution

**. . . and why**

Additional tests for convergence of series are introduced in this section.

## Integral Test

In Exploration 1 of Section 9.4, you showed that $\sum 1/n$ *diverges* by modeling it as a sum of rectangle areas that contain the area under the curve $y = 1/x$ from 1 to $\infty$. You also showed that $\sum 1/n^2$ *converges* by modeling it as a sum of rectangle areas contained by the area under the curve $y = 1/x^2$ from 1 to $\infty$. This area-based convergence test in its general form is known as the *Integral Test.*

**THEOREM 10 The Integral Test**

Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where $f$ is a continuous, positive, decreasing function of $x$ for all $x \geq N$ ($N$ a positive integer). Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x)\,dx$ either both converge or both diverge.

**Proof** We will illustrate the proof for $N = 1$ to keep the notation simple, but the illustration can be shifted horizontally to any value of $N$ without affecting the logic of the proof.

The proof is entirely contained in these two pictures (Figure 9.15):

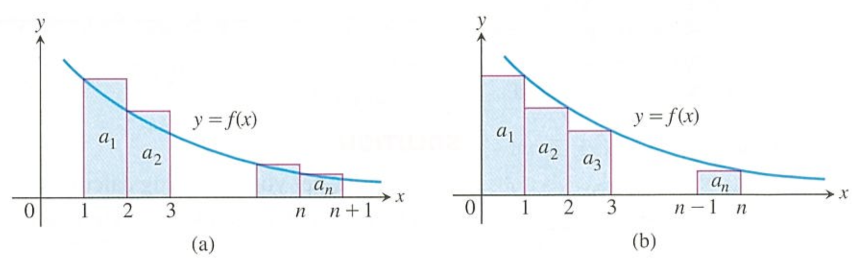



**Figure 9.15** (a) The sum $a_1 + a_2 + \cdots + a_n$ provides an upper bound for $\int_1^{n+1} f(x)\,dx$. (b) The sum $a_2 + a_3 + \cdots + a_n$ provides a lower bound for $\int_1^{n} f(x)\,dx$. (Theorem 10)

We leave it to you (in Exercise 52) to supply the words. ■

**EXAMPLE 1 Applying the Integral Test**

Does $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$ converge?

**SOLUTION**

The Integral Test applies because

$$f(x) = \frac{1}{x\sqrt{x}}$$

is a continuous, positive, decreasing function of $x$ for $x > 1$.

We have

$$\int_1^{\infty} \frac{1}{x\sqrt{x}}\,dx = \lim_{k\to\infty} \int_1^{k} x^{-3/2}\,dx = \lim_{k\to\infty} \left[-2x^{-1/2}\right]_1^k$$

$$= \lim_{k\to\infty} \left(-\frac{2}{\sqrt{k}} + 2\right) = 2.$$

Since the integral converges, so must the series. *Now try Exercise 1.*

**Caution**

The series and the integral in the Integral Test need not have the same value in the convergent case. Although the integral converges to 2 in Example 1, the series might have a quite different sum. If you use your calculator to compute or graph partial sums for the series, you can see that the 11th partial sum is already greater than 2. The *Technology Resource Manual* contains two programs, PARTSUMT, which displays partial sums in table form, and PARTSUMG, which displays partial sums graphically.

## Harmonic Series and $p$-series

The Integral Test can be used to settle the question of convergence for any series of the form $\sum_{n=1}^{\infty}(1/n^p)$, $p$ a real constant. (The series in Example 1 had this form, with $p = 3/2$.) Such a series is called a **$p$-series.**

**EXPLORATION 1 The $p$-Series Test**

1. Use the Integral Test to prove that $\sum_{n=1}^{\infty}(1/n^p)$ converges if $p > 1$.
2. Use the Integral Test to prove that $\sum_{n=1}^{\infty}(1/n^p)$ diverges if $p < 1$.
3. Use the Integral Test to prove that $\sum_{n=1}^{\infty}(1/n^p)$ diverges if $p = 1$.

**What is harmonic about the harmonic series?**

The terms in the harmonic series correspond to the nodes on a vibrating string that produce multiples of the fundamental frequency. For example, 1/2 produces the harmonic that is twice the fundamental frequency, 1/3 produces a frequency that is three times the fundamental frequency, and so on. The fundamental frequency is the lowest note or pitch we hear when a string is plucked. (Figure 9.17)

The $p$-series with $p = 1$ is the **harmonic series,** and it is probably the most famous divergent series in mathematics. The $p$-Series Test shows that the harmonic series is just *barely* divergent; if we increase $p$ to 1.000000001, for instance, the series converges!

The slowness with which the harmonic series approaches infinity is most impressive. Consider the following example.

### EXAMPLE 2 The Slow Divergence of the Harmonic Series

Approximately how many terms of the harmonic series are required to form a partial sum larger than 20?

**SOLUTION**

Before you set your graphing calculator to the task of finding this number, you might want to estimate how long the calculation might take. The graphs tell the story (Figure 9.16).

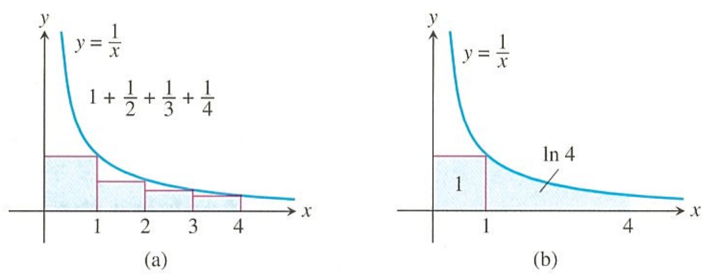



**Figure 9.16** Finding an upper bound for one of the partial sums of the harmonic series. (Example 2)

Let $H_n$ denote the $n$th partial sum of the harmonic series. Comparing the two graphs, we see that $H_4 < (1 + \ln 4)$ and (in general) that $H_n \le (1 + \ln n)$. If we wish $H_n$ to be greater than 20, then

$$1 + \ln n > H_n > 20$$
$$1 + \ln n > 20$$
$$\ln n > 19$$
$$n > e^{19}.$$

The exact value of $e^{19}$ rounds up to 178,482,301. It will take *at least* that many terms of the harmonic series to move the partial sums beyond 20. It would take your calculator several weeks to compute a partial sum of this many terms. Nonetheless, the harmonic series really does diverge! ***Now try Exercise 3.***

**Figure 9.17** On a guitar, the second harmonic note is produced when the finger is positioned halfway between the bridge and nut of the string while the string is plucked with the other hand.

## Comparison Tests

The $p$-Series Test tells everything there is to know about the convergence or divergence of series of the form $\sum (1/n^p)$. This is admittedly a rather narrow class of series, but we can test many other kinds (including those in which the $n$th term is any rational function of $n$) by *comparing* them to $p$-series.

The Direct Comparison Test (Theorem 7, Section 9.4) is one method of comparison, but the *Limit Comparison Test* is another.

**THEOREM 11 The Limit Comparison Test (LCT)**

Suppose that $a_n > 0$ and $b_n > 0$ for all $n \geq N$ ($N$ a positive integer).

**1.** If $\lim_{n\to\infty} \dfrac{a_n}{b_n} = c,\ 0 < c < \infty$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.

**2.** If $\lim_{n\to\infty} \dfrac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.

**3.** If $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

We omit the proof.

### EXAMPLE 3 Using the Limit Comparison Test

Determine whether the series converge or diverge.

**(a)** $\dfrac{3}{4} + \dfrac{5}{9} + \dfrac{7}{16} + \dfrac{9}{25} + \cdots = \sum_{n=1}^{\infty} \dfrac{2n+1}{(n+1)^2}$

**(b)** $\dfrac{1}{1} + \dfrac{1}{3} + \dfrac{1}{7} + \dfrac{1}{15} + \cdots = \sum_{n=1}^{\infty} \dfrac{1}{2^n - 1}$

**(c)** $\dfrac{8}{4} + \dfrac{11}{21} + \dfrac{14}{56} + \dfrac{17}{115} + \cdots = \sum_{n=2}^{\infty} \dfrac{3n+2}{n^3 - 2n}$

**(d)** $\sin 1 + \sin\dfrac{1}{2} + \sin\dfrac{1}{3} + \cdots = \sum_{n=1}^{\infty} \sin\left(\dfrac{1}{n}\right)$

**SOLUTION**

**(a)** For $n$ large, $\dfrac{2n+1}{(n+1)^2}$ behaves like $\dfrac{2n}{n^2} = \dfrac{2}{n}$,

so we compare terms of the given series to terms of $\sum (1/n)$ and try the LCT.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(2n+1)/(n+1)^2}{1/n}$$

$$= \lim_{n\to\infty} \frac{2n+1}{(n+1)^2} \cdot \frac{n}{1}$$

Applying l'Hôpital's rule, $\lim_{n\to\infty} \dfrac{2n^2+n}{(n+1)^2} = \lim_{n\to\infty} \dfrac{4n+1}{2(n+1)} = 2.$

Since the limit is positive and $\sum (1/n)$ diverges,

$$\sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2}$$

also diverges.

*continued*

**(b)** For $n$ large, $1/(2^n - 1)$ behaves like $1/2^n$, so we compare the given series to $\sum (1/2^n)$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{2^n - 1} \cdot \frac{2^n}{1}$$

$$= \lim_{n\to\infty} \frac{2^n}{2^n - 1}$$

$$= \lim_{n\to\infty} \frac{1}{1 - (1/2^n)} = 1$$

Since $\sum (1/2^n)$ converges (geometric, $r = 1/2$), the LCT guarantees that

$$\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$$

also converges.

**(c)** For $n$ large,

$$\frac{3n + 2}{n^3 - 2n}$$

behaves like $3/n^2$, so we compare the given series to $\sum (1/n^2)$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{3n + 2}{n^3 - 2n} \cdot \frac{n^2}{1}$$

$$= \lim_{n\to\infty} \frac{3n^3 + 2n^2}{n^3 - 2n} = 3$$

Since $\sum (1/n^2)$ converges by the $p$-Series Test,

$$\sum_{n=2}^{\infty} \frac{3n + 2}{n^3 - 2n}$$

also converges (by the LCT).

**(d)** Recall that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1,$$

so we try the LCT by comparing the given series to $\sum (1/n)$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin (1/n)}{(1/n)} = 1$$

Since $\sum (1/n)$ diverges, $\sum_{n=1}^{\infty} \sin (1/n)$ also diverges. ***Now try Exercise 5.***

As Example 3 suggests, applying the Limit Comparison Test has strong connections to analyzing end behavior in functions. In part (c) of Example 3, we could have reached the same conclusion if $a_n$ had been *any* linear polynomial in $n$ divided by *any* cubic polynomial in $n$, since any such rational function "in the end" will grow like $1/n^2$.

## Alternating Series

A series in which the terms are alternately positive and negative is an **alternating series.**

Here are three examples.

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots + \frac{(-1)^{n+1}}{n} + \cdots \tag{1}$$

$$-2 + 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots + \frac{(-1)^n 4}{2^n} + \cdots \tag{2}$$

$$1 - 2 + 3 - 4 + 5 - 6 + \cdots + (-1)^{n+1} n + \cdots \tag{3}$$

Series 1, called the **alternating harmonic series,** converges, as we will see shortly. (You may have come to this conclusion already in Exploration 2 of Section 9.4.) Series 2, a geometric series with $a = -2$, $r = -1/2$, converges to $-2/[1 + (1/2)] = -4/3$. Series 3 diverges by the $n$th-Term Test.

We prove the convergence of the alternating harmonic series by applying the following test.

**THEOREM 12 The Alternating Series Test (Leibniz's Theorem)**

The series

$$\sum_{n=1}^{\infty} (-1)^{n+1} u_n = u_1 - u_2 + u_3 - u_4 + \cdots$$

converges if all three of the following conditions are satisfied:

1. each $u_n$ is positive;
2. $u_n \geq u_{n+1}$ for all $n \geq N$, for some integer $N$;
3. $\lim_{n\to\infty} u_n \to 0$.

Figure 9.18 illustrates the convergence of the partial sums to their limit $L$.

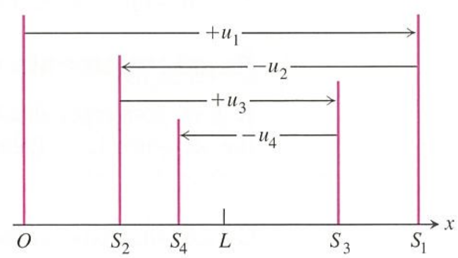



**Figure 9.18** Closing in on the sum of a convergent alternating series. (Theorem 12)

The figure that proves the Alternating Series Test actually proves more than the *fact* of convergence; it also shows the *way* that an alternating series converges when it satisfies the conditions of the test. The partial sums keep "overshooting" the limit as they go back and forth on the number line, gradually closing in as the terms tend to zero. If we stop at the $n$th partial sum, we know that the next term ($\pm u_{n+1}$) will again cause us to overshoot the limit in the positive direction or negative direction, depending on the sign carried by $u_{n+1}$. This gives us a convenient bound for the truncation error, which we state as another theorem.

**THEOREM 13 The Alternating Series Estimation Theorem**

If the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}u_n$ satisfies the conditions of Theorem 12, then the truncation error for the $n$th partial sum is less than $u_{n+1}$ and has the same sign as the first unused term.

### EXAMPLE 4 The Alternating Harmonic Series

Prove that the alternating harmonic series is convergent, but not absolutely convergent. Find a bound for the truncation error after 99 terms.

**SOLUTION**

The terms are strictly alternating in sign and decrease in absolute value from the start:

$$1 > \frac{1}{2} > \frac{1}{3} > \cdots. \qquad \text{Also,} \quad \frac{1}{n} \to 0.$$

By the Alternating Series Test,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

converges.

On the other hand, the series $\sum_{n=1}^{\infty}(1/n)$ of absolute values is the harmonic series, which diverges, so the alternating harmonic series is not absolutely convergent.

The Alternating Series Estimation Theorem guarantees that the truncation error after 99 terms is less than $u_{99+1} = 1/(99+1) = 1/100$. *Now try Exercise 23.*

**A Note on the Error Bound**

Theorem 13 does not give a *formula* for the truncation error, but a *bound* for the truncation error. The bound might be fairly conservative. For example, the first 99 terms of the alternating harmonic series add to about 0.6981721793, while the series itself has a sum of $\ln 2 \approx 0.6931471806$. That makes the actual truncation error very close to 0.005, about half the size of the bound of 0.01 given by Theorem 13.

## Absolute and Conditional Convergence

Because the alternating harmonic series is convergent but not absolutely convergent, we say it is **conditionally convergent** (or **converges conditionally**).

We take it for granted that we can rearrange the terms of a *finite* sum without affecting the sum. We can also rearrange a *finite number* of terms of an infinite series without affecting the sum. But if we rearrange an infinite number of terms of an infinite series, we can be sure of leaving the sum unaltered *only if it converges absolutely.*

**Rearrangements of Absolutely Convergent Series**

If $\sum a_n$ converges absolutely, and if $b_1, b_2, b_3, \ldots, b_n, \ldots$ is any rearrangement of the sequence $\{a_n\}$, then $\sum b_n$ converges absolutely and $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n$.

On the other hand, consider this:

**Rearrangements of Conditionally Convergent Series**

If $\sum a_n$ converges conditionally, then the terms can be rearranged to form a divergent series. The terms can also be rearranged to form a series that converges to *any* preassigned sum.

This seems incredible, but it is a logical consequence of the definition of the sum as the *limit of the sequence of partial sums.* A conditionally convergent series consists of positive terms that sum to $\infty$ and negative terms that sum to $-\infty$, so we can manipulate the partial sums to do virtually anything we wish. We illustrate the technique with the alternating harmonic series.

### EXAMPLE 5 Rearranging the Alternating Harmonic Series

Show how to rearrange the terms of

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

to form

**(a)** a divergent series; **(b)** a series that converges to $\pi$.

**SOLUTION**

The series of positive terms,

$$1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n+1} + \cdots,$$

diverges to $\infty$, while the series of negative terms,

$$-\frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \cdots - \frac{1}{2n} - \cdots,$$

diverges to $-\infty$. No matter what finite number of terms we use, the remaining positive terms or negative terms still diverge. So, we build our series as follows:

**(a)** Start by adding positive terms until the partial sum is greater than 1. Then add negative terms until the partial sum is less than $-2$. Then add positive terms until the sum is greater than 3. Then add negative terms until the sum is less than $-4$. Continue in this manner indefinitely, so that the sequence of partial sums swings arbitrarily far in both directions and hence diverges.

**(b)** Start by adding positive terms until the partial sum is greater than $\pi$. Then add negative terms until the partial sum is less than $\pi$. Then add positive terms until the sum is greater than $\pi$. Continue in this manner indefinitely, always closing in on $\pi$. Since the positive and negative terms of the original series both approach zero, the amount by which the partial sums exceed or fall short of $\pi$ approaches zero. *Now try Exercise 33.*

## Intervals of Convergence

Our purpose in this section has been to develop tests for convergence that can be used at the endpoints of the intervals of absolute convergence of power series. There are three possibilities at each endpoint: The series could diverge, it could converge absolutely, or it could converge conditionally.

### How to Test a Power Series $\sum_{n=0}^{\infty} c_n(x-a)^n$ for Convergence

1. Use the Ratio Test to find the values of $x$ for which the series converges absolutely. Ordinarily, this is an open interval

$$a - R < x < a + R.$$

In some instances, the series converges for all values of $x$. In rare cases, the series converges only at $x = a$.

2. If the interval of absolute convergence is finite, test for convergence or divergence at each endpoint. The Ratio Test fails at these points. Use a comparison test, the Integral Test, or the Alternating Series Test.

3. If the interval of absolute convergence is $a - R < x < a + R$, conclude that the series diverges (it does not even converge conditionally) for $|x - a| > R$, because for those values of $x$ the $n$th term does not approach zero.

### EXAMPLE 6 Finding Intervals of Convergence

For what values of $x$ do the following series converge?

**(a)** $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}\frac{x^{2n}}{2n} = \frac{x^2}{2} - \frac{x^4}{4} + \frac{x^6}{6} - \cdots$

**(b)** $\displaystyle\sum_{n=0}^{\infty} \frac{(10x)^n}{n!} = 1 + 10x + \frac{100x^2}{2!} + \frac{1000x^3}{3!} + \cdots$

**(c)** $\displaystyle\sum_{n=0}^{\infty} n!(x+1)^n = 1 + (x+1) + 2!(x+1)^2 + 3!(x+1)^3 + \cdots$

**(d)** $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^n}{2n} = \frac{(x-3)}{2} + \frac{(x-3)^2}{4} + \frac{(x-3)^3}{6} + \cdots$

**SOLUTION**

We apply the Ratio Test to find the interval of absolute convergence, then check the endpoints if they exist.

**(a)**
$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\frac{x^{2n+2}}{2n+2}\cdot\frac{2n}{x^{2n}}$$
$$= \lim_{n\to\infty}\left(\frac{2n}{2n+2}\right)x^2$$
$$= \lim_{n\to\infty}\left(\frac{2}{2}\right)x^2 \qquad \text{Apply l'Hôpital's rule.}$$
$$= x^2$$

The series converges absolutely for $x^2 < 1$, i.e., on the interval $(-1, 1)$. At $x = 1$, the series is

$$\sum \frac{(-1)^{n+1}}{2n},$$

which converges by the Alternating Series Test. (It is half the sum of the alternating harmonic series.) At $x = -1$, the series is the same as at $x = 1$, so it converges. The interval of convergence is $[-1, 1]$.

**(b)**
$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\frac{|10x|^{n+1}}{(n+1)!}\cdot\frac{n!}{|10x|^n}$$
$$= \lim_{n\to\infty}\frac{|10x|}{n+1} = 0$$

The series converges absolutely for all $x$.

**(c)**
$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\frac{(n+1)!|x+1|^{n+1}}{n!|x+1|^n}$$
$$= \lim_{n\to\infty}(n+1)|x+1| = \begin{cases}\infty, & x \neq -1\\ 0, & x = -1\end{cases}$$

The series converges only at $x = -1$.

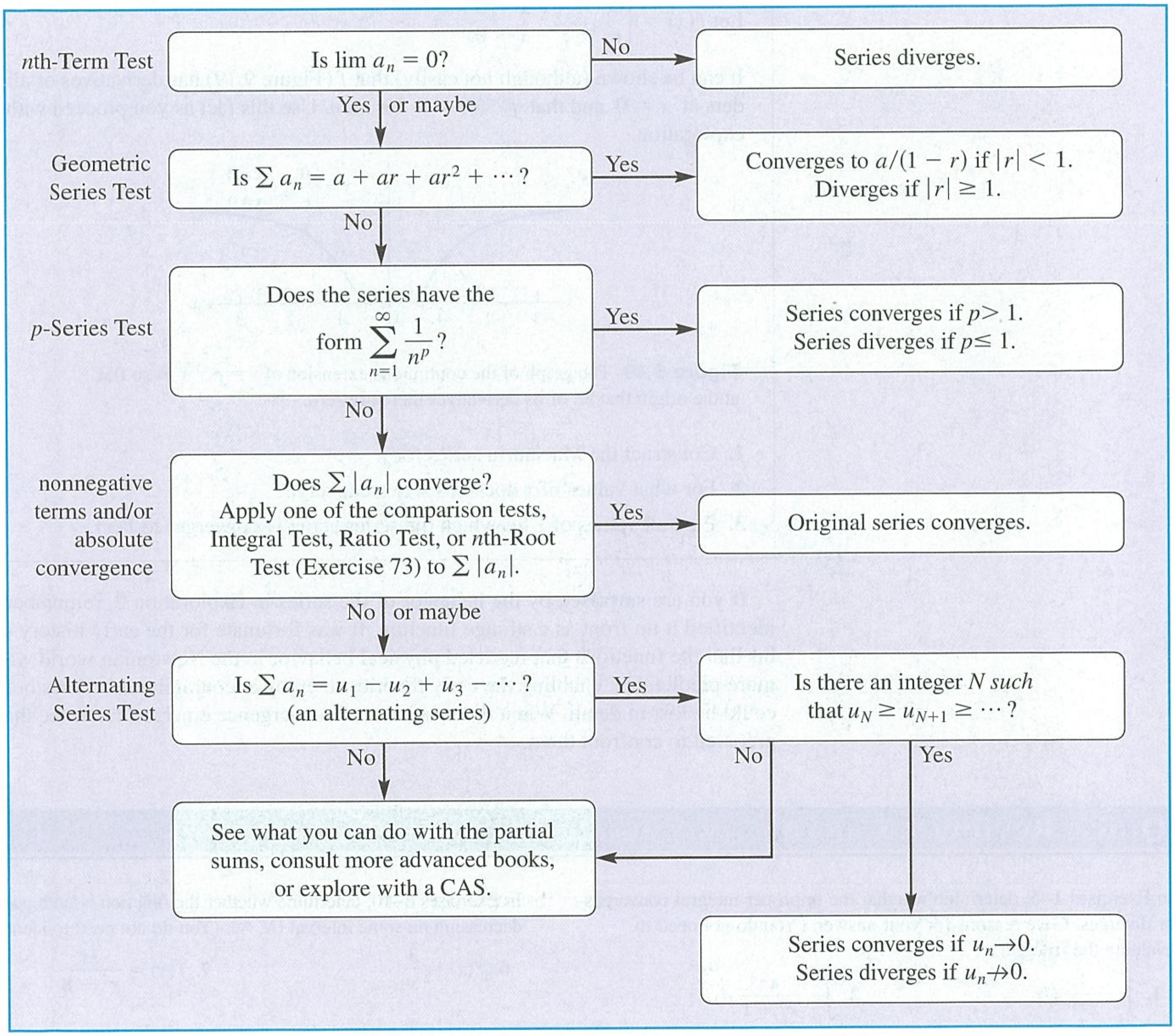
**(d)**
$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\frac{|x-3|^{n+1}}{2n+2}\cdot\frac{2n}{|x-3|^n}$$
$$= \lim_{n\to\infty}\left(\frac{2n}{2n+2}\right)|x-3| = |x-3|$$

*continued*

The series converges absolutely for $|x - 3| < 1$, i.e., on the interval $(2, 4)$. At $x = 2$, the series is $\sum (-1)^n/2n$, which converges by the Alternating Series Test. At $x = 4$, the series is $\sum 1/2n$, which diverges by limit comparison with the harmonic series. The interval of convergence is $[2, 4)$. **Now try Exercise 41.**

To facilitate testing convergence at endpoints we can use the following flowchart.

**Procedure for Determining Convergence**




## A Word of Caution

Although we can use the tests we have developed to find where a given power series converges, they do not tell us what function that power series is converging *to*. Even if the series is known to be a Maclaurin series generated by a function *f*, we cannot automatically conclude that the series converges *to the function f* on its interval of convergence. That is why it is so important to estimate the error.

For example, we can use the Ratio Test to show that the Maclaurin series for $\sin x$, $\cos x$, and $e^x$ all converge absolutely for all real numbers. However, the reason we know that they converge to $\sin x$, $\cos x$, and $e^x$ is that we used the Remainder Estimation Theorem to show that the respective truncation errors went to zero.

The following exploration shows what can happen with a strange function.

## EXPLORATION 2 The Maclaurin Series of a Strange Function

Let $f(x) = \begin{cases} 0, & x = 0 \\ e^{-1/x^2}, & x \neq 0. \end{cases}$

It can be shown (although not easily) that $f$ (Figure 9.19) has derivatives of all orders at $x = 0$ and that $f^{(n)}(0) = 0$ for all $n$. Use this fact as you proceed with the exploration.

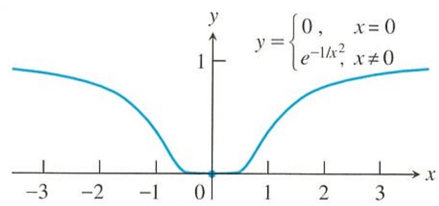



**Figure 9.19** The graph of the continuous extension of $y = e^{-1/x^2}$ is so flat at the origin that all of its derivatives there are zero.

1. Construct the Maclaurin series for $f$.
2. For what values of $x$ does this series converge?
3. Find all values of $x$ for which the series actually converges to $f(x)$.

If you are surprised by the behavior of the series in Exploration 2, remember that we identified it up front as a strange function. It was fortunate for the early history of calculus that the functions that modeled physical behavior in the Newtonian world were much more predictable, enabling the early theories to enjoy encouraging successes before they could be lost in detail. When the subtleties of convergence emerged later, the theory was prepared to confront them.

## Quick Review 9.5 *(For help, go to Sections 1.2 and 8.3.)*

In Exercises 1–5, determine whether the improper integral converges or diverges. Give reasons for your answer. (You do not need to evaluate the integral.)

**1.** $\int_1^\infty \frac{1}{x^{4/3}}\,dx$

**2.** $\int_1^\infty \frac{x^2}{x^3+1}\,dx$

**3.** $\int_1^\infty \frac{\ln x}{x}\,dx$

**4.** $\int_1^\infty \frac{1+\cos x}{x^2}\,dx$

**5.** $\int_1^\infty \frac{\sqrt{x}}{x+1}\,dx$

In Exercises 6–10, determine whether the function is both positive and decreasing on some interval $(N, \infty)$. (You do not need to identify $N$.)

**6.** $f(x) = \frac{3}{x}$

**7.** $f(x) = \frac{7x}{x^2-8}$

**8.** $f(x) = \frac{3+x^2}{3-x^2}$

**9.** $f(x) = \frac{\sin x}{x^5}$

**10.** $f(x) = \ln(1/x)$

## Section 9.5 Exercises

In Exercises 1and 2, use the Integral Test to determine convergence or divergence of the series.

**1.** $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$ **2.** $\sum_{n=1}^{\infty} n^{-3/2}$

**3.** Find the first six partial sums of $\sum_{n=1}^{\infty} \frac{1}{n}$.

**4.** If $S_k$ is the $k$-th partial sum of $\sum_{n=1}^{\infty} \frac{1}{n}$, find the first value of $k$ for which $S_k > 4$.

In Exercises 5 and 6, use the Limit Comparison Test to determine convergence or divergence of the series.

**5.** $\sum_{n=1}^{\infty} \frac{3n-1}{n^2+1}$ **6.** $\sum_{n=0}^{\infty} \frac{2^n}{3^n+1}$

In Exercises 7–22, determine whether the series converges or diverges. There may be more than one correct way to determine convergence or divergence of a given series.

**7.** $\sum_{n=1}^{\infty} \frac{5}{n+1}$ **8.** $\sum_{n=1}^{\infty} \frac{3}{\sqrt{n}}$

**9.** $\sum_{n=2}^{\infty} \frac{\ln n}{n}$ **10.** $\sum_{n=1}^{\infty} \frac{1}{2n-1}$

**11.** $\sum_{n=1}^{\infty} \frac{1}{(\ln 2)^n}$ **12.** $\sum_{n=1}^{\infty} \frac{1}{(\ln 3)^n}$

**13.** $\sum_{n=1}^{\infty} n \sin\left(\frac{1}{n}\right)$ **14.** $\sum_{n=0}^{\infty} \frac{e^n}{1+e^{2n}}$

**15.** $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$ **16.** $\sum_{n=1}^{\infty} \frac{5n^3-3n}{n^2(n+2)(n^2+5)}$

**17.** $\sum_{n=1}^{\infty} \frac{3^{n-1}+1}{3^n}$ **18.** $\sum_{n=2}^{\infty} (-1)^{n+1}\frac{1}{\ln n}$

**19.** $\sum_{n=1}^{\infty} (-1)^{n+1}\frac{10^n}{n^{10}}$ **20.** $\sum_{n=1}^{\infty} (-1)^{n+1}\frac{\sqrt{n}+1}{n+1}$

**21.** $\sum_{n=2}^{\infty} (-1)^{n+1}\frac{\ln n}{\ln n^2}$ **22.** $\sum_{n=1}^{\infty} \left(\frac{1}{n}-\frac{1}{n^2}\right)$

In Exercises 23–26, determine whether the series converges absolutely, converges conditionally, or diverges. Give reasons for your answer. Find a bound for the truncation error after 99 terms.

**23.** $\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1+n}{n^2}$ **24.** $\sum_{n=1}^{\infty} (-1)^{n+1}(0.1)^n$

**25.** $\sum_{n=2}^{\infty} (-1)^{n+1}\frac{1}{n \ln n}$ **26.** $\sum_{n=1}^{\infty} (-1)^n n^2\left(\frac{2}{3}\right)^n$

In Exercises 27–32, determine whether the series converges absolutely, converges conditionally, or diverges. Give reasons for your answers.

**27.** $\sum_{n=1}^{\infty} (-1)^{n+1}\frac{n!}{2^n}$ **28.** $\sum_{n=1}^{\infty} (-1)^{n+1}\frac{\sin n}{n^2}$

**29.** $\sum_{n=1}^{\infty} \frac{(-1)^n}{1+\sqrt{n}}$ **30.** $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n\sqrt{n}}$

**31.** $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n}$ **32.** $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}+\sqrt{n+1}}$

In Exercises 33 and 34, show how to rearrange the terms of the series from the specified exercise to form **(a)** a divergent series, and **(b)** a series that converges to 4.

**33.** Exercise 23 **34.** Exercise 25

In Exercises 35–50, find **(a)** the *interval* of convergence of the series. For what values of $x$ does the series converge **(b)** absolutely, **(c)** conditionally?

**35.** $\sum_{n=0}^{\infty} x^n$ **36.** $\sum_{n=0}^{\infty} (x+5)^n$

**37.** $\sum_{n=0}^{\infty} (-1)^n(4x+1)^n$ **38.** $\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n}$

**39.** $\sum_{n=0}^{\infty} \frac{(x-2)^n}{10^n}$ **40.** $\sum_{n=0}^{\infty} \frac{nx^n}{n+2}$

**41.** $\sum_{n=1}^{\infty} \frac{x^n}{n\sqrt{n}\,3^n}$ **42.** $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!}$

**43.** $\sum_{n=0}^{\infty} \frac{n(x+3)^n}{5^n}$ **44.** $\sum_{n=0}^{\infty} \frac{nx^n}{4^n(n^2+1)}$

**45.** $\sum_{n=0}^{\infty} \frac{\sqrt{n}x^n}{3^n}$ **46.** $\sum_{n=0}^{\infty} n!(x-4)^n$

**47.** $\sum_{n=0}^{\infty} (-2)^n(n+1)(x-1)^n$ **48.** $\sum_{n=1}^{\infty} \frac{(4x-5)^{2n+1}}{n^{3/2}}$

**49.** $\sum_{n=1}^{\infty} \frac{(x+\pi)^n}{\sqrt{n}}$ **50.** $\sum_{n=0}^{\infty} (\ln x)^n$

**51.** Not only do the figures in Example 2 show that the $n$th partial sum of the harmonic series is less than $1 + \ln n$; they also show that it is *greater* than $\ln(n+1)$. Suppose you had started summing the harmonic series with $S_1 = 1$ at the time the universe was formed, 13 billion years ago. If you had been able to add a term every *second* since then, about how large would your partial sum be today? (Assume a 365-day year.)

**52. Writing to Learn** Write out a proof of the Integral Test (Theorem 10) for $N = 1$, explaining what you see in Figure 9.15.

**53.** ***(Continuation of Exercise 52)*** Relabel the pictures for an arbitrary $N$ and explain why the same conclusions about convergence can be drawn.

**54.** In each of the following cases, decide whether the infinite series converges. Justify your answer.

**(a)** $\sum_{k=1}^{\infty} \frac{1}{\sqrt{2k+7}}$ **(b)** $\sum_{k=1}^{\infty} \left(1+\frac{1}{k}\right)^k$

**(c)** $\sum_{k=1}^{\infty} \frac{\cos k}{k^2+\sqrt{k}}$ **(d)** $\sum_{k=3}^{\infty} \frac{18}{k(\ln k)}$

In Exercises 55 and 56, find the interval of convergence of the series.

**55.** $\sum_{n=1}^{\infty} \frac{n^n(x+2)^n}{3^n n!}$ **56.** $\sum_{n=1}^{\infty} \frac{n!x^n}{n^n 5^n}$

**57.** Construct a series that diverges more slowly than the harmonic series. Justify your answer.

**58.** Let $a_k = (-1)^{k+1}\int_0^{1/k} 6(kx)^2\,dx$.

**(a)** Evaluate $a_k$.

**(b)** Show that $\sum_{k=1}^{\infty} a_k$ converges.

**(c)** Show that

$$1 \le \sum_{k=1}^{\infty} a_k \le \frac{3}{2}.$$

**59. (a)** Determine whether the series

$$A = \sum_{n=1}^{\infty} \frac{n}{3n^2+1}$$

converges or diverges. Justify your answer.

**(b)** If $S$ is the series formed by multiplying the $n$th term in $A$ by the $n$th term in $\sum_{n=1}^{\infty}(3/n)$, write an expression using summation notation for $S$ and determine whether $S$ converges or diverges.

**60. (a)** Find the Taylor series generated by $f(x) = \ln(1+x)$ at $x = 0$. Include an expression for the general term.

**(b)** For what values of $x$ does the series in part (a) converge?

**(c)** Use Theorem 13 to find a bound for the error in evaluating $\ln(3/2)$ by using only the first five nonzero terms of the series in part (a).

**(d)** Use the result found in part (a) to determine the logarithmic function whose Taylor series is

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^{2n}}{2n}.$$

**61.** Determine all values of $x$ for which the series

$$\sum_{k=0}^{\infty} \frac{2^k x^k}{\ln(k+2)}$$

converges. Justify your answer.

**62.** Consider the series $\sum_{n=2}^{\infty} \frac{1}{n^p \ln n}$, where $p \ge 0$.

**(a)** Show that the series converges for $p > 1$.

**(b)** **Writing to Learn** Determine whether the series converges or diverges for $p = 1$. Show your analysis.

**(c)** Show that the series diverges for $0 \le p < 1$.

**63.** The Maclaurin series for $1/(1+x)$ converges for $-1 < x < 1$, but when we integrate it term by term, the resulting series for $\ln|1+x|$ converges for $-1 < x \le 1$. Verify the convergence at $x = 1$.

**64.** The Maclaurin series for $1/(1+x^2)$ converges for $-1 < x < 1$, but when we integrate it term by term, the resulting series for arctan $x$ converges for $-1 \le x \le 1$. Verify the convergence at $x = 1$ and $x = -1$.

**65. (a)** The series

$$\frac{1}{3}-\frac{1}{2}+\frac{1}{9}-\frac{1}{4}+\frac{1}{27}-\frac{1}{8}+\cdots+\frac{1}{3^n}-\frac{1}{2^n}+\cdots$$

fails to satisfy one of the conditions of the Alternating Series Test. Which one?

**(b)** Find the sum of the series in part (a).

## Standardized Test Questions

You may use a graphing calculator to solve the following problems.

**66. True or False** The series

$$\sum_{n=1}^{\infty} (-1)^{n+1}\frac{x^{2n}}{2n}$$

converges at its endpoints. Justify your answer.

**67. True or False** If $S_{100}$ is used to estimate the sum of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2},$$

the estimate is an overestimate. Justify your answer.

In Exercises 68 and 69, use the series $\sum_{n=0}^{\infty} \frac{n(2x-5)^n}{n+2}$.

**68. Multiple Choice** Which of the following is the radius of convergence of the series?

**(A)** 1 **(B)** 1/2 **(C)** 3/2 **(D)** 2 **(E)** 5/2

**69. Multiple Choice** Which of the following is the interval of convergence of the series?

**(A)** $2 < x < 3$ **(B)** $4 < x < 6$ **(C)** $-\frac{1}{2} < x < \frac{1}{2}$

**(D)** $-3 < x < -2$ **(E)** $-6 < x < -4$

**70. Multiple Choice** Which of the following series converge?

I. $\sum_{n=1}^{\infty} \frac{4}{\sqrt{n}}$ II. $\sum_{n=1}^{\infty} \frac{1}{(\ln 4)^n}$ III. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

**(A)** I only **(B)** II only **(C)** III only

**(D)** I & II only **(E)** II & III only

**71. Multiple Choice** Which of the following gives the truncation error if $S_{100}$ is used to approximate the sum of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}?$$

(A) 0.0098 (B) 0.00098 (C) 0.000098

(D) 0.0000098 (E) 0.00000098

## Exploration

**72. Group Activity** Within your group, have each student construct a series that converges to one of the numbers $1, \ldots, n$. Then exchange your series with another group and try to figure out which number is matched with which series.

## Extending the Ideas

Here is a test called the *nth-Root Test.*

> ***n*th-Root Test** Let $\sum a_n$ be a series with $a_n \geq 0$ for $n \geq N$, and suppose that $\lim_{n\to\infty} \sqrt[n]{a_n} = L$. Then,
>
> **(a)** the series *converges* if $L < 1$,
>
> **(b)** the series *diverges* if $L > 1$ or $L$ is infinite,
>
> **(c)** the test is *inconclusive* if $L = 1$.

**73.** Use the $n$th-Root Test and the fact that $\lim_{n\to\infty} \sqrt[n]{n} = 1$ to test the following series for convergence or divergence.

**(a)** $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

**(b)** $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{2n-1}\right)^n$

**(c)** $\displaystyle\sum_{n=1}^{\infty} a_n$, where $a_n = \begin{cases} n/2^n, & n \text{ is odd} \\ 1/2^n, & n \text{ is even} \end{cases}$

**74.** Use the $n$th-Root Test and whatever else you need to find the intervals of convergence of the following series.

**(a)** $\displaystyle\sum_{n=0}^{\infty} \frac{(x-1)^n}{4^n}$

**(b)** $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{n \cdot 3^n}$

**(c)** $\displaystyle\sum_{n=1}^{\infty} 2^n x^n$

**(d)** $\displaystyle\sum_{n=0}^{\infty} (\ln x)^n$

# Quick Quiz for AP* Preparation: Sections 9.4 and 9.5

You may use a graphing calculator to solve the following problems.

**1. Multiple Choice** Which of the following series converge?

I. $\displaystyle\sum_{n=0}^{\infty} \frac{2}{n^2+1}$ II. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n-1}{3^n+1}$ III. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt[4]{n}}{n}$

(A) I only (B) II only (C) III only

(D) II & III only (E) I & II only

**2. Multiple Choice** Which of the following is the sum of the telescoping series

$$\sum_{n=1}^{\infty} \frac{2}{(n+1)(n+2)}?$$

(A) 1/3 (B) 1/2 (C) 3/5 (D) 2/3 (E) 1

**3. Multiple Choice** Which of the following describes the behavior of the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n}?$$

I. converges II. diverges III. converges conditionally

(A) I only (B) II only (C) III only

(D) I & III only (E) II & III only

**4. Free Response** Consider the power series

$$\sum_{n=0}^{\infty} \frac{n(2x+3)^n}{n+2}.$$

**(a)** Find all values of $x$ for which the series converges absolutely. Justify your answer.

**(b)** Find all values of $x$ for which the series converges conditionally. Justify your answer.

## Chapter 9 Key Terms



## Chapter 9 Review Exercises

The collection of exercises marked in red could be used as a chapter test.

In Exercises 1–16, find **(a)** the radius of convergence for the series and **(b)** its interval of convergence. Then identify the values of $x$ for which the series converges **(c)** absolutely and **(d)** conditionally.

**1.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-x)^n}{n!}$

**2.** $\displaystyle\sum_{n=1}^{\infty} \frac{(x+4)^n}{n3^n}$

**3.** $\displaystyle\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n (x-1)^n$

**4.** $\displaystyle\sum_{n=1}^{\infty} \frac{(x-1)^{2n-2}}{(2n-1)!}$

**5.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(3x-1)^n}{n^2}$

**6.** $\displaystyle\sum_{n=0}^{\infty} (n+1)x^{3n}$

**7.** $\displaystyle\sum_{n=0}^{\infty} \frac{(n+1)(2x+1)^n}{(2n+1)2^n}$

**8.** $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^n}$

**9.** $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$

**10.** $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n^e}x^n$

**11.** $\displaystyle\sum_{n=0}^{\infty} \frac{(n+1)x^{2n-1}}{3^n}$

**12.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n(x-1)^{2n+1}}{2n+1}$

**13.** $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{2^n}x^{2n}$

**14.** $\displaystyle\sum_{n=2}^{\infty} \frac{(10x)^n}{\ln n}$

**15.** $\displaystyle\sum_{n=1}^{\infty} (n+1)!\,x^n$

**16.** $\displaystyle\sum_{n=1}^{\infty} \left(\frac{x^2-1}{2}\right)^n$

In Exercises 17–22, the series is the value of the Maclaurin series of a function $f(x)$ at a particular point. What function and what point? What is the sum of the series?

**17.** $1 - \frac{1}{4} + \frac{1}{16} - \cdots + (-1)^n \frac{1}{4^n} + \cdots$

**18.** $\frac{2}{3} - \frac{4}{18} + \frac{8}{81} - \cdots + (-1)^{n-1} \frac{2^n}{n3^n} + \cdots$

**19.** $\pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \cdots + (-1)^n \frac{\pi^{2n+1}}{(2n+1)!} + \cdots$

**20.** $1 - \frac{\pi^2}{9 \cdot 2!} + \frac{\pi^4}{81 \cdot 4!} - \cdots + (-1)^n \frac{\pi^{2n}}{3^{2n}(2n)!} + \cdots$

**21.** $1 + \ln 2 + \frac{(\ln 2)^2}{2!} + \cdots + \frac{(\ln 2)^n}{n!} + \cdots$

**22.** $\frac{1}{\sqrt{3}} - \frac{1}{9\sqrt{3}} + \frac{1}{45\sqrt{3}} - \cdots + (-1)^{n-1} \frac{1}{(2n-1)(\sqrt{3})^{2n-1}} + \cdots$

In Exercises 23–36, find a Maclaurin series for the function.

23. $\dfrac{1}{1-6x}$

24. $\dfrac{1}{1+x^3}$

25. $x^9 - 2x^2 + 1$

26. $\dfrac{4x}{1-x}$

27. $\sin \pi x$

28. $-\sin \dfrac{2x}{3}$

29. $-x + \sin x$

30. $\dfrac{e^x + e^{-x}}{2}$

31. $\cos \sqrt{5x}$

32. $e^{(\pi x/2)}$

33. $xe^{-x^2}$

34. $\tan^{-1} 3x$

35. $\ln(1-2x)$

36. $x \ln(1-x)$

In Exercises 37–40, find the first four nonzero terms and the general term of the Taylor series generated by $f$ at $x = a$.

37. $f(x) = \dfrac{1}{3-x}, \quad a = 2$

38. $f(x) = x^3 - 2x^2 + 5, \quad a = -1$

39. $f(x) = \dfrac{1}{x}, \quad a = 3$

40. $f(x) = \sin x, \quad a = \pi$

In Exercises 41–52, determine if the series converges absolutely, converges conditionally, or diverges. Give reasons for your answer.

41. $\displaystyle\sum_{n=1}^{\infty} \frac{-5}{n}$

42. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

43. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$

44. $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n!}$

45. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$

46. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

47. $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^n}{n!}$

48. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n 3^n}{n^n}$

49. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(n^2+1)}{2n^2+n-1}$

50. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)(n+2)}}$

51. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$

52. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^n$

In Exercises 53 and 54, find the sum of the series.

53. $\displaystyle\sum_{n=3}^{\infty} \frac{1}{(2n-3)(2n-1)}$

54. $\displaystyle\sum_{n=2}^{\infty} \frac{-2}{n(n+1)}$

55. Let $f$ be a function that has derivatives of all orders for all real numbers. Assume that $f(3) = 1$, $f'(3) = 4$, $f''(3) = 6$, and $f'''(3) = 12$.

**(a)** Write the third order Taylor polynomial for $f$ at $x = 3$ and use it to approximate $f(3.2)$.

**(b)** Write the second order Taylor polynomial for $f'$ at $x = 3$ and use it to approximate $f'(2.7)$.

**(c)** Does the linearization of $f$ underestimate or overestimate the values of $f(x)$ near $x = 3$? Justify your answer.

56. Let

$$P_4(x) = 7 - 3(x-4) + 5(x-4)^2 - 2(x-4)^3 + 6(x-4)^4$$

be the Taylor polynomial of order 4 for the function $f$ at $x = 4$. Assume $f$ has derivatives of all orders for all real numbers.

**(a)** Find $f(4)$ and $f'''(4)$.

**(b)** Write the second order Taylor polynomial for $f'$ at $x = 4$ and use it to approximate $f'(4.3)$.

**(c)** Write the fourth order Taylor polynomial for $g(x) = \int_4^x f(t)\,dt$ at $x = 4$.

**(d)** Can the exact value of $f(3)$ be determined from the information given? Justify your answer.

57. **(a)** Write the first three nonzero terms and the general term of the Taylor series generated by $f(x) = 5\sin(x/2)$ at $x = 0$.

**(b)** What is the interval of convergence for the series found in (a)? Show your method.

**(c) Writing to Learn** What is the minimum number of terms of the series in (a) needed to approximate $f(x)$ on the interval $(-2, 2)$ with an error not exceeding 0.1 in magnitude? Show your method.

58. Let $f(x) = 1/(1-2x)$.

**(a)** Write the first four terms and the general term of the Taylor series generated by $f(x)$ at $x = 0$.

**(b)** What is the interval of convergence for the series found in part (a)? Show your method.

**(c)** Find $f(-1/4)$. How many terms of the series are adequate for approximating $f(-1/4)$ with an error not exceeding one percent in magnitude? Justify your answer.

59. Let $\displaystyle f(x) = \sum_{n=1}^{\infty} \frac{x^n n^n}{n!}$

for all $x$ for which the series converges.

**(a)** Find the radius of convergence of this series.

**(b)** Use the first three terms of this series to approximate $f(-1/3)$.

**(c)** Estimate the error involved in the approximation in part (b). Justify your answer.

60. Let $f(x) = 1/(x-2)$.

**(a)** Write the first four terms and the general term of the Taylor series generated by $f(x)$ at $x = 3$.

**(b)** Use the result from part (a) to find the first four terms and the general term of the series generated by $\ln|x-2|$ at $x = 3$.

**(c)** Use the series in part (b) to compute a number that differs from $\ln(3/2)$ by less than 0.05. Justify your answer.

**61.** Let $f(x) = e^{-2x^2}$.

**(a)** Find the first four nonzero terms and the general term for the power series generated by $f(x)$ at $x = 0$.

**(b)** Find the interval of convergence of the series generated by $f(x)$ at $x = 0$. Show the analysis that leads to your conclusion.

**(c) Writing to Learn** Let $g$ be the function defined by the sum of the first four nonzero terms of the series generated by $f(x)$. Show that $|f(x) - g(x)| < 0.02$ for $-0.6 \le x \le 0.6$.

**62. (a)** Find the Maclaurin series generated by $f(x) = x^2/(1 + x)$.

**(b)** Does the series converge at $x = 1$? Explain.

**63. *Evaluating Nonelementary Integrals*** Maclaurin series can be used to express nonelementary integrals in terms of series.

**(a)** Express $\int_0^x \sin t^2\, dt$ as a power series.

**(b)** According to the Alternating Series Estimation Theorem, how many terms of the series in part (a) should you use to estimate $\int_0^1 \sin x^2\, dx$ with an error of less than 0.001?

**(c)** Use NINT to approximate $\int_0^1 \sin x^2\, dx$.

**(d)** How close to the answer in part (c) do you get if you use four terms of the series in part (a)?

**64. *Estimating an Integral*** Suppose you want a quick noncalculator estimate for the value of $\int_0^1 x^2 e^x\, dx$. There are several ways to get one.

**(a)** Use the Trapezoidal rule with $n = 2$ to estimate $\int_0^1 x^2 e^x\, dx$.

**(b)** Write the first three nonzero terms of the Maclaurin series for $x^2 e^x$ to obtain the fourth order Maclaurin polynomial $P_4(x)$ for $x^2 e^x$. Use $\int_0^1 P_4(x)\, dx$ to obtain another estimate of $\int_0^1 x^2 e^x\, dx$.

**(c) Writing to Learn** The second derivative of $f(x) = x^2 e^x$ is positive for all $x > 0$. Explain why this enables you to conclude that the Trapezoidal rule estimate obtained in part (a) is too large.

**(d) Writing to Learn** All the derivatives of $f(x) = x^2 e^x$ are positive for $x > 0$. Explain why this enables you to conclude that all Maclaurin series approximations to $f(x)$ for $x$ in $[0, 1]$ will be too small. (*Hint:* $f(x) = P_n(x) + R_n(x)$.)

**(e)** Use integration by parts to evaluate $\int_0^1 x^2 e^x\, dx$.

**65. *Perpetuities*** Suppose you want to give a favorite school or charity \$1000 a year forever. This kind of gift is called a *perpetuity*. Assume you can earn 8% annually on your money, i.e., that a payment of $a_n$ today will be worth $a_n(1.08)^n$ in $n$ years.

**(a)** Show that the amount you must invest today to cover the $n$th \$1000 payment in $n$ years is $1000(1.08)^{-n}$.

**(b)** Construct an infinite series that gives the amount you must invest today to cover *all* the payments in the perpetuity.

**(c)** Show that the series in part (b) converges and find its sum. This sum is called the *present value* of the perpetuity. What does it represent?

**66. *(Continuation of Exercise 65)*** Find the present value of a \$1000-per-year perpetuity at 6% annual interest.

**67. *Expected Payoff*** How much would you expect to win playing the following game?

Toss a *fair* coin (heads and tails equally likely). Every time it comes up heads you win a dollar, but the game is over as soon as it comes up tails.

**(a)** The *expected payoff* of the game is computed by summing all possible payoffs times their respective probabilities. If the probability of tossing the first tail on the $n$th toss is $(1/2)^n$, express the expected payoff of this game as an infinite series.

**(b)** Differentiate both sides of

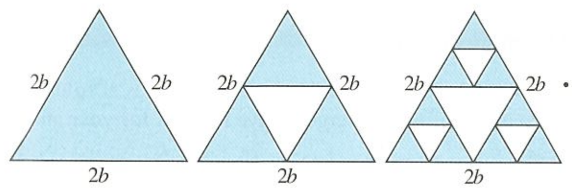
$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots$$

to get a series for $1/(1-x)^2$.

**(c)** Use the series in part (b) to get a series for $x^2/(1-x)^2$.

**(d)** Use the series in part (c) to evaluate the expected payoff of the game.

**68. *Punching out Triangles*** This exercise refers to the "right side up" equilateral triangle with sides of length $2b$ in the accompanying figure.




"Upside down" equilateral triangles are removed from the original triangle as the sequence of pictures suggests. The sum of the areas removed from the original triangle forms an infinite series.

**(a)** Find this infinite series.

**(b)** Find the sum of this infinite series and hence find the total area removed from the original triangle.

**(c)** Is every point on the original triangle removed? Explain why or why not.

**69. *Nicole Oresme's (pronounced "O-rem's") Theorem*** Prove Nicole Oresme's Theorem that

$$1 + \frac{1}{2}\cdot 2 + \frac{1}{4}\cdot 3 + \cdots + \frac{n}{2^{n-1}} + \cdots = 4.$$

(*Hint:* Differentiate both sides of the equation $1/(1-x) = 1 + \sum_{n=1}^{\infty} x^n$.)

**70. (a)** Show that

$$\sum_{n=1}^{\infty} \frac{n(n+1)}{x^n} = \frac{2x^2}{(x-1)^3}$$

for $|x| > 1$ by differentiating the identity

$$\sum_{n=1}^{\infty} x^{n+1} = \frac{x^2}{1-x}$$

twice, multiplying the result by $x$, and then replacing $x$ by $1/x$.

**(b)** Use part (a) to find the real solution greater than 1 of the equation

$$x = \sum_{n=1}^{\infty} \frac{n(n+1)}{x^n}.$$

## AP* Examination Preparation

You may use a graphing calculator to solve the following problems.

**71.** Let $f(x) = \dfrac{1}{x+1}$.

**(a)** Find the first three terms and the general term for the Taylor series for $f$ at $x = 1$.

**(b)** Find the interval of convergence for the series in part (a). Justify your answer.

**(c)** Find the third-order Taylor polynomial for $f$ at $x = 1$, and use it to approximate $f(0.5)$.

**72.** Let $f(x) = \sum_{n=0}^{\infty} \dfrac{nx^n}{2^n}$.

**(a)** Find the interval of convergence of the series. Justify your answer.

**(b)** Show that the first nine terms of the series are sufficient to approximate $f(-1)$ with an error less than 0.01.

**73.** Let $f$ be a function that has derivatives of all orders for all real numbers. Assume that $f(0) = -1$, $f'(0) = 2$, $f''(0) = -3$, and $f'''(0) = 4$.

**(a)** Write the linearization for $f$ at $x = 0$.

**(b)** Write the quadratic approximation for $f$ at $x = 0$.

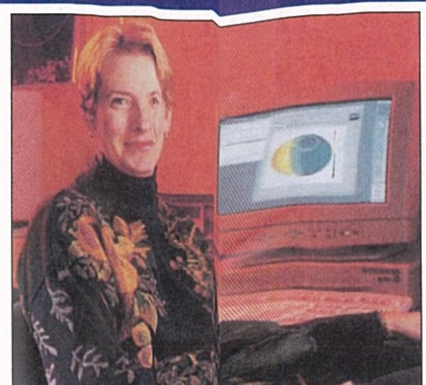
**(c)** Write the third degree Taylor approximation $P_3(x)$ for $f$ at $x = 0$.

**(d)** Use $P_3(x)$ to approximate $f(0.7)$.

## Calculus at Work

I attended the University of California at Los Angeles and received my B.S., M.S., and Ph.D. degrees in Geophysics and Space Physics. After graduating, I became a Research Physicist at SRI International in Menlo Park, California. There I studied the ionosphere, particularly from high latitudes, for the Sondre Stromfjord Incoherent Scatter Radar project.

I am currently an Associate Research Scientist at the University of Michigan, where I continue my upper atmospheric research. I study the lower thermosphere, which extends from 85 to 150 km above the earth. At this height, many physical processes interact in complex ways, making it difficult to understand the behavior of winds and temperatures and to observe atmospheric parameters. Atmospheric models are therefore vitally important in constructing a global perspective of lower thermospheric dynamics. The models use calculus in a number of ways. Polynomial expansions are used to express variables globally. Integrals are solved by summation over small steps in independent variables to determine the global wind and temperature fields. And complex processes, such as turbulence, are described by equations of motion and state. These equations are solved differentially, using series approximations where necessary.



Roberta M. Johnson
University of Michigan
Ann Arbor, MI