Chapter 3 Derivatives

Shown here is the pain reliever acetaminophen in crystalline form, photographed under a transmitted light microscope. While acetaminophen relieves pain with few side effects, it is toxic in large doses. One study found that only 30% of parents who gave acetaminophen to their children could accurately calculate and measure the correct dose.

One rule for calculating the dosage (mg) of acetaminophen for children ages 1 to 12 years old is $D(t) = 750t/(t + 12)$, where t is age in years. What is an expression for the rate of change of a child's dosage with respect to the child's age? How does the rate of change of the dosage relate to the growth rate of children? This problem can be solved with the information covered in Section 3.4.

Chapter 3 Overview

In Chapter 2, we learned how to find the slope of a tangent to a curve as the limit of the slopes of secant lines. In Example 4 of Section 2.4, we derived a formula for the slope of the tangent at an arbitrary point $(a, 1/a)$ on the graph of the function $f(x) = 1/x$ and showed that it was $-1/a^2$.

This seemingly unimportant result is more powerful than it might appear at first glance, as it gives us a simple way to calculate the instantaneous rate of change of f at any point. The study of rates of change of functions is called *differential calculus,* and the formula $-1/a^2$ was our first look at a *derivative.* The derivative was the 17th-century breakthrough that enabled mathematicians to unlock the secrets of planetary motion and gravitational attraction—of objects changing position over time. We will learn many uses for derivatives in Chapter 4, but first we will concentrate in this chapter on understanding what derivatives are and how they work.

3.1 Derivative of a Function

What you'll learn about

- Definition of Derivative
- Notation
- Relationships between the Graphs of f and f'
- Graphing the Derivative from Data
- One-sided Derivatives

. . . and why

The derivative gives the value of the slope of the tangent line to a curve at a point.

Definition of Derivative

In Section 2.4, we defined the slope of a curve $y = f(x)$ at the point where $x = a$ to be

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

When it exists, this limit is called the **derivative of f at a.** In this section, we investigate the derivative as a *function* derived from f by considering the limit at each point of the domain of f.

DEFINITION Derivative

The **derivative** of the function f with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}, \qquad (1)$$

provided the limit exists.

The domain of f', the set of points in the domain of f for which the limit exists, may be smaller than the domain of f. If $f'(x)$ exists, we say that f **has a derivative (is differentiable)** at x. A function that is differentiable at every point of its domain is a **differentiable function.**

EXAMPLE 1 Applying the Definition

Differentiate (that is, find the derivative of) $f(x) = x^3$.

continued

SOLUTION

Applying the definition, we have

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h}$$

Eq. 1 with $f(x) = x^3$, $f(x+h) = (x+h)^3$

$$= \lim_{h\to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3) - x^3}{h}$$

$(x+h)^3$ expanded

$$= \lim_{h\to 0} \frac{(3x^2 + 3xh + h^2)h}{h}$$

x^3s cancelled, h factored out

$$= \lim_{h\to 0} (3x^2 + 3xh + h^2) = 3x^2.$$

Now try Exercise 1.

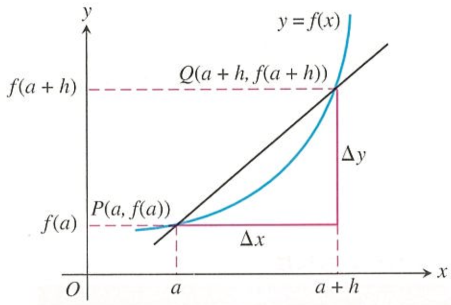


Figure 3.1 The slope of the secant line PQ is

$$\frac{\Delta y}{\Delta x} = \frac{f(a+h)-f(a)}{(a+h)-a} = \frac{f(a+h)-f(a)}{h}.$$

The derivative of $f(x)$ at a point where $x = a$ is found by taking the limit as $h \to 0$ of slopes of secant lines, as shown in Figure 3.1.

By relabeling the picture as in Figure 3.2, we arrive at a useful alternate formula for calculating the derivative. This time, the limit is taken as x approaches a.

DEFINITION (ALTERNATE) Derivative at a Point

The **derivative** of the function **f at the point $x = a$** is the limit

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}, \qquad (2)$$

provided the limit exists.

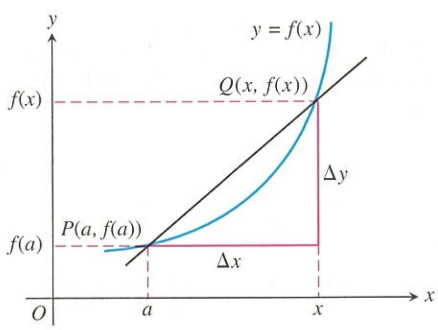


Figure 3.2 The slope of the secant line PQ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x)-f(a)}{x-a}.$$

After we find the derivative of f at a point $x = a$ using the alternate form, we can find the derivative of f as a function by applying the resulting formula to an arbitrary x in the domain of f.

EXAMPLE 2 Applying the Alternate Definition

Differentiate $f(x) = \sqrt{x}$ using the alternate definition.

SOLUTION

At the point $x = a$,

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

$$= \lim_{x\to a} \frac{\sqrt{x}-\sqrt{a}}{x-a}$$

Eq. 2 with $f(x) = \sqrt{x}$

$$= \lim_{x\to a} \frac{\sqrt{x}-\sqrt{a}}{x-a} \cdot \frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}}$$

Rationalize...

$$= \lim_{x\to a} \frac{x-a}{(x-a)(\sqrt{x}+\sqrt{a})}$$

...the numerator.

$$= \lim_{x\to a} \frac{1}{\sqrt{x}+\sqrt{a}}$$

We can now take the limit.

$$= \frac{1}{2\sqrt{a}}.$$

Applying this formula to an arbitrary $x > 0$ in the domain of f identifies the derivative as the function $f'(x) = 1/(2\sqrt{x})$ with domain $(0, \infty)$. ***Now try Exercise 5.***

Why all the notation?

The "prime" notations y' and f' come from notations that Newton used for derivatives. The d/dx notations are similar to those used by Leibniz. Each has its advantages and disadvantages.

Notation

There are many ways to denote the derivative of a function $y = f(x)$. Besides $f'(x)$, the most common notations are these:

y'	"y prime"	Nice and brief, but does not name the independent variable.
$\dfrac{dy}{dx}$	"$dy\ dx$" or "the derivative of y with respect to x"	Names both variables and uses d for derivative.
$\dfrac{df}{dx}$	"$df\ dx$" or "the derivative of f with respect to x"	Emphasizes the function's name.
$\dfrac{d}{dx}f(x)$	"$d\ dx$ of f at x" or "the derivative of f at x"	Emphasizes the idea that differentiation is an operation performed on f.

Relationships between the Graphs of f and f'

When we have the explicit formula for $f(x)$, we can derive a formula for $f'(x)$ using methods like those in Examples 1 and 2. We have already seen, however, that functions are encountered in other ways: graphically, for example, or in tables of data.

Because we can think of the derivative at a point in graphical terms as *slope,* we can get a good idea of what the graph of the function f' looks like by *estimating the slopes* at various points along the graph of f.

EXAMPLE 3 GRAPHING f' from f

Graph the derivative of the function f whose graph is shown in Figure 3.3a. Discuss the behavior of f in terms of the signs and values of f'.

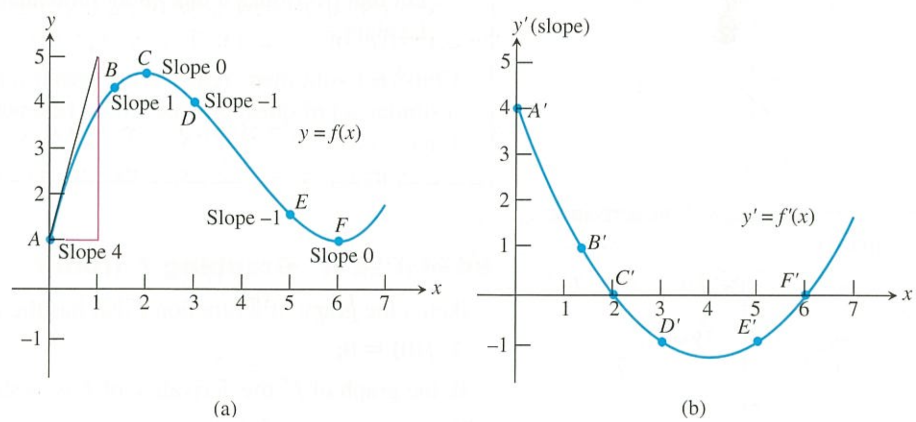


Figure 3.3 By plotting the slopes at points on the graph of $y = f(x)$, we obtain a graph of $y' = f'(x)$. The slope at point A of the graph of f in part (a) is the y-coordinate of point A' on the graph of f' in part (b), and so on. (Example 3)

SOLUTION

First, we draw a pair of coordinate axes, marking the horizontal axis in x-units and the vertical axis in slope units (Figure 3.3b). Next, we estimate the slope of the graph of f at various points, plotting the corresponding slope values using the new axes. At $A(0, f(0))$, the graph of f has slope 4, so $f'(0) = 4$. At B, the graph of f has slope 1, so $f' = 1$ at B', and so on.

continued

We complete our estimate of the graph of f' by connecting the plotted points with a smooth curve.

Although we do not have a formula for either f or f', the graph of each reveals important information about the behavior of the other. In particular, notice that f is decreasing where f' is negative and increasing where f' is positive. Where f' is zero, the graph of f has a horizontal tangent, changing from increasing to decreasing (point C) or from decreasing to increasing (point F). **Now try Exercise 23.**

EXPLORATION 1 Reading the Graphs

Suppose that the function f in Figure 3.3a represents the depth y (in inches) of water in a ditch alongside a dirt road as a function of time x (in days). How would you answer the following questions?

1. What does the graph in Figure 3.3b represent? What units would you use along the y'-axis?
2. Describe as carefully as you can what happened to the water in the ditch over the course of the 7-day period.
3. Can you describe the weather during the 7 days? When was it the wettest? When was it the driest?
4. How does the graph of the derivative help in finding when the weather was wettest or driest?
5. Interpret the significance of point C in terms of the water in the ditch. How does the significance of point C' reflect that in terms of rate of change?
6. It is tempting to say that it rains right up until the beginning of the second day, but that overlooks a fact about rainwater that is important in flood control. Explain.

Construct your own "real-world" scenario for the function in Example 3, and pose a similar set of questions that could be answered by considering the two graphs in Figure 3.3.

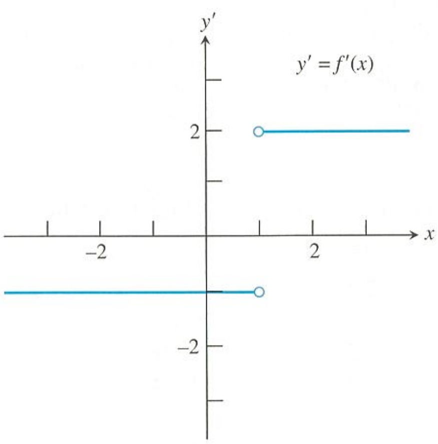


Figure 3.4 The graph of the derivative. (Example 4)

EXAMPLE 4 Graphing f from f'

Sketch the graph of a function f that has the following properties:

i. $f(0) = 0$;

ii. the graph of f', the derivative of f, is as shown in Figure 3.4;

iii. f is continuous for all x.

SOLUTION

To satisfy property (i), we begin with a point at the origin.

To satisfy property (ii), we consider what the graph of the derivative tells us about slopes. To the left of $x = 1$, the graph of f has a constant slope of -1; therefore we draw a line with slope -1 to the left of $x = 1$, making sure that it goes through the origin.

To the right of $x = 1$, the graph of f has a constant slope of 2, so it must be a line with slope 2. There are infinitely many such lines but only one—the one that meets the left side of the graph at $(1, -1)$—will satisfy the continuity requirement. The resulting graph is shown in Figure 3.5. **Now try Exercise 27.**

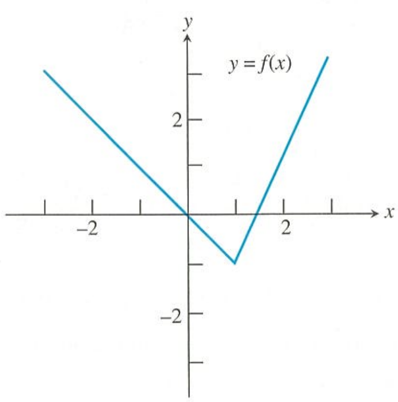


Figure 3.5 The graph of f, constructed from the graph of f' and two other conditions. (Example 4)

What's happening at $x = 1$?

Notice that f in Figure 3.5 is defined at $x = 1$, while f' is not. It is the continuity of f that enables us to conclude that $f(1) = -1$. Looking at the graph of f, can you see why f' could not possibly be defined at $x = 1$? We will explore the reason for this in Example 6.

David H. Blackwell
(1919–)

By the age of 22, David Blackwell had earned a Ph.D. in Mathematics from the University of Illinois. He taught at Howard University, where his research included statistics, Markov chains, and sequential analysis. He then went on to teach and continue his research at the University of California at Berkeley. Dr. Blackwell served as president of the American Statistical Association and was the first African American mathematician of the National Academy of Sciences.

Graphing the Derivative from Data

Discrete points plotted from sets of data do not yield a continuous curve, but we have seen that the shape and pattern of the graphed points (called a scatter plot) can be meaningful nonetheless. It is often possible to fit a curve to the points using regression techniques. If the fit is good, we could use the curve to get a graph of the derivative visually, as in Example 3. However, it is also possible to get a scatter plot of the derivative numerically, directly from the data, by computing the slopes between successive points, as in Example 5.

EXAMPLE 5 Estimating the Probability of Shared Birthdays

Suppose 30 people are in a room. What is the probability that two of them share the same birthday? Ignore the year of birth.

SOLUTION

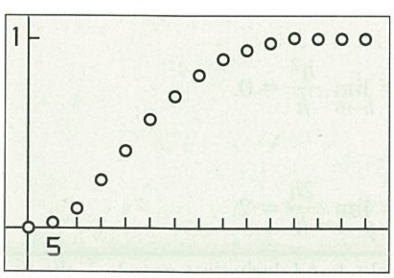
It may surprise you to learn that the probability of a shared birthday among 30 people is at least 0.706, well above two-thirds! In fact, if we assume that no one day is more likely to be a birthday than any other day, the probabilities shown in Table 3.1 are not hard to determine (see Exercise 45).

Table 3.1 Probabilities of Shared Birthdays

People in Room (x)	Probability (y)
0	0
5	0.027
10	0.117
15	0.253
20	0.411
25	0.569
30	0.706
35	0.814
40	0.891
45	0.941
50	0.970
55	0.986
60	0.994
65	0.998
70	0.999

Table 3.2 Estimates of Slopes on the Probability Curve

Midpoint of Interval (x)	Change (slope $\Delta y/\Delta x$)
2.5	0.0054
7.5	0.0180
12.5	0.0272
17.5	0.0316
22.5	0.0316
27.5	0.0274
32.5	0.0216
37.5	0.0154
42.5	0.0100
47.5	0.0058
52.5	0.0032
57.5	0.0016
62.5	0.0008
67.5	0.0002



[–5, 75] by [–0.2, 1.1]

Figure 3.6 Scatter plot of the probabilities (y) of shared birthdays among x people, for $x = 0, 5, 10, \ldots, 70$. (Example 5)

A scatter plot of the data in Table 3.1 is shown in Figure 3.6.

Notice that the probabilities grow slowly at first, then faster, then much more slowly past $x = 45$. At which x are they growing the fastest? To answer the question, we need the graph of the derivative.

Using the data in Table 3.1, we compute the slopes between successive points on the probability plot. For example, from $x = 0$ to $x = 5$ the slope is

$$\frac{0.027 - 0}{5 - 0} = 0.0054.$$

We make a new table showing the slopes, beginning with slope 0.0054 on the interval $[0, 5]$ (Table 3.2). A logical x value to use to represent the interval is its midpoint 2.5.

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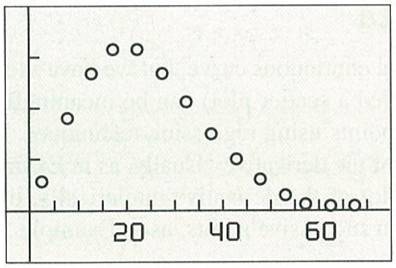


[–5, 75] by [–0.01, 0.04]

Figure 3.7 A scatter plot of the derivative data in Table 3.2. (Example 5)

A scatter plot of the derivative data in Table 3.2 is shown in Figure 3.7.

From the derivative plot, we can see that the rate of change peaks near $x = 20$. You can impress your friends with your "psychic powers" by predicting a shared birthday in a room of just 25 people (since you will be right about 57% of the time), but the derivative warns you to be cautious: a few less people can make quite a difference. On the other hand, going from 40 people to 100 people will not improve your chances much at all.

Now try Exercise 29.

Generating shared birthday probabilities: If you know a little about probability, you might try generating the probabilities in Table 3.1. Extending the Idea Exercise 45 at the end of this section shows how to generate them on a calculator.

One-Sided Derivatives

A function $y = f(x)$ is **differentiable on a closed interval $[a, b]$** if it has a derivative at every interior point of the interval, and if the limits

$$\lim_{h\to 0^+} \frac{f(a+h) - f(a)}{h} \quad \text{[the \textbf{right-hand derivative at } a\text{]}}$$

$$\lim_{h\to 0^-} \frac{f(b+h) - f(b)}{h} \quad \text{[the \textbf{left-hand derivative at } b\text{]}}$$

exist at the endpoints. In the right-hand derivative, h is positive and $a + h$ approaches a from the right. In the left-hand derivative, h is negative and $b + h$ approaches b from the left (Figure 3.8).

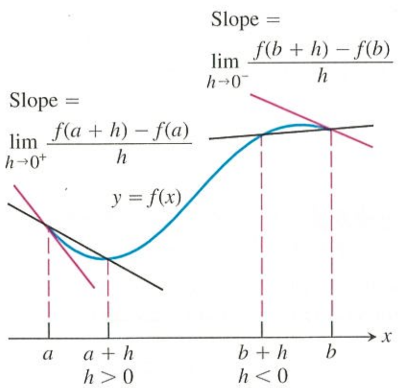


Figure 3.8 Derivatives at endpoints are one-sided limits.

Right-hand and left-hand derivatives may be defined at any point of a function's domain.

The usual relationship between one-sided and two-sided limits holds for derivatives. Theorem 3, Section 2.1, allows us to conclude that a function has a (two-sided) derivative at a point if and only if the function's right-hand and left-hand derivatives are defined and equal at that point.

EXAMPLE 6 One-Sided Derivatives can Differ at a Point

Show that the following function has left-hand and right-hand derivatives at $x = 0$, but no derivative there (Figure 3.9).

$$y = \begin{cases} x^2, & x \le 0 \\ 2x, & x > 0 \end{cases}$$

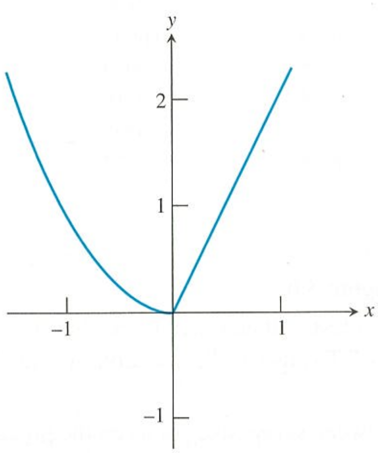


Figure 3.9 A function with different one-sided derivatives at $x = 0$. (Example 6)

SOLUTION

We verify the existence of the left-hand derivative:

$$\lim_{h\to 0^-} \frac{(0+h)^2 - 0^2}{h} = \lim_{h\to 0^-} \frac{h^2}{h} = 0.$$

We verify the existence of the right-hand derivative:

$$\lim_{h\to 0^+} \frac{2(0+h) - 0^2}{h} = \lim_{h\to 0^+} \frac{2h}{h} = 2.$$

Since the left-hand derivative equals zero and the right-hand derivative equals 2, the derivatives are not equal at $x = 0$. The function does not have a derivative at 0.

Now try Exercise 31.

Quick Review 3.1 *(For help, go to Sections 2.1 and 2.4.)*

In Exercises 1–4, evaluate the indicated limit algebraically.

1. $\lim_{h\to 0} \dfrac{(2+h)^2-4}{h}$ **2.** $\lim_{x\to 2^+} \dfrac{x+3}{2}$

3. $\lim_{y\to 0^-} \dfrac{|y|}{y}$ **4.** $\lim_{x\to 4} \dfrac{2x-8}{\sqrt{x}-2}$

5. Find the slope of the line tangent to the parabola $y = x^2 + 1$ at its vertex.

6. By considering the graph of $f(x) = x^3 - 3x^2 + 2$, find the intervals on which f is increasing.

In Exercises 7–10, let

$$f(x) = \begin{cases} x+2, & x \le 1 \\ (x-1)^2, & x > 1. \end{cases}$$

7. Find $\lim_{x\to 1^+} f(x)$ and $\lim_{x\to 1^-} f(x)$.

8. Find $\lim_{h\to 0^+} f(1+h)$.

9. Does $\lim_{x\to 1} f(x)$ exist? Explain.

10. Is f continuous? Explain.

Section 3.1 Exercises

In Exercises 1–4, use the definition

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

to find the derivative of the given function at the indicated point.

1. $f(x) = 1/x,\ a = 2$ **2.** $f(x) = x^2 + 4,\ a = 1$

3. $f(x) = 3 - x^2,\ a = -1$ **4.** $f(x) = x^3 + x,\ a = 0$

In Exercises 5–8, use the definition

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

to find the derivative of the given function at the indicated point.

5. $f(x) = 1/x,\ a = 2$ **6.** $f(x) = x^2 + 4,\ a = 1$

7. $f(x) = \sqrt{x+1},\ a = 3$ **8.** $f(x) = 2x + 3,\ a = -1$

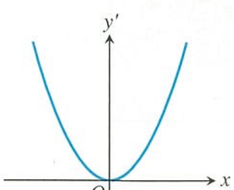
9. Find $f'(x)$ if $f(x) = 3x - 12$.

10. Find dy/dx if $y = 7x$.

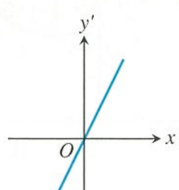
11. Find $\dfrac{d}{dx}(x^2)$.

12. Find $\dfrac{d}{dx} f(x)$ if $f(x) = 3x^2$.

In Exercises 13–16, match the graph of the function with the graph of the derivative shown here:



(a)



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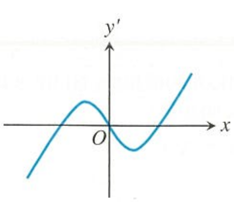

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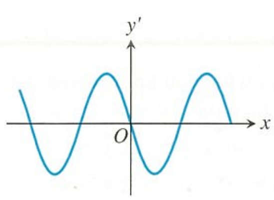

(d)

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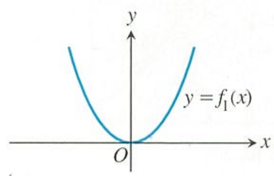

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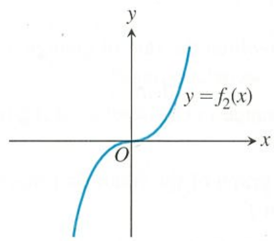

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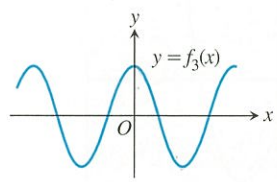

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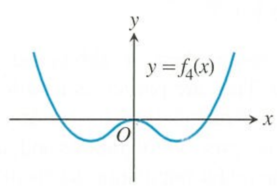


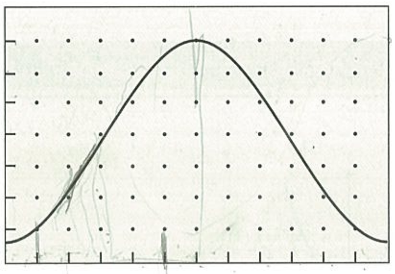
17. If $f(2) = 3$ and $f'(2) = 5$, find an equation of **(a)** the *tangent* line, and **(b)** the *normal* line to the graph of $y = f(x)$ at the point where $x = 2$.

18. Find the derivative of the function $y = 2x^2 - 13x + 5$ and use it to find an equation of the line tangent to the curve at $x = 3$.

19. Find the lines that are **(a)** tangent and **(b)** normal to the curve $y = x^3$ at the point $(1, 1)$.

20. Find the lines that are **(a)** tangent and **(b)** normal to the curve $y = \sqrt{x}$ at $x = 4$.

21. ***Daylight in Fairbanks*** The viewing window below shows the number of hours of daylight in Fairbanks, Alaska, on each day for a typical 365-day period from January 1 to December 31. Answer the following questions by estimating slopes on the graph in hours per day. For the purposes of estimation, assume that each month has 30 days.



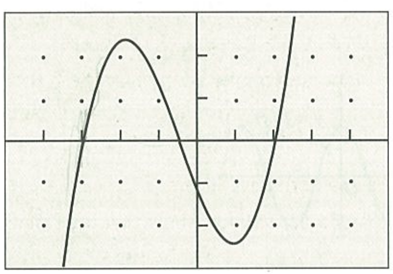
[0, 365] by [0, 24]

(a) On about what date is the amount of daylight increasing at the fastest rate? What is that rate?

(b) Do there appear to be days on which the rate of change in the amount of daylight is zero? If so, which ones?

(c) On what dates is the rate of change in the number of daylight hours positive? negative?

22. ***Graphing f′ from f*** Given the graph of the function f below, sketch a graph of the *derivative* of f.



[–5, 5] by [–3, 3]

23. The graphs in Figure 3.10a show the numbers of rabbits and foxes in a small arctic population. They are plotted as functions of time for 200 days. The number of rabbits increases at first, as the rabbits reproduce. But the foxes prey on the rabbits and, as the number of foxes increases, the rabbit population levels off and then drops. Figure 3.10b shows the graph of the derivative of the rabbit population. We made it by plotting slopes, as in Example 3.

(a) What is the value of the derivative of the rabbit population in Figure 3.10 when the number of rabbits is largest? smallest?

(b) What is the size of the rabbit population in Figure 3.10 when its derivative is largest? smallest?

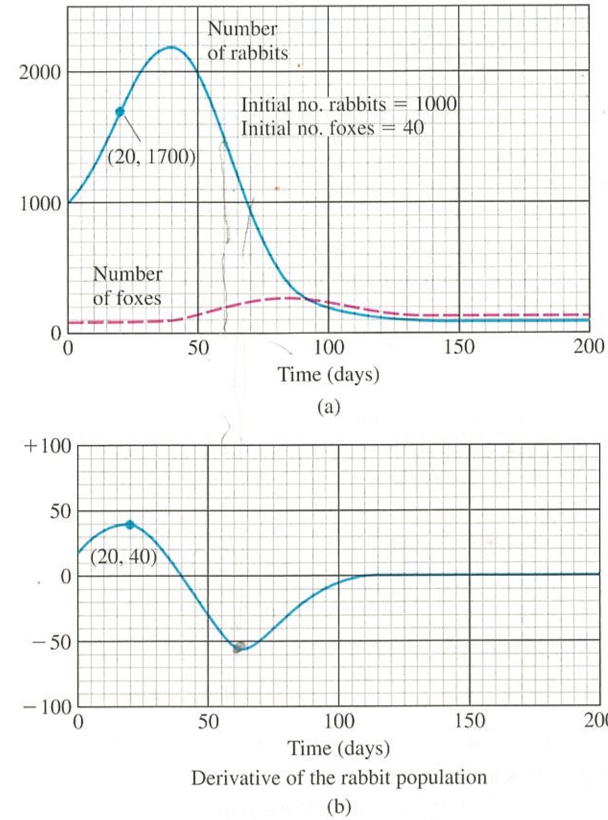
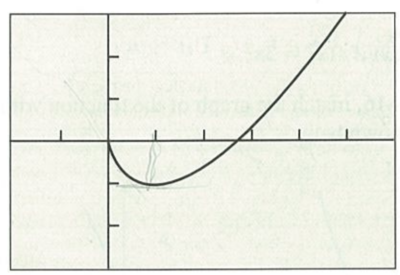


Figure 3.10 Rabbits and foxes in an arctic predator-prey food chain. *Source: Differentiation* by W. U. Walton et al., Project CALC, Education Development Center, Inc., Newton, MA, 1975, p. 86.

24. Shown below is the graph of $f(x) = x \ln x - x$. From what you know about the graphs of functions (i) through (v), pick out the one that is the *derivative* of f for $x > 0$.



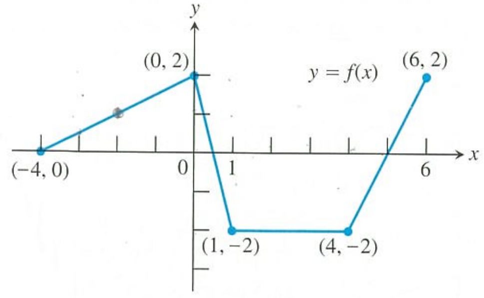
[–2, 6] by [–3, 3]

i. $y = \sin x$ **ii.** $y = \ln x$ **iii.** $y = \sqrt{x}$
iv. $y = x^2$ **v.** $y = 3x - 1$

25. From what you know about the graphs of functions (i) through (v), pick out the one that is *its own derivative.*

i. $y = \sin x$ **ii.** $y = x$ **iii.** $y = \sqrt{x}$
iv. $y = e^x$ **v.** $y = x^2$

26. The graph of the function $y = f(x)$ shown here is made of line segments joined end to end.



(a) Graph the function's derivative.

(b) At what values of x between $x = -4$ and $x = 6$ is the function not differentiable?

27. *Graphing f from f′* Sketch the graph of a continuous function f with $f(0) = -1$ and

$$f'(x) = \begin{cases} 1, & x < -1 \\ -2, & x > -1. \end{cases}$$

28. *Graphing f from f′* Sketch the graph of a continuous function f with $f(0) = 1$ and

$$f'(x) = \begin{cases} 2, & x < 2 \\ -1, & x > 2. \end{cases}$$

In Exercises 29 and 30, use the data to answer the questions.

29. *A Downhill Skier* Table 3.3 gives the approximate distance traveled by a downhill skier after t seconds for $0 \le t \le 10$. Use the method of Example 5 to sketch a graph of the derivative; then answer the following questions:

(a) What does the derivative represent?

(b) In what units would the derivative be measured?

(c) Can you guess an equation of the derivative by considering its graph?

Table 3.3 Skiing Distances

Time t (seconds)	Distance Traveled (feet)
0	0
1	3.3
2	13.3
3	29.9
4	53.2
5	83.2
6	119.8
7	163.0
8	212.9
9	269.5
10	332.7

30. *A Whitewater River* Bear Creek, a Georgia river known to kayaking enthusiasts, drops more than 770 feet over one stretch of 3.24 miles. By reading a contour map, one can estimate the elevations (y) at various distances (x) downriver from the start of the kayaking route (Table 3.4).

Table 3.4 Elevations along Bear Creek

Distance Downriver (miles)	River Elevation (feet)
0.00	1577
0.56	1512
0.92	1448
1.19	1384
1.30	1319
1.39	1255
1.57	1191
1.74	1126
1.98	1062
2.18	998
2.41	933
2.64	869
3.24	805

(a) Sketch a graph of elevation (y) as a function of distance downriver (x).

(b) Use the technique of Example 5 to get an approximate graph of the derivative, dy/dx.

(c) The average change in elevation over a given distance is called a *gradient*. In this problem, what units of measure would be appropriate for a gradient?

(d) In this problem, what units of measure would be appropriate for the derivative?

(e) How would you identify the most dangerous section of the river (ignoring rocks) by analyzing the graph in (a)? Explain.

(f) How would you identify the most dangerous section of the river by analyzing the graph in (b)? Explain.

31. Using one-sided derivatives, show that the function

$$f(x) = \begin{cases} x^2 + x, & x \le 1 \\ 3x - 2, & x > 1 \end{cases}$$

does not have a derivative at $x = 1$.

32. Using one-sided derivatives, show that the function

$$f(x) = \begin{cases} x^3, & x \le 1 \\ 3x, & x > 1 \end{cases}$$

does not have a derivative at $x = 1$.

33. Writing to Learn Graph $y = \sin x$ and $y = \cos x$ in the same viewing window. Which function could be the derivative of the other? Defend your answer in terms of the behavior of the graphs.

34. In Example 2 of this section we showed that the derivative of $y = \sqrt{x}$ is a function with domain $(0, \infty)$. However, the function $y = \sqrt{x}$ itself has domain $[0, \infty)$, so it could have a right-hand derivative at $x = 0$. Prove that it does not.

35. Writing to Learn Use the concept of the derivative to define what it might mean for two parabolas to be parallel. Construct equations for two such parallel parabolas and graph them. Are the parabolas "everywhere equidistant," and if so, in what sense?

Standardized Test Questions

You should solve the following problems without using a graphing calculator.

36. True or False If $f(x) = x^2 + x$, then $f'(x)$ exists for every real number x. Justify your answer.

37. True or False If the left-hand derivative and the right-hand derivative of f exist at $x = a$, then $f'(a)$ exists. Justify your answer.

38. Multiple Choice Let $f(x) = 4 - 3x$. Which of the following is equal to $f'(-1)$?

(A) -7 **(B)** 7 **(C)** -3 **(D)** 3 **(E)** does not exist

39. Multiple Choice Let $f(x) = 1 - 3x^2$. Which of the following is equal to $f'(1)$?

(A) -6 **(B)** -5 **(C)** 5 **(D)** 6 **(E)** does not exist

In Exercises 40 and 41, let

$$f(x) = \begin{cases} x^2 - 1, & x < 0 \\ 2x - 1, & x \geq 0. \end{cases}$$

40. Multiple Choice Which of the following is equal to the left-hand derivative of f at $x = 0$?

(A) -2 **(B)** 0 **(C)** 2 **(D)** ∞ **(E)** $-\infty$

41. Multiple Choice Which of the following is equal to the right-hand derivative of f at $x = 0$?

(A) -2 **(B)** 0 **(C)** 2 **(D)** ∞ **(E)** $-\infty$

Explorations

42. Let $f(x) = \begin{cases} x^2, & x \leq 1 \\ 2x, & x > 1. \end{cases}$

(a) Find $f'(x)$ for $x < 1$.

(b) Find $f'(x)$ for $x > 1$.

(c) Find $\lim_{x \to 1^-} f'(x)$.

(d) Find $\lim_{x \to 1^+} f'(x)$.

(e) Does $\lim_{x \to 1} f'(x)$ exist? Explain.

(f) Use the definition to find the left-hand derivative of f at $x = 1$ if it exists.

(g) Use the definition to find the right-hand derivative of f at $x = 1$ if it exists.

(h) Does $f'(1)$ exist? Explain.

43. Group Activity Using graphing calculators, have each person in your group do the following:

(a) pick two numbers a and b between 1 and 10;

(b) graph the function $y = (x - a)(x + b)$;

(c) graph the *derivative* of your function (it will be a line with slope 2);

(d) find the y-intercept of your derivative graph.

(e) Compare your answers and determine a simple way to predict the y-intercept, given the values of a and b. Test your result.

Extending the Ideas

44. Find the unique value of k that makes the function

$$f(x) = \begin{cases} x^3, & x \leq 1 \\ 3x + k, & x > 1 \end{cases}$$

differentiable at $x = 1$.

45. *Generating the Birthday Probabilities* Example 5 of this section concerns the probability that, in a group of n people, at least two people will share a common birthday. You can generate these probabilities on your calculator for values of n from 1 to 365.

Step 1: Set the values of N and P to zero:

```
0→N:0→P
                    0
```

Step 2: Type in this single, multi-step command:

```
N+1→N:1-(1-P)(366
-N)/365→P:{N,P}
```

Now each time you press the ENTER key, the command will print a new value of N (the number of people in the room) alongside P (the probability that at least two of them share a common birthday):

```
                {1 0}
     {2 .002739726}
    {3 .0082041659}
    {4 .0163559125}
    {5 .0271355737}
    {6 .0404624836}
    {7 .0562357031}
```

If you have some experience with probability, try to answer the following questions without looking at the table:

(a) If there are three people in the room, what is the probability that they all have *different* birthdays? (Assume that there are 365 possible birthdays, all of them equally likely.)

(b) If there are three people in the room, what is the probability that at least two of them share a common birthday?

(c) Explain how you can use the answer in part (b) to find the probability of a shared birthday when there are *four* people in the room. (This is how the calculator statement in Step 2 generates the probabilities.)

(d) Is it reasonable to assume that all calendar dates are equally likely birthdays? Explain your answer.

3.2 Differentiability

What you'll learn about

- How $f'(a)$ Might Fail to Exist
- Differentiability Implies Local Linearity
- Derivatives on a Calculator
- Differentiability Implies Continuity
- Intermediate Value Theorem for Derivatives

. . . and why

Graphs of differentiable functions can be approximated by their tangent lines at points where the derivative exists.

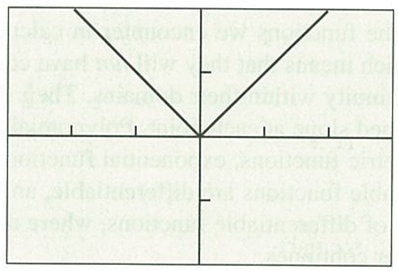
How $f'(a)$ Might Fail to Exist

A function will not have a derivative at a point $P(a, f(a))$ where the slopes of the secant lines,

$$\frac{f(x) - f(a)}{x - a},$$

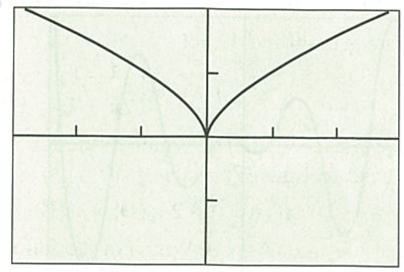
fail to approach a limit as x approaches a. Figures 3.11–3.14 illustrate four different instances where this occurs. For example, a function whose graph is otherwise smooth will fail to have a derivative at a point where the graph has

1. a *corner,* where the one-sided derivatives differ; Example: $f(x) = |x|$



[–3, 3] by [–2, 2]

Figure 3.11 There is a "corner" at $x = 0$.

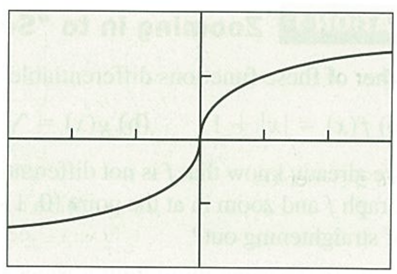


[–3, 3] by [–2, 2]

Figure 3.12 There is a "cusp" at $x = 0$.

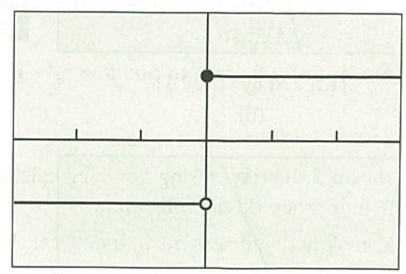
2. a *cusp,* where the slopes of the secant lines approach ∞ from one side and $-\infty$ from the other (an extreme case of a corner); Example: $f(x) = x^{2/3}$

3. a *vertical tangent,* where the slopes of the secant lines approach either ∞ or $-\infty$ from both sides (in this example, ∞); Example: $f(x) = \sqrt[3]{x}$



[–3, 3] by [–2, 2]

Figure 3.13 There is a vertical tangent line at $x = 0$.



[–3, 3] by [–2, 2]

Figure 3.14 There is a discontinuity at $x = 0$.

4. a *discontinuity* (which will cause one or both of the one-sided derivatives to be nonexistent). Example: The *Unit Step Function*

$$U(x) = \begin{cases} -1, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

In this example, the left-hand derivative fails to exist:

$$\lim_{h \to 0^-} \frac{(-1) - (1)}{h} = \lim_{h \to 0^-} \frac{-2}{h} = \infty.$$

How rough can the graph of a continuous function be?

The graph of the absolute value function fails to be differentiable at a single point. If you graph $y = \sin^{-1}(\sin(x))$ on your calculator, you will see a continuous function with an *infinite* number of points of nondifferentiability. But can a continuous function fail to be differentiable at *every* point?

The answer, surprisingly enough, is yes, as Karl Weierstrass showed in 1872. One of his formulas (there are many like it) was

$$f(x) = \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n \cos(9^n \pi x),$$

a formula that expresses f as an infinite (but converging) sum of cosines with increasingly higher frequencies. By adding wiggles to wiggles infinitely many times, so to speak, the formula produces a function whose graph is too bumpy in the limit to have a tangent anywhere!

Later in this section we will prove a theorem that states that a function *must* be continuous at a to be differentiable at a. This theorem would provide a quick and easy verification that U is not differentiable at $x = 0$.

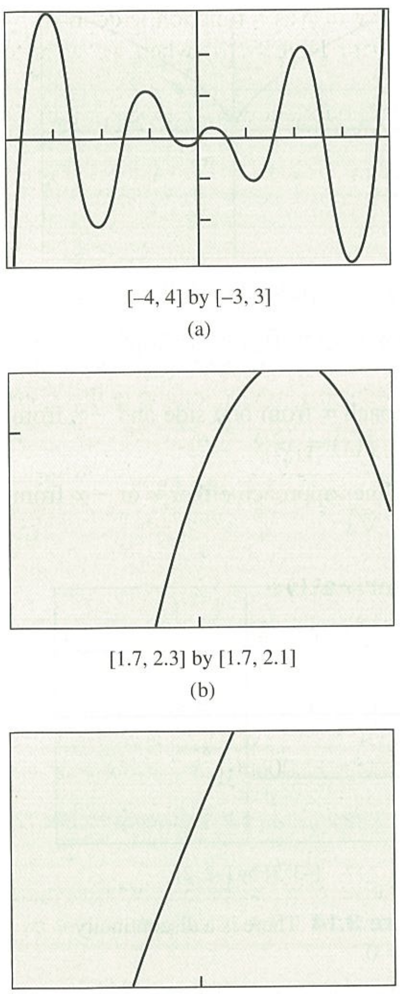
EXAMPLE 1 Finding Where a Function is not Differentiable

Find all points in the domain of $f(x) = |x - 2| + 3$ where f is not differentiable.

SOLUTION

Think graphically! The graph of this function is the same as that of $y = |x|$, translated 2 units to the right and 3 units up. This puts the corner at the point (2, 3), so this function is not differentiable at $x = 2$.

At every other point, the graph is (locally) a straight line and f has derivative $+1$ or -1 (again, just like $y = |x|$). *Now try Exercise 1.*



[–4, 4] by [–3, 3]
(a)

[1.7, 2.3] by [1.7, 2.1]
(b)

[1.93, 2.07] by [1.85, 1.95]
(c)

Figure 3.15 Three different views of the differentiable function $f(x) = x\cos(3x)$. We have zoomed in here at the point (2, 1.9).

Most of the functions we encounter in calculus are differentiable wherever they are defined, which means that they will *not* have corners, cusps, vertical tangent lines, or points of discontinuity within their domains. Their graphs will be unbroken and smooth, with a well-defined slope at each point. Polynomials are differentiable, as are rational functions, trigonometric functions, exponential functions, and logarithmic functions. Composites of differentiable functions are differentiable, and so are sums, products, integer powers, and quotients of differentiable functions, where defined. We will see why all of this is true as the chapter continues.

Differentiability Implies Local Linearity

A good way to think of differentiable functions is that they are **locally linear;** that is, a function that is differentiable at a closely resembles its own tangent line very close to a. In the jargon of graphing calculators, differentiable curves will "straighten out" when we zoom in on them at a point of differentiability. (See Figure 3.15.)

EXPLORATION 1 Zooming in to "See" Differentiability

Is either of these functions differentiable at $x = 0$?

(a) $f(x) = |x| + 1$ **(b)** $g(x) = \sqrt{x^2 + 0.0001} + 0.99$

1. We already know that f is not differentiable at $x = 0$; its graph has a corner there. Graph f and zoom in at the point (0, 1) several times. Does the corner show signs of straightening out?
2. Now do the same thing with g. Does the graph of g show signs of straightening out? We will learn a quick way to differentiate g in Section 3.6, but for now suffice it to say that it *is* differentiable at $x = 0$, and in fact has a horizontal tangent there.
3. How many zooms does it take before the graph of g looks exactly like a horizontal line?
4. Now graph f and g *together* in a standard square viewing window. They appear to be identical until you start zooming in. The differentiable function eventually straightens out, while the nondifferentiable function remains impressively unchanged.

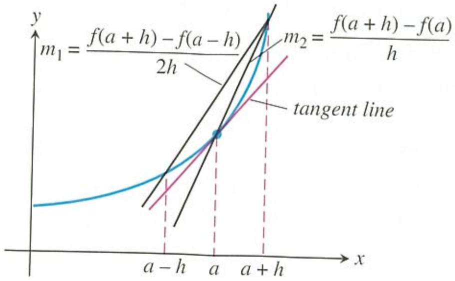


Figure 3.16 The symmetric difference quotient (slope m_1) usually gives a better approximation of the derivative for a given value of h than does the regular difference quotient (slope m_2), which is why the symmetric difference quotient is used in the numerical derivative.

Derivatives on a Calculator

Many graphing utilities can approximate derivatives numerically with good accuracy at most points of their domains.

For small values of h, the difference quotient

$$\frac{f(a+h)-f(a)}{h}$$

is often a good numerical approximation of $f'(a)$. However, as suggested by Figure 3.16, the same value of h will usually yield a *better* approximation if we use the **symmetric difference quotient**

$$\frac{f(a+h)-f(a-h)}{2h},$$

which is what our graphing calculator uses to calculate NDER $f(a)$, the **numerical derivative of f at a point a.** The **numerical derivative of f** as a function is denoted by NDER $f(x)$. Sometimes we will use NDER $(f(x), a)$ for NDER $f(a)$ when we want to emphasize both the function *and* the point.

Although the symmetric difference quotient is not the quotient used in the definition of $f'(a)$, it can be proven that

$$\lim_{h\to 0}\frac{f(a+h)-f(a-h)}{2h}$$

equals $f'(a)$ wherever $f'(a)$ exists.

You might think that an extremely small value of h would be required to give an accurate approximation of $f'(a)$, but in most cases $h = 0.001$ is more than adequate. In fact, your calculator probably assumes such a value for h unless you choose to specify otherwise (consult your *Owner's Manual*). The numerical derivatives we compute in this book will use $h = 0.001$; that is,

$$\text{NDER } f(a) = \frac{f(a+0.001)-f(a-0.001)}{0.002}.$$

EXAMPLE 2 Computing a Numerical Derivative

Compute NDER $(x^3, 2)$, the numerical derivative of x^3 at $x = 2$.

SOLUTION

Using $h = 0.001$,

$$\text{NDER }(x^3, 2) = \frac{(2.001)^3-(1.999)^3}{0.002} = 12.000001.$$

Now try Exercise 17.

In Example 1 of Section 3.1, we found the derivative of x^3 to be $3x^2$, whose value at $x = 2$ is $3(2)^2 = 12$. The numerical derivative is accurate to 5 decimal places. Not bad for the push of a button.

Example 2 gives dramatic evidence that NDER is very accurate when $h = 0.001$. Such accuracy is usually the case, although it is also possible for NDER to produce some surprisingly inaccurate results, as in Example 3.

EXAMPLE 3 Fooling the Symmetric Difference Quotient

Compute NDER $(|x|, 0)$, the numerical derivative of $|x|$ at $x = 0$.

continued

SOLUTION

We saw at the start of this section that $|x|$ is not differentiable at $x = 0$ since its right-hand and left-hand derivatives at $x = 0$ are not the same. Nonetheless,

$$\begin{aligned} \text{NDER}\,(|x|, 0) &= \lim_{h\to 0} \frac{|0 + h| - |0 - h|}{2h} \\ &= \lim_{h\to 0} \frac{|h| - |h|}{2h} \\ &= \lim_{h\to 0} \frac{0}{2h} \\ &= 0. \end{aligned}$$

The symmetric difference quotient, which works symmetrically on either side of 0, never detects the corner! Consequently, most graphing utilities will indicate (wrongly) that $y = |x|$ is differentiable at $x = 0$, with derivative 0.

Now try Exercise 23.

An Alternative to NDER

Graphing

$$y = \frac{f(x + 0.001) - f(x - 0.001)}{0.002}$$

is equivalent to graphing $y = \text{NDER}\, f(x)$ (useful if NDER is not readily available on your calculator).

In light of Example 3, it is worth repeating here that NDER $f(a)$ actually does approach $f'(a)$ *when* $f'(a)$ *exists,* and in fact approximates it quite well (as in Example 2).

EXPLORATION 2 Looking at the Symmetric Difference Quotient Analytically

Let $f(x) = x^2$ and let $h = 0.01$.

1. Find
$$\frac{f(10 + h) - f(10)}{h}.$$
How close is it to $f'(10)$?
2. Find
$$\frac{f(10 + h) - f(10 - h)}{2h}.$$
How close is it to $f'(10)$?
3. Repeat this comparison for $f(x) = x^3$.

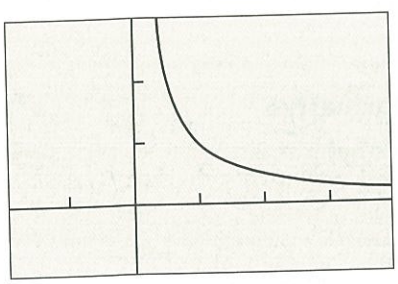
EXAMPLE 4 Graphing a Derivative Using NDER

Let $f(x) = \ln x$. Use NDER to graph $y = f'(x)$. Can you guess what function $f'(x)$ is by analyzing its graph?

SOLUTION

The graph is shown in Figure 3.17a. The shape of the graph suggests, and the table of values in Figure 3.17b supports, the conjecture that this is the graph of $y = 1/x$. We will prove in Section 3.9 (using analytic methods) that this is indeed the case.

Now try Exercise 27.



[–2, 4] by [–1, 3]

(a)

X	Y1	
.1	10	
.2	5	
.3	3.3333	
.4	2.5	
.5	2	
.6	1.6667	
.7	1.4286	
X = .1		

(b)

Figure 3.17 (a) The graph of NDER ln (x) and (b) a table of values. What graph could this be? (Example 4)

Differentiability Implies Continuity

We began this section with a look at the typical ways that a function could fail to have a derivative at a point. As one example, we indicated graphically that a discontinuity in the graph of f would cause one or both of the one-sided derivatives to be nonexistent. It is actually not difficult to give an analytic proof that continuity is an essential condition for the derivative to exist, so we include that as a theorem here.

THEOREM 1 Differentiability Implies Continuity

If f has a derivative at $x = a$, then f is continuous at $x = a$.

Proof Our task is to show that $\lim_{x \to a} f(x) = f(a)$, or, equivalently, that

$$\lim_{x \to a} [f(x) - f(a)] = 0.$$

Using the Limit Product Rule (and noting that $x - a$ is not zero), we can write

$$\begin{aligned} \lim_{x \to a} [f(x) - f(a)] &= \lim_{x \to a} \left[(x - a) \frac{f(x) - f(a)}{x - a} \right] \\ &= \lim_{x \to a} (x - a) \cdot \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \\ &= 0 \cdot f'(a) \\ &= 0. \end{aligned}$$

■

The converse of Theorem 1 is false, as we have already seen. A continuous function might have a corner, a cusp, or a vertical tangent line, and hence not be differentiable at a given point.

Intermediate Value Theorem for Derivatives

Not every function can be a derivative. A derivative must have the intermediate value property, as stated in the following theorem (the proof of which can be found in advanced texts).

THEOREM 2 Intermediate Value Theorem for Derivatives

If a and b are any two points in an interval on which f is differentiable, then f' takes on every value between $f'(a)$ and $f'(b)$.

EXAMPLE 5 Applying Theorem 2

Does any function have the Unit Step Function (see Figure 3.14) as its derivative?

SOLUTION

No. Choose some $a < 0$ and some $b > 0$. Then $U(a) = -1$ and $U(b) = 1$, but U does not take on any value between -1 and 1.

Now try Exercise 37.

The question of when a function is a derivative of some function is one of the central questions in all of calculus. The answer, found by Newton and Leibniz, would revolutionize the world of mathematics. We will see what that answer is when we reach Chapter 5.

Quick Review 3.2 *(For help, go to Sections 1.2 and 2.1.)*

In Exercises 1–5, tell whether the limit could be used to define $f'(a)$ (assuming that f is differentiable at a).

1. $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$

2. $\lim_{h\to 0} \frac{f(a+h)-f(h)}{h}$

3. $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$

4. $\lim_{x\to a} \frac{f(a)-f(x)}{a-x}$

5. $\lim_{h\to 0} \frac{f(a+h)+f(a-h)}{h}$

6. Find the domain of the function $y = x^{4/3}$.

7. Find the domain of the function $y = x^{3/4}$.

8. Find the range of the function $y = |x-2| + 3$.

9. Find the slope of the line $y - 5 = 3.2(x + \pi)$.

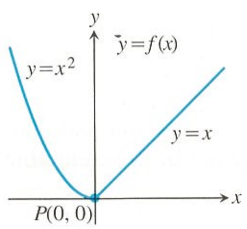
10. If $f(x) = 5x$, find

$$\frac{f(3+0.001)-f(3-0.001)}{0.002}.$$

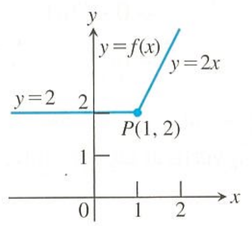
Section 3.2 Exercises

In Exercises 1–4, compare the right-hand and left-hand derivatives to show that the function is not differentiable at the point P. Find all points where f is not differentiable.

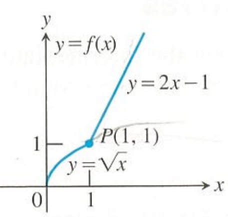
1.



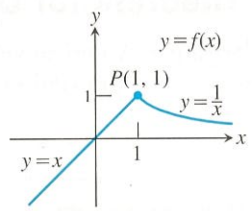
2.



3.



4.



In Exercises 5–10, the graph of a function over a closed interval D is given. At what domain points does the function appear to be

(a) differentiable? **(b)** continuous but not differentiable?

(c) neither continuous nor differentiable?

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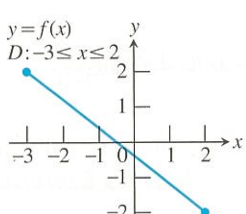

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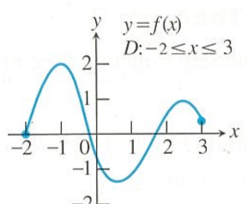

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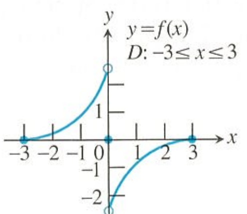

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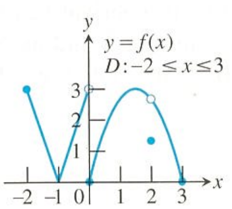

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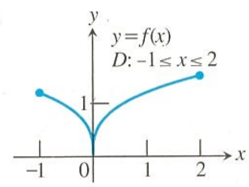

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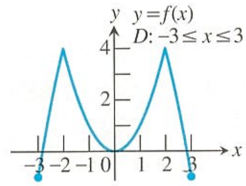


In Exercises 11–16, the function fails to be differentiable at $x = 0$. Tell whether the problem is a corner, a cusp, a vertical tangent, or a discontinuity.

11. $y = \begin{cases} \tan^{-1} x, & x \neq 0 \\ 1, & x = 0 \end{cases}$

12. $y = x^{4/5}$

13. $y = x + \sqrt{x^2} + 2$

14. $y = 3 - \sqrt[3]{x}$

15. $y = 3x - 2|x| - 1$

16. $y = \sqrt[3]{|x|}$

In Exercises 17–26, find the numerical derivative of the given function at the indicated point. Use $h = 0.001$. Is the function differentiable at the indicated point?

17. $f(x) = 4x - x^2,\ x = 0$

18. $f(x) = 4x - x^2,\ x = 3$

19. $f(x) = 4x - x^2,\ x = 1$

20. $f(x) = x^3 - 4x,\ x = 0$

21. $f(x) = x^3 - 4x,\ x = -2$

22. $f(x) = x^3 - 4x,\ x = 2$

23. $f(x) = x^{2/3},\ x = 0$

24. $f(x) = |x - 3|,\ x = 3$

25. $f(x) = x^{2/5},\ x = 0$

26. $f(x) = x^{4/5},\ x = 0$

Group Activity In Exercises 27–30, use NDER to graph the derivative of the function. If possible, identify the derivative function by looking at the graph.

27. $y = -\cos x$

28. $y = 0.25x^4$

29. $y = \frac{x|x|}{2}$

30. $y = -\ln|\cos x|$

In Exercises 31–36, find all values of x for which the function is differentiable.

31. $f(x) = \frac{x^3 - 8}{x^2 - 4x - 5}$

32. $h(x) = \sqrt[3]{3x - 6} + 5$

33. $P(x) = \sin(|x|) - 1$

34. $Q(x) = 3\cos(|x|)$

35. $g(x) = \begin{cases} (x+1)^2, & x \le 0 \\ 2x + 1, & 0 < x < 3 \\ (4 - x)^2, & x \ge 3 \end{cases}$

36. $C(x) = x|x|$

37. Show that the function

$$f(x) = \begin{cases} 0, & -1 \le x < 0 \\ 1, & 0 \le x \le 1 \end{cases}$$

is not the derivative of any function on the interval $-1 \le x \le 1$.

38. Writing to Learn Recall that the numerical derivative (NDER) can give meaningless values at points where a function is not differentiable. In this exercise, we consider the numerical derivatives of the functions $1/x$ and $1/x^2$ at $x = 0$.

(a) Explain why neither function is differentiable at $x = 0$.

(b) Find NDER at $x = 0$ for each function.

(c) By analyzing the definition of the symmetric difference quotient, explain why NDER returns wrong responses that are so different from each other for these two functions.

39. Let f be the function defined as

$$f(x) = \begin{cases} 3 - x, & x < 1 \\ ax^2 + bx, & x \ge 1 \end{cases}$$

where a and b are constants.

(a) If the function is continuous for all x, what is the relationship between a and b?

(b) Find the unique values for a and b that will make f both continuous and differentiable.

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

40. True or False If f has a derivative at $x = a$, then f is continuous at $x = a$. Justify your answer.

41. True or False If f is continuous at $x = a$, then f has a derivative at $x = a$. Justify your answer.

42. Multiple Choice Which of the following is true about the graph of $f(x) = x^{4/5}$ at $x = 0$?

(A) It has a corner.

(B) It has a cusp.

(C) It has a vertical tangent.

(D) It has a discontinuity.

(E) $f(0)$ does not exist.

43. Multiple Choice Let $f(x) = \sqrt[3]{x - 1}$. At which of the following points is $f'(a) \ne \text{NDER}(f, x, a)$?

(A) $a = 1$ **(B)** $a = -1$ **(C)** $a = 2$ **(D)** $a = -2$ **(E)** $a = 0$

In Exercises 44 and 45, let

$$f(x) = \begin{cases} 2x + 1, & x \le 0 \\ x^2 + 1, & x > 0. \end{cases}$$

44. Multiple Choice Which of the following is equal to the left-hand derivative of f at $x = 0$?

(A) $2x$ **(B)** 2 **(C)** 0 **(D)** $-\infty$ **(E)** ∞

45. Multiple Choice Which of the following is equal to the right-hand derivative of f at $x = 0$?

(A) $2x$ **(B)** 2 **(C)** 0 **(D)** $-\infty$ **(E)** ∞

Explorations

46. (a) Enter the expression "$x < 0$" into Y1 of your calculator using "<" from the TEST menu. Graph Y1 in DOT MODE in the window $[-4.7, 4.7]$ by $[-3.1, 3.1]$.

(b) Describe the graph in part (a).

(c) Enter the expression "$x \ge 0$" into Y1 of your calculator using "≥" from the TEST menu. Graph Y1 in DOT MODE in the window $[-4.7, 4.7]$ by $[-3.1, 3.1]$.

(d) Describe the graph in part (c).

47. *Graphing Piecewise Functions on a Calculator* Let

$$f(x) = \begin{cases} x^2, & x \le 0 \\ 2x, & x > 0. \end{cases}$$

(a) Enter the expression "$(X^2)(X\le0)+(2X)(X>0)$" into Y1 of your calculator and draw its graph in the window $[-4.7, 4.7]$ by $[-3, 5]$.

(b) Explain why the values of Y1 and $f(x)$ are the same.

(c) Enter the numerical derivative of Y1 into Y2 of your calculator and draw its graph in the same window. Turn off the graph of Y1.

(d) Use TRACE to calculate NDER(Y1, x, -0.1), NDER(Y1, x, 0), and NDER(Y1, x, 0.1). Compare with Section 3.1, Example 6.

Extending the Ideas

48. *Oscillation* There is another way that a function might fail to be differentiable, and that is by *oscillation*. Let

$$f(x) = \begin{cases} x \sin \dfrac{1}{x}, & x \ne 0 \\ 0, & x = 0. \end{cases}$$

(a) Show that f is continuous at $x = 0$.

(b) Show that

$$\frac{f(0 + h) - f(0)}{h} = \sin \frac{1}{h}.$$

(c) Explain why

$$\lim_{h \to 0} \frac{f(0 + h) - f(0)}{h}$$

does not exist.

(d) Does f have either a left-hand or right-hand derivative at $x = 0$?

(e) Now consider the function

$$g(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & x \ne 0 \\ 0, & x = 0. \end{cases}$$

Use the definition of the derivative to show that g is differentiable at $x = 0$ and that $g'(0) = 0$.

3.3 Rules for Differentiation

What you'll learn about

- Positive Integer Powers, Multiples, Sums, and Differences
- Products and Quotients
- Negative Integer Powers of x
- Second and Higher Order Derivatives

. . . and why

These rules help us find derivatives of functions analytically more efficiently.

Positive Integer Powers, Multiples, Sums, and Differences

The first rule of differentiation is that the derivative of every constant function is the zero function.

RULE 1 Derivative of a Constant Function

If f is the function with the constant value c, then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0.$$

Proof of Rule 1 If $f(x) = c$ is a function with a constant value c, then

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} 0 = 0. \quad ■$$

The next rule is a first step toward a rule for differentiating any polynomial.

RULE 2 Power Rule for Positive Integer Powers of x

If n is a positive integer, then

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

Proof of Rule 2 If $f(x) = x^n$, then $f(x+h) = (x+h)^n$ and the difference quotient for f is

$$\frac{(x+h)^n - x^n}{h}.$$

We can readily find the limit of this quotient as $h \to 0$ if we apply the algebraic identity

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}) \quad n \text{ a positive integer}$$

with $a = x + h$ and $b = x$. For then $(a - b) = h$ and the h's in the numerator and denominator of the quotient cancel, giving

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{(x+h)^n - x^n}{h} \\ &= \frac{h[(x+h)^{n-1} + (x+h)^{n-2}x + \cdots + (x+h)x^{n-2} + x^{n-1}]}{h} \\ &= \underbrace{(x+h)^{n-1} + (x+h)^{n-2}x + \cdots + (x+h)x^{n-2} + x^{n-1}}_{n \text{ terms, each with limit } x^{n-1} \text{ as } h \to 0}. \end{aligned}$$

Hence,

$$\frac{d}{dx}(x^n) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = nx^{n-1}. \quad ■$$

The Power Rule says: To differentiate x^n, multiply by n and subtract 1 from the exponent. For example, the derivatives of x^2, x^3, and x^4 are $2x^1$, $3x^2$, and $4x^3$, respectively.

RULE 3 The Constant Multiple Rule

If u is a differentiable function of x and c is a constant, then

$$\frac{d}{dx}(cu) = c\frac{du}{dx}.$$

Proof of Rule 3

$$\begin{aligned}\frac{d}{dx}(cu) &= \lim_{h\to 0}\frac{cu(x+h) - cu(x)}{h}\\ &= c\lim_{h\to 0}\frac{u(x+h) - u(x)}{h}\\ &= c\frac{du}{dx}\end{aligned}$$

■

Rule 3 says that if a differentiable function is multiplied by a constant, then its derivative is multiplied by the same constant. Combined with Rule 2, it enables us to find the derivative of any monomial quickly; for example, the derivative of $7x^4$ is $7(4x^3) = 28x^3$.

To find the derivatives of polynomials, we need to be able to differentiate sums and differences of monomials. We can accomplish this by applying the Sum and Difference Rule.

Denoting Functions by *u* and *v*

The functions we work with when we need a differentiation formula are likely to be denoted by letters like f and g. When we apply the formula, we do not want to find the formula using these same letters in some other way. To guard against this, we denote the functions in differentiation rules by letters like u and v that are not likely to be already in use.

RULE 4 The Sum and Difference Rule

If u and v are differentiable functions of x, then their sum and difference are differentiable at every point where u and v are differentiable. At such points,

$$\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}.$$

Proof of Rule 4

We use the difference quotient for $f(x) = u(x) + v(x)$.

$$\begin{aligned}\frac{d}{dx}[u(x) + v(x)] &= \lim_{h\to 0}\frac{[u(x+h) + v(x+h)] - [u(x) + v(x)]}{h}\\ &= \lim_{h\to 0}\left[\frac{u(x+h) - u(x)}{h} + \frac{v(x+h) - v(x)}{h}\right]\\ &= \lim_{h\to 0}\frac{u(x+h) - u(x)}{h} + \lim_{h\to 0}\frac{v(x+h) - v(x)}{h}\\ &= \frac{du}{dx} + \frac{dv}{dx}\end{aligned}$$

The proof of the rule for the difference of two functions is similar. ■

EXAMPLE 1 Differentiating a Polynomial

Find $\frac{dp}{dt}$ if $p = t^3 + 6t^2 - \frac{5}{3}t + 16$.

SOLUTION

By Rule 4 we can differentiate the polynomial term-by-term, applying Rules 1 through 3 as we go.

$$\frac{dp}{dt} = \frac{d}{dt}(t^3) + \frac{d}{dt}(6t^2) - \frac{d}{dt}\left(\frac{5}{3}t\right) + \frac{d}{dt}(16) \quad \text{Sum and Difference Rule}$$

$$= 3t^2 + 6 \cdot 2t - \frac{5}{3} + 0 \quad \text{Constant and Power Rules}$$

$$= 3t^2 + 12t - \frac{5}{3}$$

Now try Exercise 5.

EXAMPLE 2 Finding Horizontal Tangents

Does the curve $y = x^4 - 2x^2 + 2$ have any horizontal tangents? If so, where?

SOLUTION

The horizontal tangents, if any, occur where the slope dy/dx is zero. To find these points, we

(a) calculate dy/dx:

$$\frac{dy}{dx} = \frac{d}{dx}(x^4 - 2x^2 + 2) = 4x^3 - 4x.$$

(b) solve the equation $dy/dx = 0$ for x:

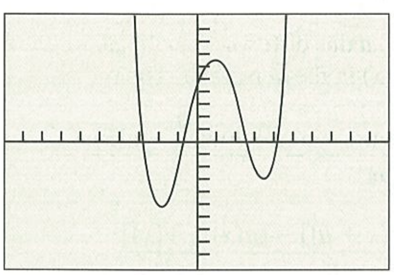
$$4x^3 - 4x = 0$$

$$4x(x^2 - 1) = 0$$

$$x = 0, 1, -1.$$

The curve has horizontal tangents at $x = 0, 1,$ and -1. The corresponding points on the curve (found from the equation $y = x^4 - 2x^2 + 2$) are $(0, 2)$, $(1, 1)$, and $(-1, 1)$. You might wish to graph the curve to see where the horizontal tangents go.

Now try Exercise 7.



[–10, 10] by [–10, 10]

Figure 3.18 The graph of $y = 0.2x^4 - 0.7x^3 - 2x^2 + 5x + 4$ has three horizontal tangents. (Example 3)

The derivative in Example 2 was easily factored, making an algebraic solution of the equation $dy/dx = 0$ correspondingly simple. When a simple algebraic solution is not possible, the solutions to $dy/dx = 0$ can still be found to a high degree of accuracy by using the SOLVE capability of your calculator.

EXAMPLE 3 Using Calculus and Calculator

As can be seen in the viewing window $[-10, 10]$ by $[-10, 10]$, the graph of $y = 0.2x^4 - 0.7x^3 - 2x^2 + 5x + 4$ has three horizontal tangents (Figure 3.18). At what points do these horizontal tangents occur?

continued

SOLUTION

First we find the derivative

$$\frac{dy}{dx} = 0.8x^3 - 2.1x^2 - 4x + 5.$$

Using the calculator solver, we find that $0.8x^3 - 2.1x^2 - 4x + 5 = 0$ when $x \approx -1.862$, 0.9484, and 3.539. We use the calculator again to evaluate the original function at these x-values and find the corresponding points to be approximately $(-1.862, -5.321)$, $(0.9484, 6.508)$, and $(3.539, -3.008)$.

Now try Exercise 11.

On Rounding Calculator Values

Notice in Example 3 that we rounded the x-values to four significant digits when we presented the answers. The calculator actually presented many more digits, but there was no practical reason for writing all of them. When we used the calculator to compute the corresponding y-values, however, we *used the x-values stored in the calculator,* not the rounded values. We then rounded the y-values to four significant digits when we presented the ordered pairs. Significant "round-off errors" can accumulate in a problem if you use rounded intermediate values for doing additional computations, so avoid rounding until the final answer.

Products and Quotients

While the derivative of the sum of two functions is the sum of their derivatives and the derivative of the difference of two functions is the difference of their derivatives, the derivative of the product of two functions is *not* the product of their derivatives.

For instance,

$$\frac{d}{dx}(x \cdot x) = \frac{d}{dx}(x^2) = 2x, \quad \text{while} \quad \frac{d}{dx}(x) \cdot \frac{d}{dx}(x) = 1 \cdot 1 = 1.$$

The derivative of a product is actually the sum of *two* products, as we now explain.

You can remember the Product Rule with the phrase "the first times the derivative of the second plus the second times the derivative of the first."

RULE 5 The Product Rule

The product of two differentiable functions u and v is differentiable, and

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Proof of Rule 5 We begin, as usual, by applying the definition.

$$\frac{d}{dx}(uv) = \lim_{h \to 0} \frac{u(x+h)v(x+h) - u(x)v(x)}{h}$$

To change the fraction into an equivalent one that contains difference quotients for the derivatives of u and v, we subtract and add $u(x+h)v(x)$ in the numerator. Then,

$$\frac{d}{dx}(uv) = \lim_{h \to 0} \frac{u(x+h)v(x+h) - u(x+h)v(x) + u(x+h)v(x) - u(x)v(x)}{h}$$

$$= \lim_{h \to 0} \left[u(x+h)\frac{v(x+h) - v(x)}{h} + v(x)\frac{u(x+h) - u(x)}{h} \right] \quad \text{Factor and separate.}$$

$$= \lim_{h \to 0} u(x+h) \cdot \lim_{h \to 0} \frac{v(x+h) - v(x)}{h} + v(x) \cdot \lim_{h \to 0} \frac{u(x+h) - u(x)}{h}.$$

As h approaches 0, $u(x+h)$ approaches $u(x)$ because u, being differentiable at x, is continuous at x. The two fractions approach the values of dv/dx and du/dx, respectively, at x. Therefore

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

■

Gottfried Wilhelm Leibniz (1646–1716)

The method of limits used in this book was not discovered until nearly a century after Newton and Leibniz, the discoverers of calculus, had died.

To Leibniz, the key idea was the *differential,* an infinitely small quantity that was almost like zero, but which—unlike zero—could be used in the denominator of a fraction. Thus, Leibniz thought of the derivative dy/dx as the quotient of two differentials, dy and dx.

The problem was explaining why these differentials sometimes became zero and sometimes did not! See Exercise 59.

Some 17th-century mathematicians were confident that the calculus of Newton and Leibniz would eventually be found to be fatally flawed because of these mysterious quantities. It was only after later generations of mathematicians had found better ways to prove their results that the calculus of Newton and Leibniz was accepted by the entire scientific community.

EXAMPLE 4 Differentiating a Product

Find $f'(x)$ if $f(x) = (x^2 + 1)(x^3 + 3)$.

SOLUTION

From the Product Rule with $u = x^2 + 1$ and $v = x^3 + 3$, we find

$$f'(x) = \frac{d}{dx}[(x^2 + 1)(x^3 + 3)] = (x^2 + 1)(3x^2) + (x^3 + 3)(2x)$$
$$= 3x^4 + 3x^2 + 2x^4 + 6x$$
$$= 5x^4 + 3x^2 + 6x.$$

Now try Exercise 13.

We could also have done Example 4 by multiplying out the original expression and then differentiating the resulting polynomial. That alternate strategy will not work, however, on a product like $x^2 \sin x$.

Just as the derivative of the product of two differentiable functions is not the product of their derivatives, the derivative of a quotient of two functions is not the quotient of their derivatives. What happens instead is this:

Using the Quotient Rule

Since order is important in subtraction, be sure to set up the numerator of the Quotient Rule correctly:

v times the derivative of *u*

minus

u times the derivative of *v*.

You can remember the Quotient Rule with the phrase "bottom times the derivative of the top minus the top times the derivative of the bottom, all over the bottom squared."

RULE 6 The Quotient Rule

At a point where $v \neq 0$, the quotient $y = u/v$ of two differentiable functions is differentiable, and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}.$$

Proof of Rule 6

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \lim_{h \to 0} \frac{\dfrac{u(x+h)}{v(x+h)} - \dfrac{u(x)}{v(x)}}{h}$$
$$= \lim_{h \to 0} \frac{v(x)u(x+h) - u(x)v(x+h)}{hv(x+h)v(x)}$$

To change the last fraction into an equivalent one that contains the difference quotients for the derivatives of u and v, we subtract and add $v(x)u(x)$ in the numerator. This allows us to continue with

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \lim_{h \to 0} \frac{v(x)u(x+h) - v(x)u(x) + v(x)u(x) - u(x)v(x+h)}{hv(x+h)v(x)}$$
$$= \lim_{h \to 0} \frac{v(x)\dfrac{u(x+h) - u(x)}{h} - u(x)\dfrac{v(x+h) - v(x)}{h}}{v(x+h)v(x)}.$$

Taking the limits in both the numerator and denominator now gives us the Quotient Rule. ■

EXAMPLE 5 Supporting Computations Graphically

Differentiate $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$. Support graphically.

continued

SOLUTION

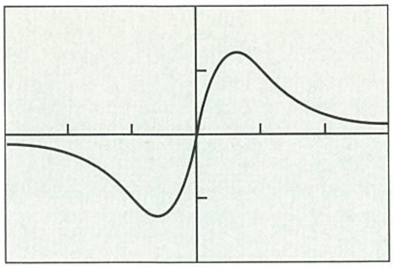
We apply the Quotient Rule with $u = x^2 - 1$ and $v = x^2 + 1$:

$$f'(x) = \frac{(x^2+1)\cdot 2x - (x^2-1)\cdot 2x}{(x^2+1)^2} \qquad \frac{v(du/dx) - u(dv/dx)}{v^2}$$

$$= \frac{2x^3 + 2x - 2x^3 + 2x}{(x^2+1)^2}$$

$$= \frac{4x}{(x^2+1)^2}.$$

The graphs of $y_1 = f'(x)$ calculated above and of $y_2 = \text{NDER}\,f(x)$ are shown in Figure 3.19. The fact that they appear to be identical provides strong graphical support that our calculations are indeed correct. ***Now try Exercise 19.***



[–3, 3] by [–2, 2]

Figure 3.19 The graph of

$$y = \frac{4x}{(x^2+1)^2}$$

and the graph of

$$y = \text{NDER}\left(\frac{x^2-1}{x^2+1}\right)$$

appear to be the same. (Example 5)

EXAMPLE 6 Working with Numerical Values

Let $y = uv$ be the product of the functions u and v. Find $y'(2)$ if

$$u(2) = 3, \quad u'(2) = -4, \quad v(2) = 1, \quad \text{and} \quad v'(2) = 2.$$

SOLUTION

From the Product Rule, $y' = (uv)' = uv' + vu'$. In particular,

$$y'(2) = u(2)v'(2) + v(2)u'(2)$$
$$= (3)(2) + (1)(-4)$$
$$= 2.$$

Now try Exercise 23.

Negative Integer Powers of x

The rule for differentiating negative powers of x is the same as Rule 2 for differentiating positive powers of x, although our proof of Rule 2 does not work for negative values of n. We can now extend the Power Rule to negative integer powers by a clever use of the Quotient Rule.

RULE 7 Power Rule for Negative Integer Powers of x

If n is a negative integer and $x \neq 0$, then

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

Proof of Rule 7 If n is a negative integer, then $n = -m$, where m is a positive integer. Hence, $x^n = x^{-m} = 1/x^m$, and

$$\frac{d}{dx}(x^n) = \frac{d}{dx}\left(\frac{1}{x^m}\right) = \frac{x^m \cdot \frac{d}{dx}(1) - 1\cdot\frac{d}{dx}(x^m)}{(x^m)^2}$$

$$= \frac{0 - mx^{m-1}}{x^{2m}}$$

$$= -mx^{-m-1}$$

$$= nx^{n-1}.$$

■

EXAMPLE 7 Using the Power Rule

Find an equation for the line tangent to the curve

$$y = \frac{x^2 + 3}{2x}$$

at the point (1, 2). Support your answer graphically.

SOLUTION

We could find the derivative by the Quotient Rule, but it is easier to first simplify the function as a sum of two powers of x.

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^2}{2x} + \frac{3}{2x}\right)$$
$$= \frac{d}{dx}\left(\frac{1}{2}x + \frac{3}{2}x^{-1}\right)$$
$$= \frac{1}{2} - \frac{3}{2}x^{-2}$$

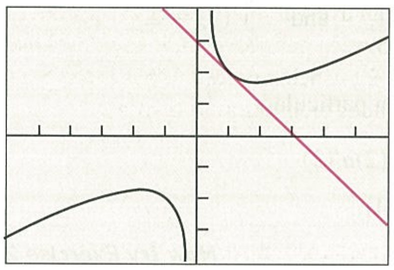
The slope at $x = 1$ is

$$\left.\frac{dy}{dx}\right|_{x=1} = \left[\frac{1}{2} - \frac{3}{2}x^{-2}\right]_{x=1} = \frac{1}{2} - \frac{3}{2} = -1.$$

The line through (1, 2) with slope $m = -1$ is

$$y - 2 = (-1)(x - 1)$$
$$y = -x + 1 + 2$$
$$y = -x + 3.$$

We graph $y = (x^2 + 3)/2x$ and $y = -x + 3$ (Figure 3.20), observing that the line appears to be tangent to the curve at (1, 2). Thus, we have graphical support that our computations are correct. *Now try Exercise 27.*



[–6, 6] by [–4, 4]

Figure 3.20 The line $y = -x + 3$ appears to be tangent to the graph of

$$y = \frac{x^2 + 3}{2x}$$

at the point (1, 2). (Example 7)

Second and Higher Order Derivatives

The derivative $y' = dy/dx$ is called the *first derivative* of y with respect to x. The first derivative may itself be a differentiable function of x. If so, its derivative,

$$y'' = \frac{dy'}{dx} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2},$$

is called the *second derivative* of y with respect to x. If y'' ("y double-prime") is differentiable, its derivative,

$$y''' = \frac{dy''}{dx} = \frac{d^3y}{dx^3},$$

is called the *third derivative* of y with respect to x. The names continue as you might expect they would, except that the multiple-prime notation begins to lose its usefulness after about three primes. We use

$$y^{(n)} = \frac{d}{dx}y^{(n-1)} \quad \text{"y super n"}$$

to denote the **nth derivative** of y with respect to x. (We also use d^ny/dx^n.) Do not confuse $y^{(n)}$ with the nth power of y, which is y^n.

Technology Tip

HIGHER ORDER DERIVATIVES WITH NDER

Some graphers will allow the *nesting* of the NDER function,

NDER2 f = NDER(NDER f),

but such nesting, in general, is safe only to the second derivative. Beyond that, the error buildup in the algorithm makes the results unreliable.

EXAMPLE 8 Finding Higher Order Derivatives

Find the first four derivatives of $y = x^3 - 5x^2 + 2$.

SOLUTION

The first four derivatives are:

First derivative:	$y' = 3x^2 - 10x;$
Second derivative:	$y'' = 6x - 10;$
Third derivative:	$y''' = 6;$
Fourth derivative:	$y^{(4)} = 0.$

This function has derivatives of all orders, the fourth and higher order derivatives all being zero. *Now try Exercise 33.*

EXAMPLE 9 Finding Instantaneous Rate of Change

An orange farmer currently has 200 trees yielding an average of 15 bushels of oranges per tree. She is expanding her farm at the rate of 15 trees per year, while improved husbandry is improving her average annual yield by 1.2 bushels per tree. What is the current (instantaneous) rate of increase of her total annual production of oranges?

SOLUTION

Let the functions t and y be defined as follows.

$$t(x) = \text{the number of trees } x \text{ years from now.}$$

$$y(x) = \text{yield per tree } x \text{ years from now.}$$

Then $p(x) = t(x)y(x)$ is the total production of oranges in year x. We know the following values.

$$t(0) = 200, \qquad y(0) = 15$$

$$t'(0) = 15, \qquad y'(0) = 1.2$$

We need to find $p'(0)$, where $p = ty$.

$$\begin{aligned} p'(0) &= t(0)y'(0) + y(0)t'(0) \\ &= (200)(1.2) + (15)(15) \\ &= 465 \end{aligned}$$

The rate we seek is 465 bushels per year. *Now try Exercise 51.*

Quick Review 3.3 *(For help, go to Sections 1.2 and 3.1.)*

In Exercises 1–6, write the expression as a sum of powers of x.

1. $(x^2 - 2)(x^{-1} + 1)$

2. $\left(\dfrac{x}{x^2 + 1}\right)^{-1}$

3. $3x^2 - \dfrac{2}{x} + \dfrac{5}{x^2}$

4. $\dfrac{3x^4 - 2x^3 + 4}{2x^2}$

5. $(x^{-1} + 2)(x^{-2} + 1)$

6. $\dfrac{x^{-1} + x^{-2}}{x^{-3}}$

7. Find the positive roots of the equation

$$2x^3 - 5x^2 - 2x + 6 = 0$$

and evaluate the function $y = 500x^6$ at each root. Round your answers to the nearest integer, but only in the final step.

8. If $f(x) = 7$ for all real numbers x, find

(a) $f(10)$.

(b) $f(0)$.

(c) $f(x + h)$.

(d) $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$.

9. Find the derivatives of these functions with respect to x.

(a) $f(x) = \pi$ **(b)** $f(x) = \pi^2$ **(c)** $f(x) = \pi^{15}$

10. Find the derivatives of these functions with respect to x using the definition of the derivative.

(a) $f(x) = \frac{x}{\pi}$ **(b)** $f(x) = \frac{\pi}{x}$

Section 3.3 Exercises

In Exercises 1–6, find dy/dx.

1. $y = -x^2 + 3$

2. $y = \frac{x^3}{3} - x$

3. $y = 2x + 1$

4. $y = x^2 + x + 1$

5. $y = \frac{x^3}{3} + \frac{x^2}{2} + x$

6. $y = 1 - x + x^2 - x^3$

In Exercises 7–12, find the horizontal tangents of the curve.

7. $y = x^3 - 2x^2 + x + 1$

8. $y = x^3 - 4x^2 + x + 2$

9. $y = x^4 - 4x^2 + 1$

10. $y = 4x^3 - 6x^2 - 1$

11. $y = 5x^3 - 3x^5$

12. $y = x^4 - 7x^3 + 2x^2 + 15$

13. Let $y = (x + 1)(x^2 + 1)$. Find dy/dx **(a)** by applying the Product Rule, and **(b)** by multiplying the factors first and then differentiating.

14. Let $y = (x^2 + 3)/x$. Find dy/dx **(a)** by using the Quotient Rule, and **(b)** by first dividing the terms in the numerator by the denominator and then differentiating.

In Exercises 15–22, find dy/dx. Support your answer graphically.

15. $(x^3 + x + 1)(x^4 + x^2 + 1)$

16. $(x^2 + 1)(x^3 + 1)$

17. $y = \frac{2x + 5}{3x - 2}$

18. $y = \frac{x^2 + 5x - 1}{x^2}$

19. $y = \frac{(x - 1)(x^2 + x + 1)}{x^3}$

20. $y = (1 - x)(1 + x^2)^{-1}$

21. $y = \frac{x^2}{1 - x^3}$

22. $y = \frac{(x + 1)(x + 2)}{(x - 1)(x - 2)}$

23. Suppose u and v are functions of x that are differentiable at $x = 0$, and that $u(0) = 5$, $u'(0) = -3$, $v(0) = -1$, $v'(0) = 2$. Find the values of the following derivatives at $x = 0$.

(a) $\frac{d}{dx}(uv)$

(b) $\frac{d}{dx}\left(\frac{u}{v}\right)$

(c) $\frac{d}{dx}\left(\frac{v}{u}\right)$

(d) $\frac{d}{dx}(7v - 2u)$

24. Suppose u and v are functions of x that are differentiable at $x = 2$ and that $u(2) = 3$, $u'(2) = -4$, $v(2) = 1$, and $v'(2) = 2$. Find the values of the following derivatives at $x = 2$.

(a) $\frac{d}{dx}(uv)$

(b) $\frac{d}{dx}\left(\frac{u}{v}\right)$

(c) $\frac{d}{dx}\left(\frac{v}{u}\right)$

(d) $\frac{d}{dx}(3u - 2v + 2uv)$

25. Which of the following numbers is the slope of the line tangent to the curve $y = x^2 + 5x$ at $x = 3$?

i. 24 **ii.** $-5/2$ **iii.** 11 **iv.** 8

26. Which of the following numbers is the slope of the line $3x - 2y + 12 = 0$?

i. 6 **ii.** 3 **iii.** 3/2 **iv.** 2/3

In Exercises 27 and 28, find an equation for the line tangent to the curve at the given point.

27. $y = \frac{x^3 + 1}{2x}$, $x = 1$

28. $y = \frac{x^4 + 2}{x^2}$, $x = -1$

In Exercises 29–32, find dy/dx.

29. $y = 4x^{-2} - 8x + 1$

30. $y = \frac{x^{-4}}{4} - \frac{x^{-3}}{3} + \frac{x^{-2}}{2} - x^{-1} + 3$

31. $y = \frac{\sqrt{x} - 1}{\sqrt{x} + 1}$

32. $y = 2\sqrt{x} - \frac{1}{\sqrt{x}}$

In Exercises 33–36, find the first four derivatives of the function.

33. $y = x^4 + x^3 - 2x^2 + x - 5$

34. $y = x^2 + x + 3$

35. $y = x^{-1} + x^2$

36. $y = \frac{x + 1}{x}$

In Exercises 37–42, support your answer graphically.

37. Find an equation of the line perpendicular to the tangent to the curve $y = x^3 - 3x + 1$ at the point (2, 3).

38. Find the tangents to the curve $y = x^3 + x$ at the points where the slope is 4. What is the smallest slope of the curve? At what value of x does the curve have this slope?

39. Find the points on the curve $y = 2x^3 - 3x^2 - 12x + 20$ where the tangent is parallel to the x-axis.

40. Find the x- and y-intercepts of the line that is tangent to the curve $y = x^3$ at the point $(-2, -8)$.

41. Find the tangents to *Newton's serpentine*,

$$y = \frac{4x}{x^2 + 1},$$

at the origin and the point (1, 2).

42. Find the tangent to the *witch of Agnesi*,

$$y = \frac{8}{4 + x^2},$$

at the point (2, 1).

43. Use the definition of derivative (given in Section 3.1, Equation 1) to show that

(a) $\frac{d}{dx}(x) = 1.$

(b) $\frac{d}{dx}(-u) = -\frac{du}{dx}.$

44. Use the Product Rule to show that

$$\frac{d}{dx}(c \cdot f(x)) = c \cdot \frac{d}{dx} f(x)$$

for any constant c.

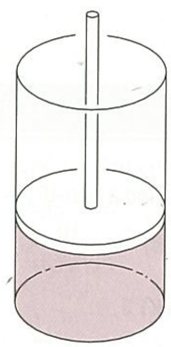
45. Devise a rule for $\frac{d}{dx}\left(\frac{1}{f(x)}\right)$.

When we work with functions of a single variable in mathematics, we often call the independent variable x and the dependent variable y. Applied fields use many different letters, however. Here are some examples.

46. *Cylinder Pressure* If gas in a cylinder is maintained at a constant temperature T, the pressure P is related to the volume V by a formula of the form

$$P = \frac{nRT}{V - nb} - \frac{an^2}{V^2},$$

in which a, b, n, and R are constants. Find dP/dV.



47. *Free Fall* When a rock falls from rest near the surface of the earth, the distance it covers during the first few seconds is given by the equation

$$s = 4.9t^2.$$

In this equation, s is the distance in meters and t is the elapsed time in seconds. Find ds/dt and d^2s/dt^2.

Group Activity In Exercises 48–52, *work in groups of two or three* to solve the problems.

48. *The Body's Reaction to Medicine* The reaction of the body to a dose of medicine can often be represented by an equation of the form

$$R = M^2\left(\frac{C}{2} - \frac{M}{3}\right),$$

where C is a positive constant and M is the amount of medicine absorbed in the blood. If the reaction is a change in blood pressure, R is measured in millimeters of mercury. If the reaction is a change in temperature, R is measured in degrees, and so on.

Find dR/dM. This derivative, as a function of M, is called the sensitivity of the body to medicine. In Chapter 4, we shall see how to find the amount of medicine to which the body is most sensitive. *Source: Some Mathematical Models in Biology,* Revised Edition, December 1967, PB-202 364, p. 221; distributed by N.T.I.S., U.S. Department of Commerce.

49. Writing to Learn Recall that the area A of a circle with radius r is πr^2 and that the circumference C is $2\pi r$. Notice that $dA/dr = C$. Explain in terms of geometry why the instantaneous rate of change of the area with respect to the radius should equal the circumference.

50. Writing to Learn Recall that the volume V of a sphere of radius r is $(4/3)\pi r^3$ and that the surface area A is $4\pi r^2$. Notice that $dV/dr = A$. Explain in terms of geometry why the instantaneous rate of change of the volume with respect to the radius should equal the surface area.

51. *Orchard Farming* An apple farmer currently has 156 trees yielding an average of 12 bushels of apples per tree. He is expanding his farm at a rate of 13 trees per year, while improved husbandry is improving his average annual yield by 1.5 bushels per tree. What is the current (instantaneous) rate of increase of his total annual production of apples? Answer in appropriate units of measure.

52. *Picnic Pavilion Rental* The members of the Blue Boar society always divide the pavilion rental fee for their picnics equally among the members. Currently there are 65 members and the pavilion rents for $250. The pavilion cost is increasing at a rate of $10 per year, while the Blue Boar membership is increasing at a rate of 6 members per year. What is the current (instantaneous) rate of change in each member's share of the pavilion rental fee? Answer in appropriate units of measure.

Standardized Test Questions

You should solve the following problems without using a graphing calculator.

53. True or False $\frac{d}{dx}(\pi^3) = 3\pi^2$. Justify your answer.

54. True or False The graph of $f(x) = 1/x$ has no horizontal tangents. Justify your answer.

55. **Multiple Choice** Let $y = uv$ be the product of the functions u and v. Find $y'(1)$ if $u(1) = 2$, $u'(1) = 3$, $v(1) = -1$, and $v'(1) = 1$.

(A) -4 (B) -1 (C) 1 (D) 4 (E) 7

56. **Multiple Choice** Let $f(x) = x - \frac{1}{x}$. Find $f''(x)$.

(A) $1 + \frac{1}{x^2}$ (B) $1 - \frac{1}{x^2}$ (C) $\frac{2}{x^3}$

(D) $-\frac{2}{x^3}$ (E) does not exist

57. **Multiple Choice** Which of the following is $\frac{d}{dx}\left(\frac{x+1}{x-1}\right)$?

(A) $\frac{2}{(x-1)^2}$ (B) 0 (C) $-\frac{x^2+1}{x^2}$

(D) $2x - \frac{1}{x^2} - 1$ (E) $-\frac{2}{(x-1)^2}$

58. **Multiple Choice** Assume $f(x) = (x^2 - 1)(x^2 + 1)$. Which of the following gives the number of horizontal tangents of f?

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Extending the Ideas

59. ***Leibniz's Proof of the Product Rule*** Here's how Leibniz explained the Product Rule in a letter to his colleague John Wallis:

It is useful to consider quantities infinitely small such that when their ratio is sought, they may not be considered zero, but which are rejected as often as they occur with quantities incomparably greater. Thus if we have $x + dx$, dx is rejected. Similarly we cannot have xdx and $dxdx$ standing together, as xdx is incomparably greater than $dxdx$. Hence if we are to differentiate uv, we write

$$\begin{aligned} d(uv) &= (u + du)(v + dv) - uv \\ &= uv + vdu + udv + dudv - uv \\ &= vdu + udv. \end{aligned}$$

Answer the following questions about Leibniz's proof.

(a) What does Leibniz mean by a quantity being "rejected"?

(b) What happened to $dudv$ in the last step of Leibniz's proof?

(c) Divide both sides of Leibniz's formula

$$d(uv) = vdu + udv$$

by the differential dx. What formula results?

(d) Why would the critics of Leibniz's time have objected to dividing both sides of the equation by dx?

(e) Leibniz had a similar simple (but not-so-clean) proof of the Quotient Rule. Can you reconstruct it?

Quick Quiz for AP* Preparation: Sections 3.1–3.3

You may use a graphing calculator to solve the following problems.

1. **Multiple Choice** Let $f(x) = |x + 1|$. Which of the following statements about f are true?

I. f is continuous at $x = -1$.

II. f is differentiable at $x = -1$.

III. f has a corner at $x = -1$.

(A) I only (B) II only (C) III only

(D) I and III only (E) I and II only

2. **Multiple Choice** If the line normal to the graph of f at the point $(1, 2)$ passes through the point $(-1, 1)$, then which of the following gives the value of $f'(1)$ = ?

(A) -2 (B) 2 (C) $-1/2$ (D) $1/2$ (E) 3

3. **Multiple Choice** Find dy/dx if $y = \frac{4x-3}{2x+1}$.

(A) $\frac{10}{(4x-3)^2}$ (B) $-\frac{10}{(4x-3)^2}$ (C) $\frac{10}{(2x+1)^2}$

(D) $-\frac{10}{(2x+1)^2}$ (E) 2

4. **Free Response** Let $f(x) = x^4 - 4x^2$.

(a) Find all the points where f has horizontal tangents.

(b) Find an equation of the tangent line at $x = 1$.

(c) Find an equation of the normal line at $x = 1$.

3.4 Velocity and Other Rates of Change

What you'll learn about

- Instantaneous Rates of Change
- Motion along a Line
- Sensitivity to Change
- Derivatives in Economics

. . . and why

Derivatives give the rates at which things change in the world.

Instantaneous Rates of Change

In this section we examine some applications in which derivatives as functions are used to represent the rates at which things change in the world around us. It is natural to think of change as change with respect to time, but other variables can be treated in the same way. For example, a physician may want to know how change in dosage affects the body's response to a drug. An economist may want to study how the cost of producing steel varies with the number of tons produced.

If we interpret the difference quotient

$$\frac{f(x+h)-f(x)}{h}$$

as the average rate of change of the function f over the interval from x to $x+h$, we can interpret its limit as h approaches 0 to be the rate at which f is changing at the point x.

DEFINITION Instantaneous Rate of Change

The **(instantaneous) rate of change** of f with respect to x at a is the derivative

$$f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h},$$

provided the limit exists.

It is conventional to use the word *instantaneous* even when x does not represent time. The word, however, is frequently omitted in practice. When we say *rate of change,* we mean *instantaneous rate of change.*

EXAMPLE 1 Enlarging Circles

(a) Find the rate of change of the area A of a circle with respect to its radius r.

(b) Evaluate the rate of change of A at $r = 5$ and at $r = 10$.

(c) If r is measured in inches and A is measured in square inches, what units would be appropriate for dA/dr?

SOLUTION

The area of a circle is related to its radius by the equation $A = \pi r^2$.

(a) The (instantaneous) rate of change of A with respect to r is

$$\frac{dA}{dr} = \frac{d}{dr}(\pi r^2) = \pi \cdot 2r = 2\pi r.$$

(b) At $r = 5$, the rate is 10π (about 31.4). At $r = 10$, the rate is 20π (about 62.8).

Notice that the rate of change gets bigger as r gets bigger. As can be seen in Figure 3.21, the same change in radius brings about a bigger change in area as the circles grow radially away from the center.

(c) The appropriate units for dA/dr are square inches (of area) per inch (of radius).

Now try Exercise 1.

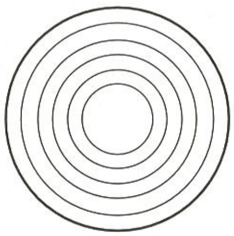


Figure 3.21 The same change in radius brings about a larger change in area as the circles grow radially away from the center. (Example 1, Exploration 1)

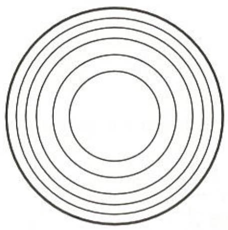


Figure 3.22 Which is the more appropriate model for the growth of rings in a tree—the circles here or those in Figure 3.21? (Exploration 1)

EXPLORATION 1 **Growth Rings on a Tree**

The phenomenon observed in Example 1, that the rate of change in area of a circle with respect to its radius gets larger as the radius gets larger, is reflected in nature in many ways. When trees grow, they add layers of wood directly under the inner bark during the growing season, then form a darker, protective layer for protection during the winter. This results in concentric rings that can be seen in a cross-sectional slice of the trunk. The age of the tree can be determined by counting the rings.

1. Look at the concentric rings in Figure 3.21 and Figure 3.22. Which is a better model for the pattern of growth rings in a tree? Is it likely that a tree could find the nutrients and light necessary to increase its amount of growth every year?
2. Considering how trees grow, explain why the change in *area* of the rings remains relatively constant from year to year.
3. If the change in area is constant, and if

$$\frac{dA}{dr} = \frac{\text{change in area}}{\text{change in radius}} = 2\pi r,$$

explain why the change in radius must get smaller as r gets bigger.

Motion along a Line

Suppose that an object is moving along a coordinate line (say an s-axis) so that we know its position s on that line as a function of time t:

$$s = f(t).$$

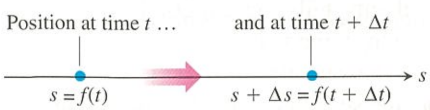


Figure 3.23 The positions of an object moving along a coordinate line at time t and shortly later at time $t + \Delta t$.

The **displacement** of the object over the time interval from t to $t + \Delta t$ is

$$\Delta s = f(t + \Delta t) - f(t)$$

(Figure 3.23) and the **average velocity** of the object over that time interval is

$$v_{\text{av}} = \frac{\text{displacement}}{\text{travel time}} = \frac{\Delta s}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

To find the object's velocity at the exact instant t, we take the limit of the average velocity over the interval from t to $t + \Delta t$ as Δt shrinks to zero. The limit is the derivative of f with respect to t.

DEFINITION **Instantaneous Velocity**

The **(instantaneous) velocity** is the derivative of the position function $s = f(t)$ with respect to time. At time t the velocity is

$$v(t) = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

EXAMPLE 2 Finding the Velocity of a Race Car

Figure 3.24 shows the time-to-distance graph of a 1996 Riley & Scott Mk III-Olds WSC race car. The slope of the secant PQ is the average velocity for the 3-second interval from $t = 2$ to $t = 5$ sec, in this case, about 100 ft/sec or 68 mph. The slope of the tangent at P is the speedometer reading at $t = 2$ sec, about 57 ft/sec or 39 mph. The acceleration for the period shown is a nearly constant 28.5 ft/sec during each second, which is about $0.89g$ where g is the acceleration due to gravity. The race car's top speed is an estimated 190 mph. *Source: Road and Track,* March 1997.

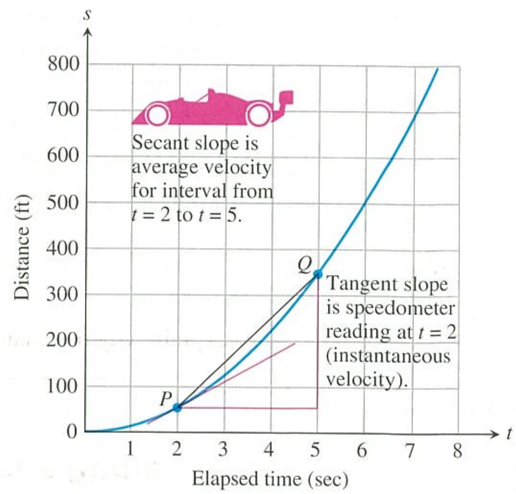


Figure 3.24 The time-to-distance graph for Example 2. *Now try Exercise 7.*

Besides telling how fast an object is moving, velocity tells the direction of motion. When the object is moving forward (when s is increasing), the velocity is positive; when the object is moving backward (when s is decreasing), the velocity is negative.

If we drive to a friend's house and back at 30 mph, the speedometer will show 30 on the way over but will not show -30 on the way back, even though our distance from home is decreasing. The speedometer always shows *speed,* which is the absolute value of velocity. Speed measures the rate of motion regardless of direction.

DEFINITION Speed

Speed is the absolute value of velocity.

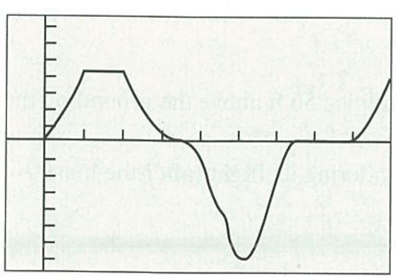
$$\text{Speed} = |v(t)| = \left|\frac{ds}{dt}\right|$$

EXAMPLE 3 Reading a Velocity Graph

A student walks around in front of a motion detector that records her velocity at 1-second intervals for 36 seconds. She stores the data in her graphing calculator and uses it to generate the time-velocity graph shown in Figure 3.25. Describe her motion as a function of time by reading the velocity graph. When is her *speed* a maximum?

SOLUTION

The student moves forward for the first 14 seconds, moves backward for the next 12 seconds, stands still for 6 seconds, and then moves forward again. She achieves her maximum speed at $t \approx 20$, while moving backward. *Now try Exercise 9.*



[–4, 36] by [–7.5, 7.5]

Figure 3.25 A student's velocity graph from data recorded by a motion detector. (Example 3)

The rate at which a body's velocity changes is called the body's *acceleration.* The acceleration measures how quickly the body picks up or loses speed.

DEFINITION Acceleration

Acceleration is the derivative of velocity with respect to time. If a body's velocity at time t is $v(t) = ds/dt$, then the body's acceleration at time t is

$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

The earliest questions that motivated the discovery of calculus were concerned with velocity and acceleration, particularly the motion of freely falling bodies under the force of gravity. (See Examples 1 and 2 in Section 2.1.) The mathematical description of this type of motion captured the imagination of many great scientists, including Aristotle, Galileo, and Newton. Experimental and theoretical investigations revealed that the distance a body released from rest falls freely is proportional to the square of the amount of time it has fallen. We express this by saying that

$$s = \frac{1}{2}gt^2,$$

where s is distance, g is the acceleration due to Earth's gravity, and t is time. The value of g in the equation depends on the units used to measure s and t. With t in seconds (the usual unit), we have the following values:

Free-fall Constants (Earth)

English units: $g = 32 \dfrac{\text{ft}}{\text{sec}^2}, \quad s = \dfrac{1}{2}(32)t^2 = 16t^2$ (s in feet)

Metric units: $g = 9.8 \dfrac{\text{m}}{\text{sec}^2}, \quad s = \dfrac{1}{2}(9.8)t^2 = 4.9t^2$ (s in meters)

The abbreviation ft/sec^2 is read "feet per second squared" or "feet per second per second," and m/sec^2 is read "meters per second squared."

EXAMPLE 4 Modeling Vertical Motion

A dynamite blast propels a heavy rock straight up with a launch velocity of 160 ft/sec (about 109 mph) (Figure 3.26a). It reaches a height of $s = 160t - 16t^2$ ft after t seconds.

(a) How high does the rock go?

(b) What is the velocity and speed of the rock when it is 256 ft above the ground on the way up? on the way down?

(c) What is the acceleration of the rock at any time t during its flight (after the blast)?

(d) When does the rock hit the ground?

SOLUTION

In the coordinate system we have chosen, s measures height from the ground up, so velocity is positive on the way up and negative on the way down.

continued

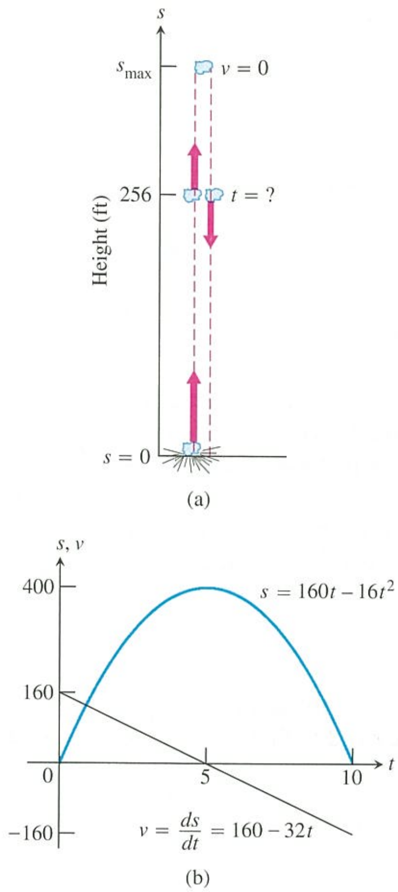


Figure 3.26 (a) The rock in Example 4. (b) The graphs of s and v as functions of time t, showing that s is largest when $v = ds/dt = 0$. (The graph of s is *not* the path of the rock; it is a plot of height as a function of time.) (Example 4)

(a) The instant when the rock is at its highest point is the one instant during the flight when the velocity is 0. At any time t, the velocity is

$$v = \frac{ds}{dt} = \frac{d}{dt}(160t - 16t^2) = 160 - 32t \text{ ft/sec.}$$

The velocity is zero when $160 - 32t = 0$, or at $t = 5$ sec.

The maximum height is the height of the rock at $t = 5$ sec. That is,

$$s_{max} = s(5) = 160(5) - 16(5)^2 = 400 \text{ ft.}$$

See Figure 3.26b.

(b) To find the velocity when the height is 256 ft, we determine the two values of t for which $s(t) = 256$ ft.

$$s(t) = 160t - 16t^2 = 256$$
$$16t^2 - 160t + 256 = 0$$
$$16(t^2 - 10t + 16) = 0$$
$$(t - 2)(t - 8) = 0$$
$$t = 2 \text{ sec} \quad \text{or} \quad t = 8 \text{ sec}$$

The velocity of the rock at each of these times is

$$v(2) = 160 - 32(2) = 96 \text{ ft/sec,}$$
$$v(8) = 160 - 32(8) = -96 \text{ ft/sec.}$$

At both instants, the speed of the rock is 96 ft/sec.

(c) At any time during its flight after the explosion, the rock's acceleration is

$$a = \frac{dv}{dt} = \frac{d}{dt}(160 - 32t) = -32 \text{ ft/sec}^2.$$

The acceleration is always downward. When the rock is rising, it is slowing down; when it is falling, it is speeding up.

(d) The rock hits the ground at the positive time for which $s = 0$. The equation $160t - 16t^2 = 0$ has two solutions: $t = 0$ and $t = 10$. The blast initiated the flight of the rock from ground level at $t = 0$. The rock returned to the ground 10 seconds later.

Now try Exercise 13.

EXAMPLE 5 Studying Particle Motion

A particle moves along a line so that its position at any time $t \geq 0$ is given by the function $s(t) = t^2 - 4t + 3$, where s is measured in meters and t is measured in seconds.

(a) Find the displacement of the particle during the first 2 seconds.

(b) Find the average velocity of the particle during the first 4 seconds.

(c) Find the instantaneous velocity of the particle when $t = 4$.

(d) Find the acceleration of the particle when $t = 4$.

(e) Describe the motion of the particle. At what values of t does the particle change directions?

(f) Use parametric graphing to view the motion of the particle on the horizontal line $y = 2$.

continued

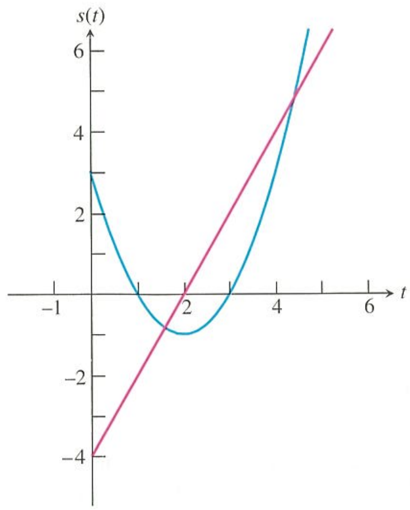
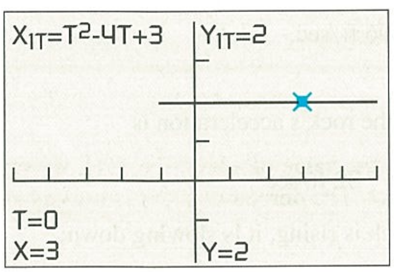


Figure 3.27 The graphs of $s(t) = t^2 - 4t + 3, t \geq 0$ (blue) and its derivative $v(t) = 2t - 4, t \geq 0$ (red). (Example 5)



$[-5, 5]$ by $[-2, 4]$

Figure 3.28 The graph of $X_{1T} = T^2 - 4T + 3, Y_{1T} = 2$ in parametric mode. (Example 5)

SOLUTION

(a) The displacement is given by $s(2) - s(0) = (-1) - 3 = -4$. This value means that the particle is 4 units left of where it started.

(b) The average velocity we seek is

$$\frac{s(4) - s(0)}{4 - 0} = \frac{3 - 3}{4} = 0 \text{ m/sec.}$$

(c) The velocity $v(t)$ at any time t is $v(t) = ds/dt = 2t - 4$. So $v(4) = 4$ m/sec

(d) The acceleration $a(t)$ at any time t is $a(t) = dv/dt = 2$ m /sec^2. So $a(4) = 2$.

(e) The graphs of $s(t) = t^2 - 4t + 3$ for $t \geq 0$ and its derivative $v(t) = 2t - 4$ shown in Figure 3.27 will help us analyze the motion.

For $0 \leq t < 2$, $v(t) < 0$, so the particle is moving to the left. Notice that $s(t)$ is decreasing. The particle starts ($t = 0$) at $s = 3$ and moves left, arriving at the origin $t = 1$ when $s = 0$. The particle continues moving to the left until it reaches the point $s = -1$ at $t = 2$.

At $t = 2$, $v = 0$, so the particle is at rest.

For $t > 2$, $v(t) > 0$, so the particle is moving to the right. Notice that $s(t)$ is increasing. In this interval, the particle starts at $s = -1$, moving to the right through the origin and continuing to the right for the rest of time.

The particle changes direction at $t = 2$ when $v = 0$.

(f) Enter $X_{1T} = T^2 - 4T + 3$, $Y_{1T} = 2$ in parametric mode and graph in the window $[-5, 5]$ by $[-2, 4]$ with Tmin = 0, Tmax = 10 (it really should be ∞), and Xscl=Yscl=1. (Figure 3.28) By using TRACE you can follow the path of the particle. You will learn more ways to visualize motion in Explorations 2 and 3.

Now try Exercise 19.

EXPLORATION 2 Modeling Horizontal Motion

The position (x-coordinate) of a particle moving on the horizontal line $y = 2$ is given by $x(t) = 4t^3 - 16t^2 + 15t$ for $t \geq 0$.

1. Graph the parametric equations $x_1(t) = 4t^3 - 16t^2 + 15t, y_1(t) = 2$ in $[-4, 6]$ by $[-3, 5]$. Use TRACE to support that the particle starts at the point (0, 2), moves to the right, then to the left, and finally to the right. At what times does the particle reverse direction?
2. Graph the parametric equations $x_2(t) = x_1(t)$, $y_2(t) = t$ in the same viewing window. Explain how this graph shows the back and forth motion of the particle. Use this graph to find when the particle reverses direction.
3. Graph the parametric equations $x_3(t) = t$, $y_3(t) = x_1(t)$ in the same viewing window. Explain how this graph shows the back and forth motion of the particle. Use this graph to find when the particle reverses direction.
4. Use the methods in parts 1, 2, and 3 to represent and describe the *velocity* of the particle.

EXPLORATION 3 **Seeing Motion on a Graphing Calculator**

The graphs in Figure 3.26b give us plenty of information about the flight of the rock in Example 4, but neither graph shows the path of the rock in flight. We can simulate the moving rock by graphing the parametric equations

$$x_1(t) = 3(t < 5) + 3.1(t \geq 5), \quad y_1(t) = 160t - 16t^2$$

in dot mode.

This will show the upward flight of the rock along the vertical line $x = 3$, and the downward flight of the rock along the line $x = 3.1$.

1. To see the flight of the rock from beginning to end, what should we use for tMin and tMax in our graphing window?
2. Set xMin $= 0$, xMax $= 6$, and yMin $= -10$. Use the results from Example 4 to determine an appropriate value for yMax. (You will want the entire flight of the rock to fit within the vertical range of the screen.)
3. Set tStep initially at 0.1. (A higher number will make the simulation move faster. A lower number will slow it down.)
4. Can you explain why the grapher actually slows down when the rock would slow down, and speeds up when the rock would speed up?

Sensitivity to Change

When a small change in x produces a large change in the value of a function $f(x)$, we say that the function is relatively **sensitive** to changes in x. The derivative $f'(x)$ is a measure of this sensitivity.

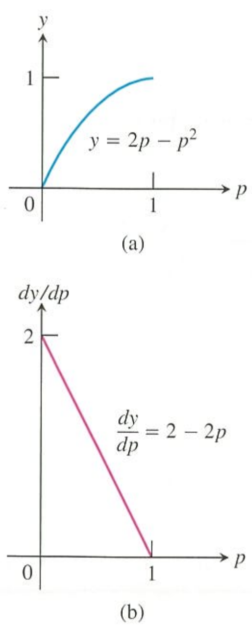


Figure 3.29 (a) The graph of $y = 2p - p^2$ describing the proportion of smooth-skinned peas. (b) The graph of dy/dp. (Example 6)

EXAMPLE 6 Sensitivity to Change

The Austrian monk Gregor Johann Mendel (1822–1884), working with garden peas and other plants, provided the first scientific explanation of hybridization. His careful records showed that if p (a number between 0 and 1) is the relative frequency of the gene for smooth skin in peas (dominant) and $(1 - p)$ is the relative frequency of the gene for wrinkled skin in peas (recessive), then the proportion of smooth-skinned peas in the next generation will be

$$y = 2p(1 - p) + p^2 = 2p - p^2.$$

Compare the graphs of y and dy/dp to determine what values of y are more sensitive to a change in p. The graph of y versus p in Figure 3.29a suggests that the value of y is more sensitive to a change in p when p is small than it is to a change in p when p is large. Indeed, this is borne out by the derivative graph in Figure 3.29b, which shows that dy/dp is close to 2 when p is near 0 and close to 0 when p is near 1.

Now try Exercise 25.

Derivatives in Economics

Engineers use the terms *velocity* and *acceleration* to refer to the derivatives of functions describing motion. Economists, too, have a specialized vocabulary for rates of change and derivatives. They call them *marginals*.

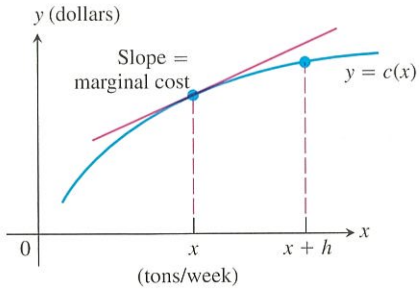


Figure 3.30 Weekly steel production: $c(x)$ is the cost of producing x tons per week. The cost of producing an additional h tons per week is $c(x + h) - c(x)$.

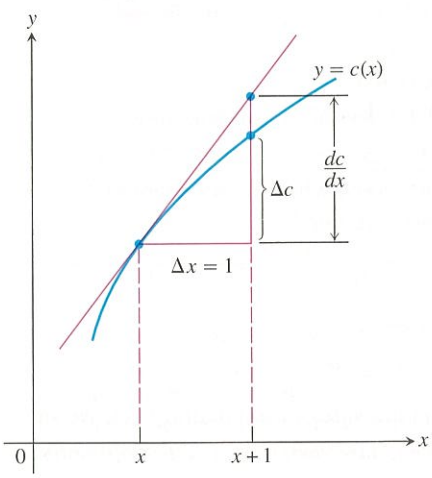


Figure 3.31 Because dc/dx is the slope of the tangent at x, the marginal cost dc/dx approximates the extra cost Δc of producing $\Delta x = 1$ more unit.

In a manufacturing operation, *the cost of production* $c(x)$ is a function of x, the number of units produced. The *marginal cost of production* is the rate of change of cost with respect to the level of production, so it is dc/dx.

Suppose $c(x)$ represents the dollars needed to produce x tons of steel in one week. It costs more to produce $x + h$ tons per week, and the cost difference divided by h is the average cost of producing each additional ton.

$$\frac{c(x+h) - c(x)}{h} = \begin{cases} \text{the average cost of each of the} \\ \text{additional } h \text{ tons produced} \end{cases}$$

The limit of this ratio as $h \to 0$ is the **marginal cost** of producing more steel per week when the current production is x tons (Figure 3.30).

$$\frac{dc}{dx} = \lim_{h \to 0} \frac{c(x+h) - c(x)}{h} = \text{marginal cost of production}$$

Sometimes the marginal cost of production is loosely defined to be the extra cost of producing one more unit,

$$\frac{\Delta c}{\Delta x} = \frac{c(x+1) - c(x)}{1},$$

which is approximated by the value of dc/dx at x. This approximation is acceptable if the slope of c does not change quickly near x, for then the difference quotient is close to its limit dc/dx even if $\Delta x = 1$ (Figure 3.31). The approximation works best for large values of x.

EXAMPLE 7 Marginal Cost and Marginal Revenue

Suppose it costs

$$c(x) = x^3 - 6x^2 + 15x$$

dollars to produce x radiators when 8 to 10 radiators are produced, and that

$$r(x) = x^3 - 3x^2 + 12x$$

gives the dollar revenue from selling x radiators. Your shop currently produces 10 radiators a day. Find the marginal cost and **marginal revenue.**

SOLUTION

The marginal cost of producing one more radiator a day when 10 are being produced is $c'(10)$.

$$c'(x) = \frac{d}{dx}(x^3 - 6x^2 + 15x) = 3x^2 - 12x + 15$$

$$c'(10) = 3(100) - 12(10) + 15 = 195 \text{ dollars}$$

The marginal revenue is

$$r'(x) = \frac{d}{dx}(x^3 - 3x^2 + 12x) = 3x^2 - 6x + 12,$$

so,

$$r'(10) = 3(100) - 6(10) + 12 = 252 \text{ dollars.}$$

Now try Exercises 27 and 28.

Quick Review 3.4 *(For help, go to Sections 1.2, 3.1, and 3.3.)*

In Exercises 1–10, answer the questions about the graph of the quadratic function $y = f(x) = -16x^2 + 160x - 256$ by analyzing the equation algebraically. Then support your answers graphically.

1. Does the graph open upward or downward?

2. What is the y-intercept?

3. What are the x-intercepts?

4. What is the range of the function?

5. What point is the vertex of the parabola?

6. At what x-values does $f(x) = 80$?

7. For what x-value does $dy/dx = 100$?

8. On what interval is $dy/dx > 0$?

9. Find $\lim_{h \to 0} \dfrac{f(3+h) - f(3)}{h}$.

10. Find d^2y/dx^2 at $x = 7$.

Section 3.4 Exercises

1. **(a)** Write the volume V of a cube as a function of the side length s.

(b) Find the (instantaneous) rate of change of the volume V with respect to a side s.

(c) Evaluate the rate of change of V at $s = 1$ and $s = 5$.

(d) If s is measured in inches and V is measured in cubic inches, what units would be appropriate for dV/ds?

2. **(a)** Write the area A of a circle as a function of the circumference C.

(b) Find the (instantaneous) rate of change of the area A with respect to the circumference C.

(c) Evaluate the rate of change of A at $C = \pi$ and $C = 6\pi$.

(d) If C is measured in inches and A is measured in square inches, what units would be appropriate for dA/dC?

3. **(a)** Write the area A of an equilateral triangle as a function of the side length s.

(b) Find the (instantaneous) rate of change of the area A with respect to a side s.

(c) Evaluate the rate of change of A at $s = 2$ and $s = 10$.

(d) If s is measured in inches and A is measured in square inches, what units would be appropriate for dA/ds?

4. A square of side length s is inscribed in a circle of radius r.

(a) Write the area A of the square as a function of the radius r of the circle.

(b) Find the (instantaneous) rate of change of the area A with respect to the radius r of the circle.

(c) Evaluate the rate of change of A at $r = 1$ and $r = 8$.

(d) If r is measured in inches and A is measured in square inches, what units would be appropriate for dA/dr?

Group Activity In Exercises 5 and 6, the coordinates s of a moving body for various values of t are given. **(a)** Plot s versus t on coordinate paper, and sketch a smooth curve through the given points. **(b)** Assuming that this smooth curve represents the motion of the body, estimate the velocity at $t = 1.0$, $t = 2.5$, and $t = 3.5$.

5.

t (sec)	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
s (ft)	12.5	26	36.5	44	48.5	50	48.5	44	36.5

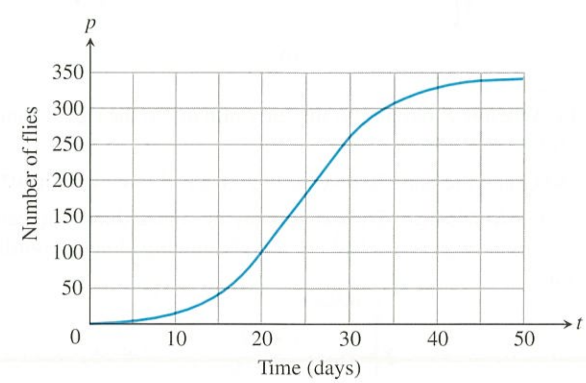
6.

t (sec)	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
s (ft)	3.5	−4	−8.5	−10	−8.5	−4	3.5	14	27.5

7. **Group Activity** ***Fruit Flies*** *(Example 2, Section 2.4 continued)* Populations starting out in closed environments grow slowly at first, when there are relatively few members, then more rapidly as the number of reproducing individuals increases and resources are still abundant, then slowly again as the population reaches the carrying capacity of the environment.

(a) Use the graphical technique of Section 3.1, Example 3, to graph the derivative of the fruit fly population introduced in Section 2.4. The graph of the population is reproduced below. What units should be used on the horizontal and vertical axes for the derivative's graph?

(b) During what days does the population seem to be increasing fastest? slowest?



8. *Draining a Tank* The number of gallons of water in a tank t minutes after the tank has started to drain is $Q(t) = 200(30 - t)^2$. How fast is the water running out at the end of 10 min? What is the average rate at which the water flows out during the first 10 min?

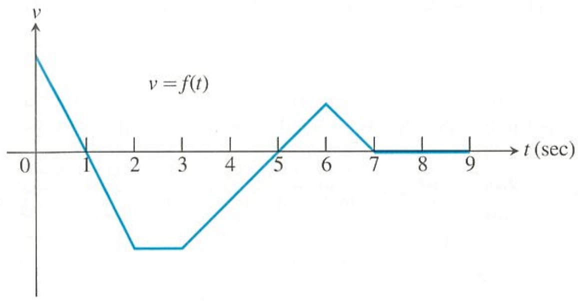
9. *Particle Motion* The accompanying figure shows the velocity $v = f(t)$ of a particle moving on a coordinate line.

(a) When does the particle move forward? move backward? speed up? slow down?

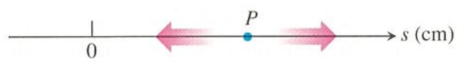
(b) When is the particle's acceleration positive? negative? zero?

(c) When does the particle move at its greatest speed?

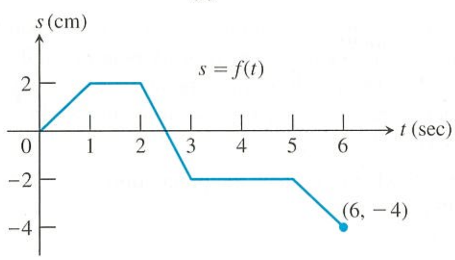
(d) When does the particle stand still for more than an instant?



10. *Particle Motion* A particle P moves on the number line shown in part (a) of the accompanying figure. Part (b) shows the position of P as a function of time t.



(a)

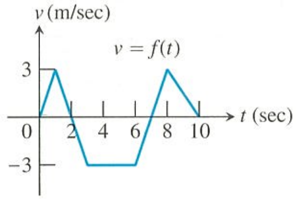


(b)

(a) When is P moving to the left? moving to the right? standing still?

(b) Graph the particle's velocity and speed (where defined).

11. *Particle Motion* The accompanying figure shows the velocity $v = ds/dt = f(t)$ (m/sec) of a body moving along a coordinate line.



(a) When does the body reverse direction?

(b) When (approximately) is the body moving at a constant speed?

(c) Graph the body's speed for $0 \le t \le 10$.

(d) Graph the acceleration, where defined.

12. *Thoroughbred Racing* A racehorse is running a 10-furlong race. (A furlong is 220 yards, although we will use furlongs and seconds as our units in this exercise.) As the horse passes each furlong marker (F), a steward records the time elapsed (t) since the beginning of the race, as shown in the table below:

F	0	1	2	3	4	5	6	7	8	9	10
t	0	20	33	46	59	73	86	100	112	124	135

(a) How long does it take the horse to finish the race?

(b) What is the average speed of the horse over the first 5 furlongs?

(c) What is the approximate speed of the horse as it passes the 3-furlong marker?

(d) During which portion of the race is the horse running the fastest?

(e) During which portion of the race is the horse accelerating the fastest?

13. *Lunar Projectile Motion* A rock thrown vertically upward from the surface of the moon at a velocity of 24 m/sec (about 86 km/h) reaches a height of $s = 24t - 0.8t^2$ meters in t seconds.

(a) Find the rock's velocity and acceleration as functions of time. (The acceleration in this case is the acceleration of gravity on the moon.)

(b) How long did it take the rock to reach its highest point?

(c) How high did the rock go?

(d) When did the rock reach half its maximum height?

(e) How long was the rock aloft?

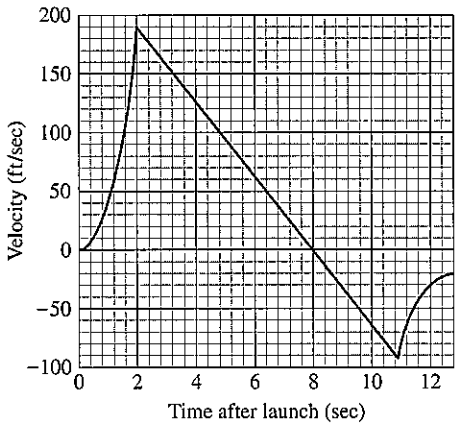
14. *Free Fall* The equations for free fall near the surfaces of Mars and Jupiter (s in meters, t in seconds) are: Mars, $s = 1.86t^2$; Jupiter, $s = 11.44t^2$. How long would it take a rock falling from rest to reach a velocity of 16.6 m/sec (about 60 km/h) on each planet?

15. *Projectile Motion* On Earth, in the absence of air, the rock in Exercise 13 would reach a height of $s = 24t - 4.9t^2$ meters in t seconds. How high would the rock go?

16. *Speeding Bullet* A bullet fired straight up from the moon's surface would reach a height of $s = 832t - 2.6t^2$ ft after t sec. On Earth, in the absence of air, its height would be $s = 832t - 16t^2$ ft after t sec. How long would it take the bullet to get back down in each case?

17. *Parametric Graphing* Devise a grapher simulation of the problem situation in Exercise 16. Use it to support the answers obtained analytically.

18. *Launching a Rocket* When a model rocket is launched, the propellant burns for a few seconds, accelerating the rocket upward. After burnout, the rocket coasts upward for a while and then begins to fall. A small explosive charge pops out a parachute shortly after the rocket starts downward. The parachute slows the rocket to keep it from breaking when it lands. This graph shows velocity data from the flight.



Use the graph to answer the following.

(a) How fast was the rocket climbing when the engine stopped?

(b) For how many seconds did the engine burn?

(c) When did the rocket reach its highest point? What was its velocity then?

(d) When did the parachute pop out? How fast was the rocket falling then?

(e) How long did the rocket fall before the parachute opened?

(f) When was the rocket's acceleration greatest? When was the acceleration constant?

19. *Particle Motion* A particle moves along a line so that its position at any time $t \geq 0$ is given by the function

$$s(t) = t^2 - 3t + 2,$$

where s is measured in meters and t is measured in seconds.

(a) Find the displacement during the first 5 seconds.

(b) Find the average velocity during the first 5 seconds.

(c) Find the instantaneous velocity when $t = 4$.

(d) Find the acceleration of the particle when $t = 4$.

(e) At what values of t does the particle change direction?

(f) Where is the particle when s is a minimum?

20. *Particle Motion* A particle moves along a line so that its position at any time $t \geq 0$ is given by the function $s(t) = -t^3 + 7t^2 - 14t + 8$ where s is measured in meters and t is measured in seconds.

(a) Find the instantaneous velocity at any time t.

(b) Find the acceleration of the particle at any time t.

(c) When is the particle at rest?

(d) Describe the motion of the particle. At what values of t does the particle change directions?

21. *Particle Motion* A particle moves along a line so that its position at any time $t \geq 0$ is given by the function $s(t) = (t-2)^2(t-4)$ where s is measured in meters and t is measured in seconds.

(a) Find the instantaneous velocity at any time t.

(b) Find the acceleration of the particle at any time t.

(c) When is the particle at rest?

(d) Describe the motion of the particle. At what values of t does the particle change directions?

22. *Particle Motion* A particle moves along a line so that its position at any time $t \geq 0$ is given by the function $s(t) = t^3 - 6t^2 + 8t + 2$ where s is measured in meters and t is measured in seconds.

(a) Find the instantaneous velocity at any time t.

(b) Find the acceleration of the particle at any time t.

(c) When is the particle at rest?

(d) Describe the motion of the particle. At what values of t does the particle change directions?

23. *Particle Motion* The position of a body at time t sec is $s = t^3 - 6t^2 + 9t$ m. Find the body's acceleration each time the velocity is zero.

24. *Finding Speed* A body's velocity at time t sec is $v = 2t^3 - 9t^2 + 12t - 5$ m/sec. Find the body's speed each time the acceleration is zero.

25. *Draining a Tank* It takes 12 hours to drain a storage tank by opening the valve at the bottom. The depth y of fluid in the tank t hours after the valve is opened is given by the formula

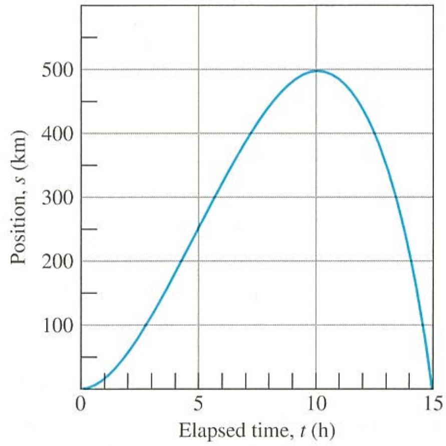
$$y = 6\left(1 - \frac{t}{12}\right)^2 \text{ m}.$$

(a) Find the rate dy/dt (m/h) at which the water level is changing at time t.

(b) When is the fluid level in the tank falling fastest? slowest? What are the values of dy/dt at these times?

(c) Graph y and dy/dt together and discuss the behavior of y in relation to the signs and values of dy/dt.

26. *Moving Truck* The graph here shows the position s of a truck traveling on a highway. The truck starts at $t = 0$ and returns 15 hours later at $t = 15$.



(a) Use the technique described in Section 3.1, Example 3, to graph the truck's velocity $v = ds/dt$ for $0 \le t \le 15$. Then repeat the process, with the velocity curve, to graph the truck's acceleration dv/dt.

(b) Suppose $s = 15t^2 - t^3$. Graph ds/dt and d^2s/dt^2, and compare your graphs with those in part (a).

27. *Marginal Cost* Suppose that the dollar cost of producing x washing machines is $c(x) = 2000 + 100x - 0.1x^2$.

(a) Find the average cost of producing 100 washing machines.

(b) Find the marginal cost when 100 machines are produced.

(c) Show that the marginal cost when 100 washing machines are produced is approximately the cost of producing one more washing machine after the first 100 have been made, by calculating the latter cost directly.

28. *Marginal Revenue* Suppose the weekly revenue in dollars from selling x custom-made office desks is

$$r(x) = 2000\left(1 - \frac{1}{x+1}\right).$$

(a) Draw the graph of r. What values of x make sense in this problem situation?

(b) Find the marginal revenue when x desks are sold.

(c) Use the function $r'(x)$ to estimate the increase in revenue that will result from increasing sales from 5 desks a week to 6 desks a week.

(d) Writing to Learn Find the limit of $r'(x)$ as $x \to \infty$. How would you interpret this number?

29. *Finding Profit* The monthly profit (in thousands of dollars) of a software company is given by

$$P(x) = \frac{10}{1 + 50 \cdot 2^{5-0.1x}},$$

where x is the number of software packages sold.

(a) Graph $P(x)$.

(b) What values of x make sense in the problem situation?

(c) Use NDER to graph $P'(x)$. For what values of x is P relatively sensitive to changes in x?

(d) What is the profit when the marginal profit is greatest?

(e) What is the marginal profit when 50 units are sold? 100 units, 125 units, 150 units, 175 units, and 300 units?

(f) What is $\lim_{x\to\infty} P(x)$? What is the maximum profit possible?

(g) Writing to Learn Is there a practical explanation to the maximum profit answer? Explain your reasoning.

30. In Step 1 of Exploration 2, at what time is the particle at the point (5, 2)?

31. Group Activity The graphs in Figure 3.32 show as functions of time t the position s, velocity $v = ds/dt$, and acceleration $a = d^2s/dt^2$ of a body moving along a coordinate line. Which graph is which? Give reasons for your answers.

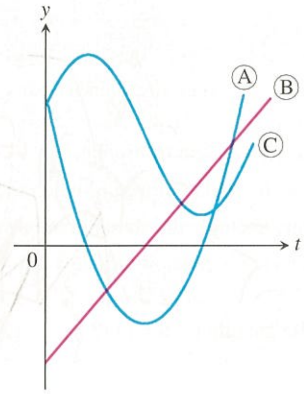


Figure 3.32 The graphs for Exercise 31.

32. Group Activity The graphs in Figure 3.33 show as functions of time t the position s, the velocity $v = ds/dt$, and the acceleration $a = d^2s/dt^2$ of a body moving along a coordinate line. Which graph is which? Give reasons for your answers.

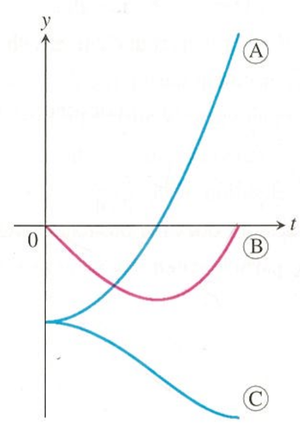


Figure 3.33 The graphs for Exercise 32.

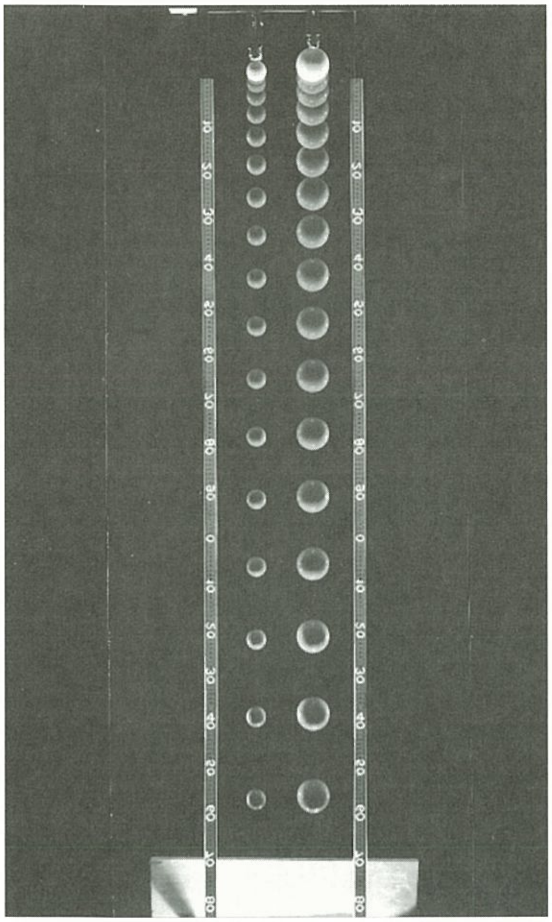


Figure 3.34 Two balls falling from rest. (Exercise 38)

33. *Pisa by Parachute* *(continuation of Exercise 18)* A few years ago, Mike McCarthy parachuted 179 ft from the top of the Tower of Pisa. Make a rough sketch to show the shape of the graph of his downward velocity during the jump.

34. *Inflating a Balloon* The volume $V = (4/3)\pi r^3$ of a spherical balloon changes with the radius.

(a) At what rate does the volume change with respect to the radius when $r = 2$ ft?

(b) By approximately how much does the volume increase when the radius changes from 2 to 2.2 ft?

35. *Volcanic Lava Fountains* Although the November 1959 Kilauea Iki eruption on the island of Hawaii began with a line of fountains along the wall of the crater, activity was later confined to a single vent in the crater's floor, which at one point shot lava 1900 ft straight into the air (a world record). What was the lava's exit velocity in feet per second? in miles per hour? [*Hint:* If v_0 is the exit velocity of a particle of lava, its height t seconds later will be $s = v_0 t - 16t^2$ feet. Begin by finding the time at which $ds/dt = 0$. Neglect air resistance.]

36. Writing to Learn Suppose you are looking at a graph of velocity as a function of time. How can you estimate the acceleration at a given point in time?

37. *Particle Motion* The position (x-coordinate) of a particle moving on the line $y = 2$ is given by $x(t) = 2t^3 - 13t^2 + 22t - 5$ where t is time in seconds.

(a) Describe the motion of the particle for $t \geq 0$.

(b) When does the particle speed up? slow down?

(c) When does the particle change direction?

(d) When is the particle at rest?

(e) Describe the velocity and speed of the particle.

(f) When is the particle at the point (5, 2)?

38. *Falling Objects* The multiflash photograph in Figure 3.34 shows two balls falling from rest. The vertical rulers are marked in centimeters. Use the equation $s = 490t^2$ (the free-fall equation for s in centimeters and t in seconds) to answer the following questions.

(a) How long did it take the balls to fall the first 160 cm? What was their average velocity for the period?

(b) How fast were the balls falling when they reached the 160-cm mark? What was their acceleration then?

(c) About how fast was the light flashing (flashes per second)?

39. Writing to Learn Explain how the Sum and Difference Rule (Rule 4 in Section 3.3) can be used to derive a formula for *marginal profit* in terms of marginal revenue and marginal cost.

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

40. True or False The speed of a particle at $t = a$ is given by the value of the velocity at $t = a$. Justify your answer.

41. True or False The acceleration of a particle is the second derivative of the position function. Justify your answer.

42. Multiple Choice Find the instantaneous rate of change of $f(x) = x^2 - 2/x + 4$ at $x = -1$.

(A) -7 **(B)** -4 **(C)** 0 **(D)** 4 **(E)** 7

43. Multiple Choice Find the instantaneous rate of change of the volume of a cube with respect to a side length x.

(A) x **(B)** $3x$ **(C)** $6x$ **(D)** $3x^2$ **(E)** x^3

In Exercises 44 and 45, a particle moves along a line so that its position at any time $t \geq 0$ is given by $s(t) = 2 + 7t - t^2$.

44. Multiple Choice At which of the following times is the particle moving to the left?

(A) $t = 0$ **(B)** $t = 1$ **(C)** $t = 2$ **(D)** $t = 7/2$ **(E)** $t = 4$

45. Multiple Choice When is the particle at rest?

(A) $t = 1$ **(B)** $t = 2$ **(C)** $t = 7/2$ **(D)** $t = 4$ **(E)** $t = 5$

Explorations

46. *Bacterium Population* When a bactericide was added to a nutrient broth in which bacteria were growing, the bacterium population continued to grow for a while but then stopped growing and began to decline. The size of the population at time t (hours) was $b(t) = 10^6 + 10^4 t - 10^3 t^2$. Find the growth rates at $t = 0$, $t = 5$, and $t = 10$ hours.

47. ***Finding f from f′*** Let $f'(x) = 3x^2$.

(a) Compute the derivatives of $g(x) = x^3$, $h(x) = x^3 - 2$, and $t(x) = x^3 + 3$.

(b) Graph the numerical derivatives of g, h, and t.

(c) Describe a *family* of functions, $f(x)$, that have the property that $f'(x) = 3x^2$.

(d) Is there a function f such that $f'(x) = 3x^2$ and $f(0) = 0$? If so, what is it?

(e) Is there a function f such that $f'(x) = 3x^2$ and $f(0) = 3$? If so, what is it?

48. ***Airplane Takeoff*** Suppose that the distance an aircraft travels along a runway before takeoff is given by $D = (10/9)t^2$, where D is measured in meters from the starting point and t is measured in seconds from the time the brakes are released. If the aircraft will become airborne when its speed reaches 200 km/h, how long will it take to become airborne, and what distance will it have traveled by that time?

Extending the Ideas

49. ***Even and Odd Functions***

(a) Show that if f is a differentiable even function, then f' is an odd function.

(b) Show that if f is a differentiable odd function, then f' is an even function.

50. ***Extended Product Rule*** Derive a formula for the derivative of the product fgh of three differentiable functions.

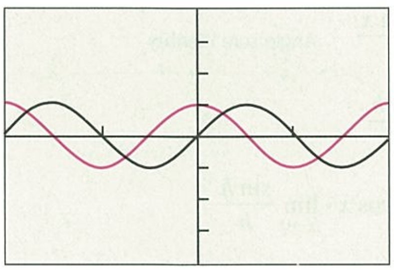
3.5 Derivatives of Trigonometric Functions

What you'll learn about

- Derivative of the Sine Function
- Derivative of the Cosine Function
- Simple Harmonic Motion
- Jerk
- Derivatives of the Other Basic Trigonometric Functions

. . . and why

The derivatives of sines and cosines play a key role in describing periodic change.



$[-2\pi, 2\pi]$ by $[-4, 4]$

Figure 3.35 Sine and its derivative. What is the derivative? (Exploration 1)

Derivative of the Sine Function

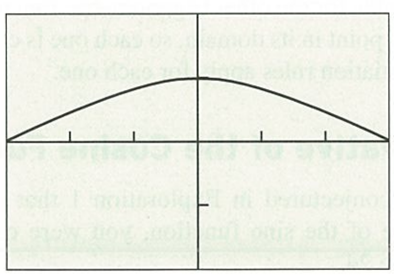
Trigonometric functions are important because so many of the phenomena we want information about are periodic (heart rhythms, earthquakes, tides, weather). It is known that continuous periodic functions can always be expressed in terms of sines and cosines, so the derivatives of sines and cosines play a key role in describing periodic change. This section introduces the derivatives of the six basic trigonometric functions.

EXPLORATION 1 **Making a Conjecture with NDER**

In the window $[-2\pi, 2\pi]$ by $[-4, 4]$, graph $y_1 = \sin x$ and $y_2 = \text{NDER}(\sin x)$ (Figure 3.35).

1. When the graph of $y_1 = \sin x$ is increasing, what is true about the graph of $y_2 = \text{NDER}(\sin x)$?
2. When the graph of $y_1 = \sin x$ is decreasing, what is true about the graph of $y_2 = \text{NDER}(\sin x)$?
3. When the graph of $y_1 = \sin x$ stops increasing and starts decreasing, what is true about the graph of $y_2 = \text{NDER}(\sin x)$?
4. At the places where $\text{NDER}(\sin x) = \pm 1$, what appears to be the slope of the graph of $y_1 = \sin x$?
5. Make a conjecture about what function the derivative of sine might be. Test your conjecture by graphing your function and $\text{NDER}(\sin x)$ in the same viewing window.
6. Now let $y_1 = \cos x$ and $y_2 = \text{NDER}(\cos x)$. Answer questions (1) through (5) *without* looking at the graph of $\text{NDER}(\cos x)$ until you are ready to test your conjecture about what function the derivative of cosine might be.

If you conjectured that the derivative of the sine function is the cosine function, then you are right. We will confirm this analytically, but first we appeal to technology one more time to evaluate two limits needed in the proof (see Figure 3.36 below and Figure 3.37 on the next page):



$[-3, 3]$ by $[-2, 2]$

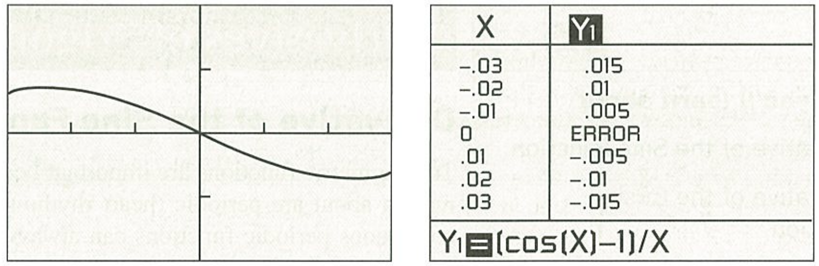
(a)

X	Y1	
−.03	.99985	
−.02	.99993	
−.01	.99998	
0	ERROR	
.01	.99998	
.02	.99993	
.03	.99985	

Y1 = sin(X)/X

(b)

Figure 3.36 (a) Graphical and (b) tabular support that $\lim_{h \to 0} \frac{\sin(h)}{h} = 1$.



[–3, 3] by [–2, 2]

(a)

(b)

Figure 3.37 (a) Graphical and (b) tabular support that $\lim_{h\to 0} \frac{\cos(h)-1}{h} = 0$.

Confirm Analytically

(Also, see Section 2.1, Exercise 75.) Now, let $y = \sin x$. Then

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \qquad \text{Angle sum identity} \\
&= \lim_{h\to 0} \frac{(\sin x)(\cos h - 1) + \cos x \sin h}{h} \\
&= \lim_{h\to 0} \sin x \cdot \lim_{h\to 0} \frac{(\cos h - 1)}{h} + \lim_{h\to 0} \cos x \cdot \lim_{h\to 0} \frac{\sin h}{h} \\
&= \sin x \cdot 0 + \cos x \cdot 1 \\
&= \cos x.
\end{aligned}$$

In short, the derivative of the sine is the cosine.

$$\frac{d}{dx}\sin x = \cos x$$

Now that we know that the sine function is differentiable, we know that sine and its derivative obey all the rules for differentiation. We also know that $\sin x$ is continuous. The same holds for the other trigonometric functions in this section. Each one is differentiable at every point in its domain, so each one is continuous at every point in its domain, and the differentiation rules apply for each one.

Derivative of the Cosine Function

If you conjectured in Exploration 1 that the derivative of the cosine function is the negative of the sine function, you were correct. You can confirm this analytically in Exercise 24.

$$\frac{d}{dx}\cos x = -\sin x$$

Radian Measure in Calculus

In case you have been wondering why calculus uses radian measure when the rest of the world seems to measure angles in degrees, you are now ready to understand the answer. The derivative of sin *x* is cos *x* *only* if *x* is measured in radians! If you look at the analytic confirmation, you will note that the derivative comes down to

$$\cos x \text{ times } \lim_{h\to 0} \frac{\sin h}{h}.$$

We saw that

$$\lim_{h\to 0} \frac{\sin h}{h} = 1$$

in Figure 3.36, but only because the graph in Figure 3.36 is in *radian mode*. If you look at the limit of the same function in *degree* mode you will get a very different limit (and hence a different derivative for sine). See Exercise 50.

EXAMPLE 1 Revisiting the Differentiation Rules

Find the derivatives of **(a)** $y = x^2 \sin x$ and **(b)** $u = \cos x/(1 - \sin x)$.

SOLUTION

(a) $$\frac{dy}{dx} = x^2 \cdot \frac{d}{dx}(\sin x) + \sin x \cdot \frac{d}{dx}(x^2) \qquad \text{Product Rule}$$
$$= x^2 \cos x + 2x \sin x$$

(b) $$\frac{du}{dx} = \frac{(1 - \sin x) \cdot \frac{d}{dx}(\cos x) - \cos x \cdot \frac{d}{dx}(1 - \sin x)}{(1 - \sin x)^2} \qquad \text{Quotient Rule}$$
$$= \frac{(1 - \sin x)(-\sin x) - \cos x\,(0 - \cos x)}{(1 - \sin x)^2}$$
$$= \frac{-\sin x + \sin^2 x + \cos^2 x}{(1 - \sin x)^2}$$
$$= \frac{1 - \sin x}{(1 - \sin x)^2} \qquad \sin^2 x + \cos^2 x = 1$$
$$= \frac{1}{1 - \sin x}$$

Now try Exercises 5 and 9.

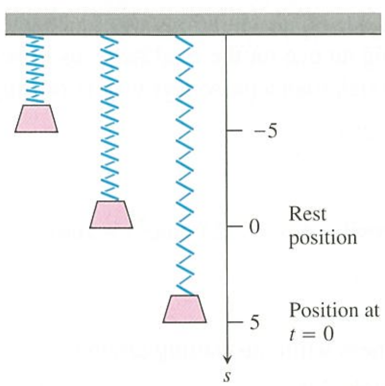


Figure 3.38 The weighted spring in Example 2.

Simple Harmonic Motion

The motion of a weight bobbing up and down on the end of a spring is an example of **simple harmonic motion.** Example 2 describes a case in which there are no opposing forces like friction or buoyancy to slow down the motion.

EXAMPLE 2 The Motion of a Weight on a Spring

A weight hanging from a spring (Figure 3.38) is stretched 5 units beyond its rest position ($s = 0$) and released at time $t = 0$ to bob up and down. Its position at any later time t is

$$s = 5 \cos t.$$

What are its velocity and acceleration at time t? Describe its motion.

SOLUTION We have:

Position: $s = 5 \cos t;$

Velocity: $v = \frac{ds}{dt} = \frac{d}{dt}(5 \cos t) = -5 \sin t;$

Acceleration: $a = \frac{dv}{dt} = \frac{d}{dt}(-5 \sin t) = -5 \cos t.$

Notice how much we can learn from these equations:

1. As time passes, the weight moves down and up between $s = -5$ and $s = 5$ on the s-axis. The amplitude of the motion is 5. The period of the motion is 2π.

2. The velocity $v = -5 \sin t$ attains its greatest magnitude, 5, when $\cos t = 0$, as the graphs show in Figure 3.39. Hence the speed of the weight, $|v| = 5\,|\sin t|$, is greatest when $\cos t = 0$, that is, when $s = 0$ (the rest position). The speed of the weight is zero when $\sin t = 0$. This occurs when $s = 5 \cos t = \pm 5$, at the endpoints of the interval of motion.

3. The acceleration value is always the exact opposite of the position value. When the weight is above the rest position, gravity is pulling it back down; when the weight is below the rest position, the spring is pulling it back up.

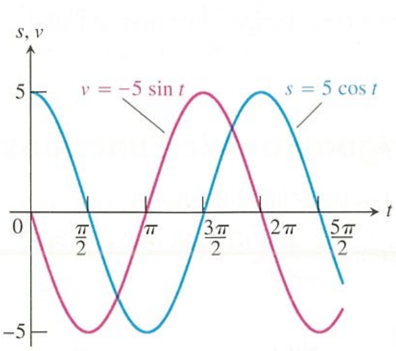


Figure 3.39 Graphs of the position and velocity of the weight in Example 2.

continued

4. The acceleration, $a = -5 \cos t$, is zero only at the rest position where $\cos t = 0$ and the force of gravity and the force from the spring offset each other. When the weight is anywhere else, the two forces are unequal and acceleration is nonzero. The acceleration is greatest in magnitude at the points farthest from the rest position, where $\cos t = \pm 1$. *Now try Exercise 11.*

Jerk

A sudden change in acceleration is called a "jerk." When a ride in a car or a bus is jerky, it is not that the accelerations involved are necessarily large but that the changes in acceleration are abrupt. Jerk is what spills your soft drink. The derivative responsible for jerk is the *third* derivative of position.

DEFINITION Jerk

Jerk is the derivative of acceleration. If a body's position at time t is $s(t)$, the body's jerk at time t is

$$j(t) = \frac{da}{dt} = \frac{d^3s}{dt^3}.$$

Recent tests have shown that motion sickness comes from accelerations whose changes in magnitude or direction take us by surprise. Keeping an eye on the road helps us to see the changes coming. A driver is less likely to become sick than a passenger who is reading in the back seat.

EXAMPLE 3 A Couple of Jerks

(a) The jerk caused by the constant acceleration of gravity ($g = -32$ ft/sec^2) is zero:

$$j = \frac{d}{dt}(g) = 0.$$

This explains why we don't experience motion sickness while just sitting around.

(b) The jerk of the simple harmonic motion in Example 2 is

$$j = \frac{da}{dt} = \frac{d}{dt}(-5 \cos t)$$
$$= 5 \sin t.$$

It has its greatest magnitude when $\sin t = \pm 1$. This does not occur at the extremes of the displacement, but at the rest position, where the acceleration changes direction and sign. *Now try Exercise 19.*

Derivatives of the Other Basic Trigonometric Functions

Because $\sin x$ and $\cos x$ are differentiable functions of x, the related functions

$$\tan x = \frac{\sin x}{\cos x}, \qquad \sec x = \frac{1}{\cos x},$$

$$\cot x = \frac{\cos x}{\sin x}, \qquad \csc x = \frac{1}{\sin x}$$

are differentiable at every value of x for which they are defined. Their derivatives (Exercises 25 and 26) are given by the following formulas.

$$\frac{d}{dx}\tan x = \sec^2 x, \qquad \frac{d}{dx}\sec x = \sec x \tan x$$
$$\frac{d}{dx}\cot x = -\csc^2 x, \qquad \frac{d}{dx}\csc x = -\csc x \cot x$$

EXAMPLE 4 Finding Tangent and Normal Lines

Find equations for the lines that are tangent and normal to the graph of

$$f(x) = \frac{\tan x}{x}$$

at $x = 2$. Support graphically.

SOLUTION

Solve Numerically Since we will be using a calculator approximation for $f(2)$ anyway, this is a good place to use NDER.

We compute $(\tan 2)/2$ on the calculator and store it as k. The slope of the tangent line at $(2, k)$ is

$$\text{NDER}\left(\frac{\tan x}{x}, 2\right),$$

which we compute and store as m. The equation of the tangent line is $y - k = m(x - 2)$, or

$$y = mx + k - 2m.$$

Only after we have found m and $k - 2m$ do we round the coefficients, giving the tangent line as

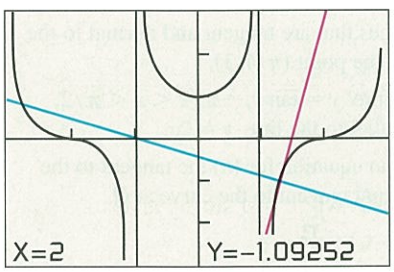
$$y = 3.43x - 7.96.$$

The equation of the normal line is

$$y - k = -\frac{1}{m}(x - 2), \text{ or}$$
$$y = -\frac{1}{m}x + k + \frac{2}{m}.$$

Again we wait until the end to round the coefficients, giving the normal line as

$$y = -0.291x - 0.51.$$



$[-3\pi/2, 3\pi/2]$ by $[-3, 3]$

Figure 3.40 Graphical support for Example 4.

Support Graphically Figure 3.40, showing the original function and the two lines, supports our computations. *Now try Exercise 23.*

EXAMPLE 5 A Trigonometric Second Derivative

Find y'' if $y = \sec x$.

SOLUTION

$$\begin{aligned}
y &= \sec x \\
y' &= \sec x \tan x \\
y'' &= \frac{d}{dx}(\sec x \tan x) \\
&= \sec x \frac{d}{dx}(\tan x) + \tan x \frac{d}{dx}(\sec x) \\
&= \sec x (\sec^2 x) + \tan x (\sec x \tan x) \\
&= \sec^3 x + \sec x \tan^2 x
\end{aligned}$$

Now try Exercise 36.

Quick Review 3.5 *(For help, go to Sections 1.6, 3.1, and 3.4.)*

1. Convert 135 degrees to radians.

2. Convert 1.7 radians to degrees.

3. Find the exact value of $\sin(\pi/3)$ without a calculator.

4. State the domain and the range of the cosine function.

5. State the domain and the range of the tangent function.

6. If $\sin a = -1$, what is $\cos a$?

7. If $\tan a = -1$, what are two possible values of $\sin a$?

8. Verify the identity:
$$\frac{1-\cos h}{h} = \frac{\sin^2 h}{h(1+\cos h)}.$$

9. Find an equation of the line tangent to the curve $y = 2x^3 - 7x^2 + 10$ at the point $(3, 1)$.

10. A particle moves along a line with velocity $v = 2t^3 - 7t^2 + 10$ for time $t \geq 0$. Find the acceleration of the particle at $t = 3$.

Section 3.5 Exercises

In Exercises 1–10, find dy/dx. Use your grapher to support your analysis if you are unsure of your answer.

1. $y = 1 + x - \cos x$

2. $y = 2\sin x - \tan x$

3. $y = \dfrac{1}{x} + 5\sin x$

4. $y = x\sec x$

5. $y = 4 - x^2\sin x$

6. $y = 3x + x\tan x$

7. $y = \dfrac{4}{\cos x}$

8. $y = \dfrac{x}{1+\cos x}$

9. $y = \dfrac{\cot x}{1+\cot x}$

10. $y = \dfrac{\cos x}{1+\sin x}$

In Exercises 11 and 12, a weight hanging from a spring (see Figure 3.38) bobs up and down with position function $s = f(t)$ (s in meters, t in seconds). What are its velocity and acceleration at time t? Describe its motion.

11. $s = 5\sin t$

12. $s = 7\cos t$

In Exercises 13–16, a body is moving in simple harmonic motion with position function $s = f(t)$ (s in meters, t in seconds).

(a) Find the body's velocity, speed, and acceleration at time t.

(b) Find the body's velocity, speed, and acceleration at time $t = \pi/4$.

(c) Describe the motion of the body.

13. $s = 2 + 3\sin t$

14. $s = 1 - 4\cos t$

15. $s = 2\sin t + 3\cos t$

16. $s = \cos t - 3\sin t$

In Exercises 17–20, a body is moving in simple harmonic motion with position function $s = f(t)$ (s in meters, t in seconds). Find the jerk at time t.

17. $s = 2\cos t$

18. $s = 1 + 2\cos t$

19. $s = \sin t - \cos t$

20. $s = 2 + 2\sin t$

21. Find equations for the lines that are tangent and normal to the graph of $y = \sin x + 3$ at $x = \pi$.

22. Find equations for the lines that are tangent and normal to the graph of $y = \sec x$ at $x = \pi/4$.

23. Find equations for the lines that are tangent and normal to the graph of $y = x^2\sin x$ at $x = 3$.

24. Use the definition of the derivative to prove that $(d/dx)(\cos x) = -\sin x$. (You will need the limits found at the beginning of this section.)

25. Assuming that $(d/dx)(\sin x) = \cos x$ and $(d/dx)(\cos x) = -\sin x$, prove each of the following.

(a) $\dfrac{d}{dx}\tan x = \sec^2 x$ **(b)** $\dfrac{d}{dx}\sec x = \sec x\tan x$

26. Assuming that $(d/dx)(\sin x) = \cos x$ and $(d/dx)(\cos x) = -\sin x$, prove each of the following.

(a) $\dfrac{d}{dx}\cot x = -\csc^2 x$ **(b)** $\dfrac{d}{dx}\csc x = -\csc x\cot x$

27. Show that the graphs of $y = \sec x$ and $y = \cos x$ have horizontal tangents at $x = 0$.

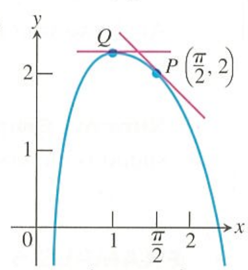
28. Show that the graphs of $y = \tan x$ and $y = \cot x$ have no horizontal tangents.

29. Find equations for the lines that are tangent and normal to the curve $y = \sqrt{2}\cos x$ at the point $(\pi/4, 1)$.

30. Find the points on the curve $y = \tan x$, $-\pi/2 < x < \pi/2$, where the tangent is parallel to the line $y = 2x$.

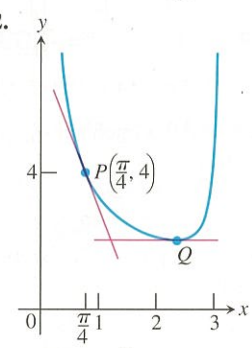
In Exercises 31 and 32, find an equation for **(a)** the tangent to the curve at P and **(b)** the horizontal tangent to the curve at Q.

31.



$y = 4 + \cot x - 2\csc x$

32.



$y = 1 + \sqrt{2}\csc x + \cot x$

Group Activity In Exercises 33 and 34, a body is moving in simple harmonic motion with position $s = f(t)$ (s in meters, t in seconds).

(a) Find the body's velocity, speed, acceleration, and jerk at time t.

(b) Find the body's velocity, speed, acceleration, and jerk at time $t = \pi/4$ sec.

(c) Describe the motion of the body.

33. $s = 2 - 2 \sin t$

34. $s = \sin t + \cos t$

35. Find y'' if $y = \csc x$.

36. Find y'' if $y = \theta \tan \theta$.

37. Writing to Learn Is there a value of b that will make

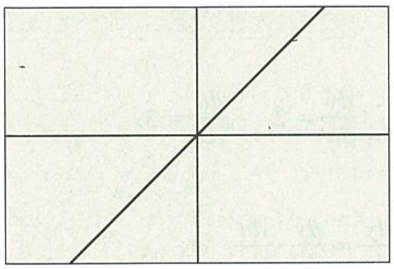
$$g(x) = \begin{cases} x + b, & x < 0 \\ \cos x, & x \geq 0 \end{cases}$$

continuous at $x = 0$? differentiable at $x = 0$? Give reasons for your answers.

38. Find $\dfrac{d^{999}}{dx^{999}}(\cos x)$.

39. Find $\dfrac{d^{725}}{dx^{725}}(\sin x)$.

40. *Local Linearity* This is the graph of the function $y = \sin x$ close to the origin. Since $\sin x$ is differentiable, this graph resembles a line. Find an equation for this line.



41. *(Continuation of Exercise 40)* For values of x close to 0, the linear equation found in Exercise 40 gives a good approximation of $\sin x$.

(a) Use this fact to estimate sin (0.12).

(b) Find sin (0.12) with a calculator. How close is the approximation in part (a)?

42. Use the identity $\sin 2x = 2 \sin x \cos x$ to find the derivative of $\sin 2x$. Then use the identity $\cos 2x = \cos^2 x - \sin^2 x$ to express that derivative in terms of $\cos 2x$.

43. Use the identity $\cos 2x = \cos x \cos x - \sin x \sin x$ to find the derivative of $\cos 2x$. Express the derivative in terms of $\sin 2x$.

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

In Exercises 44 and 45, a spring is bobbing up and down on the end of a spring according to $s(t) = -3 \sin t$.

44. True or False The spring is traveling upward at $t = 3\pi/4$. Justify your answer.

45. True or False The velocity and speed of the particle are the same at $t = \pi/4$. Justify your answer.

46. Multiple Choice Which of the following is an equation of the tangent line to $y = \sin x + \cos x$ at $x = \pi$?

(A) $y = -x + \pi - 1$ **(B)** $y = -x + \pi + 1$

(C) $y = -x - \pi + 1$ **(D)** $y = -x - \pi - 1$

(E) $y = x - \pi + 1$

47. Multiple Choice Which of the following is an equation of the normal line to $y = \sin x + \cos x$ at $x = \pi$?

(A) $y = -x + \pi - 1$ **(B)** $y = x - \pi - 1$ **(C)** $y = x - \pi + 1$

(D) $y = x + \pi + 1$ **(E)** $y = x + \pi - 1$

48. Multiple Choice Find y'' if $y = x \sin x$.

(A) $-x \sin x$ **(B)** $x \cos x + \sin x$ **(C)** $-x \sin x + 2 \cos x$

(D) $x \sin x$ **(E)** $-\sin x + \cos x$

49. Multiple Choice A body is moving in simple harmonic motion with position $s = 3 + \sin t$. At which of the following times is the velocity zero?

(A) $t = 0$ **(B)** $t = \pi/4$ **(C)** $t = \pi/2$

(D) $t = \pi$ **(E)** none of these

Exploration

50. *Radians vs. Degrees* What happens to the derivatives of $\sin x$ and $\cos x$ if x is measured in degrees instead of radians? To find out, take the following steps.

(a) With your grapher in degree mode, graph

$$f(h) = \frac{\sin h}{h}$$

and estimate $\lim_{h \to 0} f(h)$. Compare your estimate with $\pi/180$. Is there any reason to believe the limit should be $\pi/180$?

(b) With your grapher in degree mode, estimate

$$\lim_{h \to 0} \frac{\cos h - 1}{h}.$$

(c) Now go back to the derivation of the formula for the derivative of $\sin x$ in the text and carry out the steps of the derivation using degree-mode limits. What formula do you obtain for the derivative?

(d) Derive the formula for the derivative of $\cos x$ using degree-mode limits.

(e) The disadvantages of the degree-mode formulas become apparent as you start taking derivatives of higher order. What are the second and third degree-mode derivatives of $\sin x$ and $\cos x$?

Extending the Ideas

51. Use analytic methods to show that

$$\lim_{h \to 0} \frac{\cos h - 1}{h} = 0.$$

[*Hint:* Multiply numerator and denominator by $(\cos h + 1)$.]

52. Find A and B in $y = A \sin x + B \cos x$ so that $y'' - y = \sin x$.

3.6 Chain Rule

What you'll learn about

- Derivative of a Composite Function
- "Outside-Inside" Rule
- Repeated Use of the Chain Rule
- Slopes of Parametrized Curves
- Power Chain Rule

. . . and why

The Chain Rule is the most widely used differentiation rule in mathematics.

Derivative of a Composite Function

We now know how to differentiate $\sin x$ and $x^2 - 4$, but how do we differentiate a composite like $\sin(x^2 - 4)$? The answer is with the Chain Rule, which is probably the most widely used differentiation rule in mathematics. This section describes the rule and how to use it.

EXAMPLE 1 Relating Derivatives

The function $y = 6x - 10 = 2(3x - 5)$ is the composite of the functions $y = 2u$ and $u = 3x - 5$. How are the derivatives of these three functions related?

SOLUTION

We have

$$\frac{dy}{dx} = 6, \quad \frac{dy}{du} = 2, \quad \frac{du}{dx} = 3.$$

Since $6 = 2 \cdot 3$,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Now try Exercise 1.

Is it an accident that $dy/dx = dy/du \cdot du/dx$?

If we think of the derivative as a rate of change, our intuition allows us to see that this relationship is reasonable. For $y = f(u)$ and $u = g(x)$, if y changes twice as fast as u and u changes three times as fast as x, then we expect y to change six times as fast as x. This is much like the effect of a multiple gear train (Figure 3.41).

Let us try again on another function.

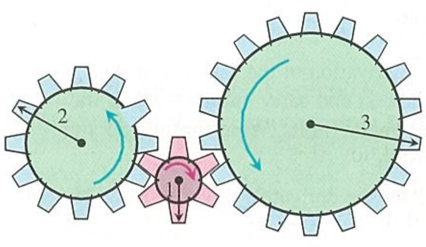


Figure 3.41 When gear A makes x turns, gear B makes u turns, and gear C makes y turns. By comparing circumferences or counting teeth, we see that $y = u/2$ and $u = 3x$, so $y = 3x/2$. Thus $dy/du = 1/2$, $du/dx = 3$, and $dy/dx = 3/2 = (dy/du)(du/dx)$.

EXAMPLE 2 Relating Derivatives

The polynomial $y = 9x^4 + 6x^2 + 1 = (3x^2 + 1)^2$ is the composite of $y = u^2$ and $u = 3x^2 + 1$. Calculating derivatives, we see that

$$\begin{aligned}\frac{dy}{du} \cdot \frac{du}{dx} &= 2u \cdot 6x \\ &= 2(3x^2 + 1) \cdot 6x \\ &= 36x^3 + 12x.\end{aligned}$$

Also,

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(9x^4 + 6x^2 + 1) \\ &= 36x^3 + 12x.\end{aligned}$$

Once again,

$$\frac{dy}{du} \cdot \frac{du}{dx} = \frac{dy}{dx}.$$

Now try Exercise 5.

The derivative of the composite function $f(g(x))$ at x is the derivative of f at $g(x)$ times the derivative of g at x (Figure 3.42). This is known as the Chain Rule.

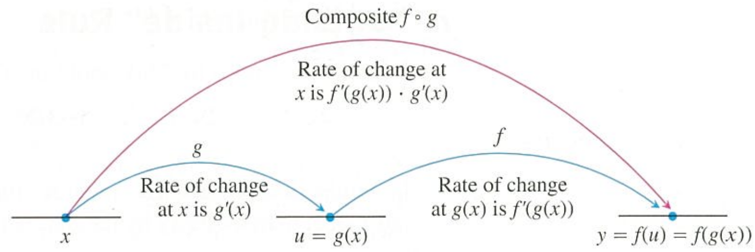


Figure 3.42 Rates of change multiply: the derivative of $f \circ g$ at x is the derivative of f at the point $g(x)$ times the derivative of g at x.

RULE 8 The Chain Rule

If f is differentiable at the point $u = g(x)$, and g is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where dy/du is evaluated at $u = g(x)$.

It would be tempting to try to prove the Chain Rule by writing

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x}$$

(a true statement about fractions with nonzero denominators) and taking the limit as $\Delta x \to 0$. This is essentially what is happening, and it would work as a proof if we knew that Δu, the change in u, was nonzero; but we do not know this. A small change in x could conceivably produce no change in u. An air-tight proof of the Chain Rule can be constructed through a different approach, but we will omit it here.

EXAMPLE 3 Applying the Chain Rule

An object moves along the x-axis so that its position at any time $t \geq 0$ is given by $x(t) = \cos(t^2 + 1)$. Find the velocity of the object as a function of t.

SOLUTION

We know that the velocity is dx/dt. In this instance, x is a composite function: $x = \cos(u)$ and $u = t^2 + 1$. We have

$$\frac{dx}{du} = -\sin(u) \qquad x = \cos(u)$$

$$\frac{du}{dt} = 2t. \qquad u = t^2 + 1$$

By the Chain Rule,

$$\begin{aligned} \frac{dx}{dt} &= \frac{dx}{du} \cdot \frac{du}{dt} \\ &= -\sin(u) \cdot 2t \\ &= -\sin(t^2 + 1) \cdot 2t \\ &= -2t \sin(t^2 + 1). \end{aligned}$$

Now try Exercise 9.

"Outside-Inside" Rule

It sometimes helps to think about the Chain Rule this way: If $y = f(g(x))$, then

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x).$$

In words, differentiate the "outside" function f and evaluate it at the "inside" function $g(x)$ left alone; then multiply by the derivative of the "inside function."

EXAMPLE 4 Differentiating from the Outside in

Differentiate $\sin(x^2 + x)$ with respect to x.

SOLUTION

$$\frac{d}{dx}\sin\underbrace{(x^2 + x)}_{\text{inside}} = \cos\underbrace{(x^2 + x)}_{\substack{\text{inside} \\ \text{left alone}}} \cdot \underbrace{(2x + 1)}_{\substack{\text{derivative of} \\ \text{the inside}}}$$

Now try Exercise 13.

Repeated Use of the Chain Rule

We sometimes have to use the Chain Rule two or more times to find a derivative. Here is an example:

EXAMPLE 5 A Three-Link "Chain"

Find the derivative of $g(t) = \tan(5 - \sin 2t)$.

SOLUTION

Notice here that tan is a function of $5 - \sin 2t$, while sin is a function of $2t$, which is itself a function of t. Therefore, by the Chain Rule,

$$\begin{aligned} g'(t) &= \frac{d}{dt}(\tan(5 - \sin 2t)) \\ &= \sec^2(5 - \sin 2t) \cdot \frac{d}{dt}(5 - \sin 2t) && \text{Derivative of } \tan u \text{ with } u = 5 - \sin 2t \\ &= \sec^2(5 - \sin 2t) \cdot \left(0 - \cos 2t \cdot \frac{d}{dt}(2t)\right) && \text{Derivative of } 5 - \sin u \text{ with } u = 2t \\ &= \sec^2(5 - \sin 2t) \cdot (-\cos 2t) \cdot 2 \\ &= -2(\cos 2t)\sec^2(5 - \sin 2t). \end{aligned}$$

Now try Exercise 23.

Slopes of Parametrized Curves

A parametrized curve $(x(t), y(t))$ is *differentiable at t* if x and y are differentiable at t. At a point on a differentiable parametrized curve where y is also a differentiable function of x, the derivatives dy/dt, dx/dt, and dy/dx are related by the Chain Rule:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}.$$

If $dx/dt \neq 0$, we may divide both sides of this equation by dx/dt to solve for dy/dx.

Finding dy/dx Parametrically

If all three derivatives exist and $dx/dt \neq 0$,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}. \tag{3}$$

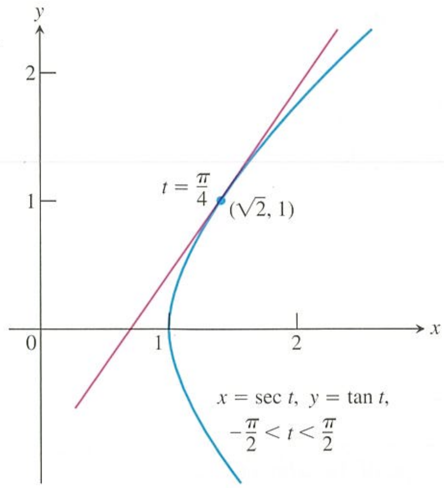


Figure 3.43 The hyperbola branch in Example 6. Equation 3 applies for every point on the graph except (1, 0). Can you state why Equation 3 fails at (1, 0)?

EXAMPLE 6 Differentiating with a Parameter

Find the line tangent to the right-hand hyperbola branch defined parametrically by

$$x = \sec t, \quad y = \tan t, \quad -\frac{\pi}{2} < t < \frac{\pi}{2}$$

at the point $(\sqrt{2}, 1)$, where $t = \pi/4$ (Figure 3.43).

SOLUTION

All three of the derivatives in Equation 3 exist and $dx/dt = \sec t \tan t \neq 0$ at the indicated point. Therefore, Equation 3 applies and

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy/dt}{dx/dt} \\ &= \frac{\sec^2 t}{\sec t \tan t} \\ &= \frac{\sec t}{\tan t} \\ &= \csc t. \end{aligned}$$

Setting $t = \pi/4$ gives

$$\left.\frac{dy}{dx}\right|_{t=\pi/4} = \csc(\pi/4) = \sqrt{2}.$$

The equation of the tangent line is

$$\begin{aligned} y - 1 &= \sqrt{2}(x - \sqrt{2}) \\ y &= \sqrt{2}x - 2 + 1 \\ y &= \sqrt{2}x - 1. \end{aligned}$$

Now try Exercise 41.

Power Chain Rule

If f is a differentiable function of u, and u is a differentiable function of x, then substituting $y = f(u)$ into the Chain Rule formula

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

leads to the formula

$$\frac{d}{dx} f(u) = f'(u)\frac{du}{dx}.$$

Here's an example of how it works: If n is an integer and $f(u) = u^n$, the Power Rules (Rules 2 and 7) tell us that $f'(u) = nu^{n-1}$. If u is a differentiable function of x, then we can use the Chain Rule to extend this to the **Power Chain Rule:**

$$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}. \qquad \frac{d}{du}(u^n) = nu^{n-1}$$

EXAMPLE 7 Finding Slope

(a) Find the slope of the line tangent to the curve $y = \sin^5 x$ at the point where $x = \pi/3$.

(b) Show that the slope of every line tangent to the curve $y = 1/(1 - 2x)^3$ is positive.

SOLUTION

(a)
$$\frac{dy}{dx} = 5\sin^4 x \cdot \frac{d}{dx}\sin x \quad \text{Power Chain Rule with } u = \sin x,\ n = 5$$
$$= 5\sin^4 x \cos x$$

The tangent line has slope

$$\left.\frac{dy}{dx}\right|_{x=\pi/3} = 5\left(\frac{\sqrt{3}}{2}\right)^4\left(\frac{1}{2}\right) = \frac{45}{32}.$$

(b)
$$\frac{dy}{dx} = \frac{d}{dx}(1 - 2x)^{-3}$$
$$= -3(1 - 2x)^{-4} \cdot \frac{d}{dx}(1 - 2x) \quad \text{Power Chain Rule with } u = (1 - 2x),\ n = -3$$
$$= -3(1 - 2x)^{-4} \cdot (-2)$$
$$= \frac{6}{(1 - 2x)^4}$$

At any point (x, y) on the curve, $x \neq 1/2$ and the slope of the tangent line is

$$\frac{dy}{dx} = \frac{6}{(1 - 2x)^4},$$

the quotient of two positive numbers. *Now try Exercise 53.*

EXAMPLE 8 Radians Versus Degrees

It is important to remember that the formulas for the derivatives of both $\sin x$ and $\cos x$ were obtained under the assumption that x is measured in radians, *not* degrees. The Chain Rule gives us new insight into the difference between the two. Since $180° = \pi$ radians, $x° = \pi x/180$ radians. By the Chain Rule,

$$\frac{d}{dx}\sin(x°) = \frac{d}{dx}\sin\left(\frac{\pi x}{180}\right) = \frac{\pi}{180}\cos\left(\frac{\pi x}{180}\right) = \frac{\pi}{180}\cos(x°).$$

See Figure 3.44.

The factor $\pi/180$, annoying in the first derivative, would compound with repeated differentiation. We see at a glance the compelling reason for the use of radian measure.

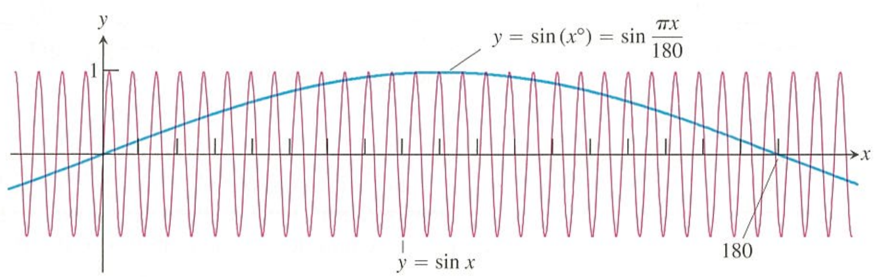


Figure 3.44 $\sin(x°)$ oscillates only $\pi/180$ times as often as $\sin x$ oscillates. Its maximum slope is $\pi/180$. (Example 8)

Quick Review 3.6 *(For help, go to Sections 1.2 and 1.6.)*

In Exercises 1–5, let $f(x) = \sin x$, $g(x) = x^2 + 1$, and $h(x) = 7x$. Write a simplified expression for the composite function.

1. $f(g(x))$
2. $f(g(h(x)))$
3. $(g \circ h)(x)$
4. $(h \circ g)(x)$
5. $f\left(\frac{g(x)}{h(x)}\right)$

In Exercises 6–10, let $f(x) = \cos x$, $g(x) = \sqrt{x+2}$, and $h(x) = 3x^2$. Write the given function as a composite of two or more of f, g, and h. For example, $\cos 3x^2$ is $f(h(x))$.

6. $\sqrt{\cos x + 2}$
7. $\sqrt{3\cos^2 x + 2}$
8. $3\cos x + 6$
9. $\cos 27x^4$
10. $\cos\sqrt{2 + 3x^2}$

Section 3.6 Exercises

In Exercises 1–8, use the given substitution and the Chain Rule to find dy/dx.

1. $y = \sin(3x + 1)$, $u = 3x + 1$
2. $y = \sin(7 - 5x)$, $u = 7 - 5x$
3. $y = \cos(\sqrt{3}x)$, $u = \sqrt{3}x$
4. $y = \tan(2x - x^3)$, $u = 2x - x^3$
5. $y = \left(\frac{\sin x}{1 + \cos x}\right)^2$, $u = \frac{\sin x}{1 + \cos x}$
6. $y = 5\cot\left(\frac{2}{x}\right)$, $u = \frac{2}{x}$
7. $y = \cos(\sin x)$, $u = \sin x$
8. $y = \sec(\tan x)$, $u = \tan x$

In Exercises 9–12, an object moves along the x-axis so that its position at any time $t \ge 0$ is given by $x(t) = s(t)$. Find the velocity of the object as a function of t.

9. $s = \cos\left(\frac{\pi}{2} - 3t\right)$
10. $s = t\cos(\pi - 4t)$
11. $s = \frac{4}{3\pi}\sin 3t + \frac{4}{5\pi}\cos 5t$
12. $s = \sin\left(\frac{3\pi}{2}t\right) + \cos\left(\frac{7\pi}{4}t\right)$

In Exercises 13–24, find dy/dx. If you are unsure of your answer, use NDER to support your computation.

13. $y = (x + \sqrt{x})^{-2}$
14. $y = (\csc x + \cot x)^{-1}$
15. $y = \sin^{-5} x - \cos^3 x$
16. $y = x^3(2x - 5)^4$
17. $y = \sin^3 x \tan 4x$
18. $y = 4\sqrt{\sec x + \tan x}$
19. $y = \frac{3}{\sqrt{2x + 1}}$
20. $y = \frac{x}{\sqrt{1 + x^2}}$
21. $y = \sin^2(3x - 2)$
22. $y = (1 + \cos 2x)^2$
23. $y = (1 + \cos^2 7x)^3$
24. $y = \sqrt{\tan 5x}$

In Exercises 25–28 find $dr/d\theta$.

25. $r = \tan(2 - \theta)$
26. $r = \sec 2\theta \tan 2\theta$
27. $r = \sqrt{\theta \sin\theta}$
28. $r = 2\theta\sqrt{\sec\theta}$

In Exercises 29–32, find y''.

29. $y = \tan x$
30. $y = \cot x$
31. $y = \cot(3x - 1)$
32. $y = 9\tan(x/3)$

In Exercises 33–38, find the value of $(f \circ g)'$ at the given value of x.

33. $f(u) = u^5 + 1$, $u = g(x) = \sqrt{x}$, $x = 1$
34. $f(u) = 1 - \frac{1}{u}$, $u = g(x) = \frac{1}{1 - x}$, $x = -1$
35. $f(u) = \cot\frac{\pi u}{10}$, $u = g(x) = 5\sqrt{x}$, $x = 1$
36. $f(u) = u + \frac{1}{\cos^2 u}$, $u = g(x) = \pi x$, $x = \frac{1}{4}$
37. $f(u) = \frac{2u}{u^2 + 1}$, $u = g(x) = 10x^2 + x + 1$, $x = 0$
38. $f(u) = \left(\frac{u - 1}{u + 1}\right)^2$, $u = g(x) = \frac{1}{x^2} - 1$, $x = -1$

What happens if you can write a function as a composite in different ways? Do you get the same derivative each time? The Chain Rule says you should. Try it with the functions in Exercises 39 and 40.

39. Find dy/dx if $y = \cos(6x + 2)$ by writing y as a composite with
    (a) $y = \cos u$ and $u = 6x + 2$.
    (b) $y = \cos 2u$ and $u = 3x + 1$.
40. Find dy/dx if $y = \sin(x^2 + 1)$ by writing y as a composite with
    (a) $y = \sin(u + 1)$ and $u = x^2$.
    (b) $y = \sin u$ and $u = x^2 + 1$.

In Exercises 41–48, find the equation of the line tangent to the curve at the point defined by the given value of t.

41. $x = 2\cos t$, $y = 2\sin t$, $t = \pi/4$
42. $x = \sin 2\pi t$, $y = \cos 2\pi t$, $t = -1/6$
43. $x = \sec^2 t - 1$, $y = \tan t$, $t = -\pi/4$
44. $x = \sec t$, $y = \tan t$, $t = \pi/6$
45. $x = t$, $y = \sqrt{t}$, $t = 1/4$
46. $x = 2t^2 + 3$, $y = t^4$, $t = -1$
47. $x = t - \sin t$, $y = 1 - \cos t$, $t = \pi/3$
48. $x = \cos t$, $y = 1 + \sin t$, $t = \pi/2$

49. Let $x = t^2 + t$, and let $y = \sin t$.

(a) Find dy/dx as a function of t.

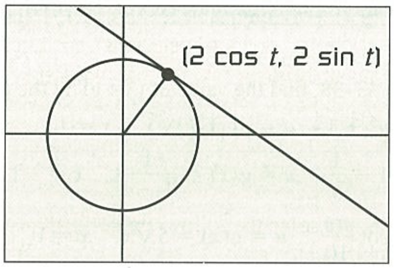
(b) Find $\frac{d}{dt}\left(\frac{dy}{dx}\right)$ as a function of t.

(c) Find $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ as a function of t.

Use the Chain Rule and your answer from part (b).

(d) Which of the expressions in parts (b) and (c) is d^2y/dx^2?

50. A circle of radius 2 and center (0, 0) can be parametrized by the equations $x = 2 \cos t$ and $y = 2 \sin t$. Show that for any value of t, the line tangent to the circle at $(2 \cos t, 2 \sin t)$ is perpendicular to the radius.



51. Let $s = \cos \theta$. Evaluate ds/dt when $\theta = 3\pi/2$ and $d\theta/dt = 5$.

52. Let $y = x^2 + 7x - 5$. Evaluate dy/dt when $x = 1$ and $dx/dt = 1/3$.

53. What is the largest value possible for the slope of the curve $y = \sin (x/2)$?

54. Write an equation for the tangent to the curve $y = \sin mx$ at the origin.

55. Find the lines that are tangent and normal to the curve $y = 2 \tan (\pi x/4)$ at $x = 1$. Support your answer graphically.

56. ***Working with Numerical Values*** Suppose that functions f and g and their derivatives have the following values at $x = 2$ and $x = 3$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
2	8	2	1/3	−3
3	3	−4	2π	5

Evaluate the derivatives with respect to x of the following combinations at the given value of x.

(a) $2f(x)$ at $x = 2$
(b) $f(x) + g(x)$ at $x = 3$
(c) $f(x) \cdot g(x)$ at $x = 3$
(d) $f(x)/g(x)$ at $x = 2$
(e) $f(g(x))$ at $x = 2$
(f) $\sqrt{f(x)}$ at $x = 2$
(g) $1/g^2(x)$ at $x = 3$
(h) $\sqrt{f^2(x) + g^2(x)}$ at $x = 2$

57. ***Extension of Example 8*** Show that $\frac{d}{dx} \cos (x°)$ is $-\frac{\pi}{180} \sin (x°)$.

58. ***Working with Numerical Values*** Suppose that the functions f and g and their derivatives with respect to x have the following values at $x = 0$ and $x = 1$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
0	1	1	5	1/3
1	3	−4	−1/3	−8/3

Evaluate the derivatives with respect to x of the following combinations at the given value of x.

(a) $5f(x) - g(x)$, $x = 1$
(b) $f(x)g^3(x)$, $x = 0$
(c) $\frac{f(x)}{g(x) + 1}$, $x = 1$
(d) $f(g(x))$, $x = 0$
(e) $g(f(x))$, $x = 0$
(f) $(g(x) + f(x))^{-2}$, $x = 1$
(g) $f(x + g(x))$, $x = 0$

59. ***Orthogonal Curves*** Two curves are said to cross at right angles if their tangents are perpendicular at the crossing point. The technical word for "crossing at right angles" is **orthogonal.** Show that the curves $y = \sin 2x$ and $y = -\sin (x/2)$ are orthogonal at the origin. Draw both graphs and both tangents in a square viewing window.

60. **Writing to Learn** Explain why the Chain Rule formula

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

is not simply the well-known rule for multiplying fractions.

61. ***Running Machinery Too Fast*** Suppose that a piston is moving straight up and down and that its position at time t seconds is

$$s = A \cos (2\pi bt),$$

with A and b positive. The value of A is the amplitude of the motion, and b is the frequency (number of times the piston moves up and down each second). What effect does doubling the frequency have on the piston's velocity, acceleration, and jerk? (Once you find out, you will know why machinery breaks when you run it too fast.)

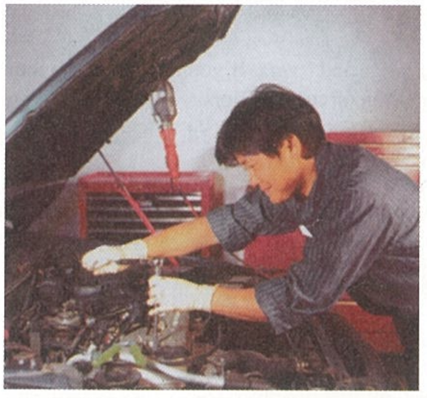


Figure 3.45 The internal forces in the engine get so large that they tear the engine apart when the velocity is too great.

62. Group Activity ***Temperatures in Fairbanks, Alaska.*** The graph in Figure 3.46 shows the average Fahrenheit temperature in Fairbanks, Alaska, during a typical 365-day year. The equation that approximates the temperature on day x is

$$y = 37 \sin\left[\frac{2\pi}{365}(x - 101)\right] + 25.$$

(a) On what day is the temperature increasing the fastest?

(b) About how many degrees per day is the temperature increasing when it is increasing at its fastest?

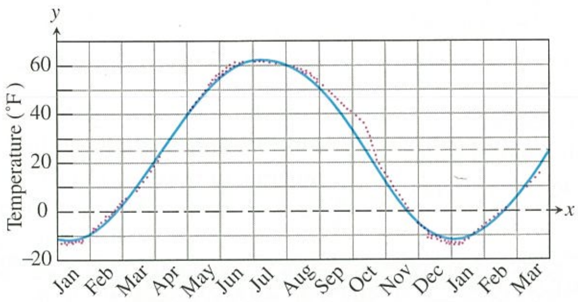


Figure 3.46 Normal mean air temperatures at Fairbanks, Alaska, plotted as data points, and the approximating sine function (Exercise 62).

63. *Particle Motion* The position of a particle moving along a coordinate line is $s = \sqrt{1 + 4t}$, with s in meters and t in seconds. Find the particle's velocity and acceleration at $t = 6$ sec.

64. *Constant Acceleration* Suppose the velocity of a falling body is $v = k\sqrt{s}$ m/sec (k a constant) at the instant the body has fallen s meters from its starting point. Show that the body's acceleration is constant.

65. *Falling Meteorite* The velocity of a heavy meteorite entering the earth's atmosphere is inversely proportional to $\sqrt{s}$ when it is s kilometers from the earth's center. Show that the meteorite's acceleration is inversely proportional to s^2.

66. *Particle Acceleration* A particle moves along the x-axis with velocity $dx/dt = f(x)$. Show that the particle's acceleration is $f(x)f'(x)$.

67. *Temperature and the Period of a Pendulum* For oscillations of small amplitude (short swings), we may safely model the relationship between the period T and the length L of a simple pendulum with the equation

$$T = 2\pi\sqrt{\frac{L}{g}},$$

where g is the constant acceleration of gravity at the pendulum's location. If we measure g in centimeters per second squared, we measure L in centimeters and T in seconds. If the pendulum is made of metal, its length will vary with temperature, either increasing or decreasing at a rate that is roughly proportional to L. In symbols, with u being temperature and k the proportionality constant,

$$\frac{dL}{du} = kL.$$

Assuming this to be the case, show that the rate at which the period changes with respect to temperature is $kT/2$.

68. Writing to Learn ***Chain Rule*** Suppose that $f(x) = x^2$ and $g(x) = |x|$. Then the composites

$$(f \circ g)(x) = |x|^2 = x^2 \quad \text{and} \quad (g \circ f)(x) = |x^2| = x^2$$

are both differentiable at $x = 0$ even though g itself is not differentiable at $x = 0$. Does this contradict the Chain Rule? Explain.

69. *Tangents* Suppose that $u = g(x)$ is differentiable at $x = 1$ and that $y = f(u)$ is differentiable at $u = g(1)$. If the graph of $y = f(g(x))$ has a horizontal tangent at $x = 1$, can we conclude anything about the tangent to the graph of g at $x = 1$ or the tangent to the graph of f at $u = g(1)$? Give reasons for your answer.

Standardized Test Questions

You should solve the following problems without using a graphing calculator.

70. True or False $\frac{d}{dx}(\sin x) = \cos x$, if x is measured in degrees or radians. Justify your answer.

71. True or False The slope of the normal line to the curve $x = 3\cos t$, $y = 3\sin t$ at $t = \pi/4$ is -1. Justify your answer.

72. Multiple Choice Which of the following is dy/dx if $y = \tan(4x)$?

(A) $4\sec(4x)\tan(4x)$ **(B)** $\sec(4x)\tan(4x)$ **(C)** $4\cot(4x)$
(D) $\sec^2(4x)$ **(E)** $4\sec^2(4x)$

73. Multiple Choice Which of the following is dy/dx if $y = \cos^2(x^3 + x^2)$?

(A) $-2(3x^2 + 2x)$
(B) $-(3x^2 + 2x)\cos(x^3 + x^2)\sin(x^3 + x^2)$
(C) $-2(3x^2 + 2x)\cos(x^3 + x^2)\sin(x^3 + x^2)$
(D) $2(3x^2 + 2x)\cos(x^3 + x^2)\sin(x^3 + x^2)$
(E) $2(3x^2 + 2x)$

In Exercises 74 and 75, use the curve defined by the parametric equations $x = t - \cos t$, $y = -1 + \sin t$.

74. Multiple Choice Which of the following is an equation of the tangent line to the curve at $t = 0$?

(A) $y = x$ **(B)** $y = -x$ **(C)** $y = x + 2$
(D) $y = x - 2$ **(E)** $y = -x - 2$

75. Multiple Choice At which of the following values of t is $dy/dx = 0$?

(A) $t = \pi/4$ **(B)** $t = \pi/2$ **(C)** $t = 3\pi/4$
(D) $t = \pi$ **(E)** $t = 2\pi$

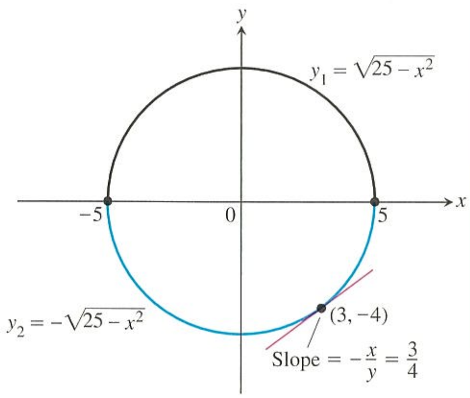


Figure 3.49 The circle combines the graphs of two functions. The graph of y_2 is the lower semicircle and passes through (3, −4). (Example 2)

EXAMPLE 2 Finding Slope on a Circle

Find the slope of the circle $x^2 + y^2 = 25$ at the point $(3, -4)$.

SOLUTION

The circle is not the graph of a single function of x, but it is the union of the graphs of two differentiable functions, $y_1 = \sqrt{25 - x^2}$ and $y_2 = -\sqrt{25 - x^2}$ (Figure 3.49). The point $(3, -4)$ lies on the graph of y_2, so it is possible to find the slope by calculating explicitly:

$$\left.\frac{dy_2}{dx}\right|_{x=3} = -\left.\frac{-2x}{2\sqrt{25 - x^2}}\right|_{x=3} = -\frac{-6}{2\sqrt{25 - 9}} = \frac{3}{4}.$$

But we can also find this slope more easily by differentiating both sides of the equation of the circle implicitly with respect to x:

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25) \quad \text{Differentiate both sides with respect to } x.$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}.$$

The slope at $(3, -4)$ is

$$-\left.\frac{x}{y}\right|_{(3,-4)} = -\frac{3}{-4} = \frac{3}{4}.$$

The implicit solution, besides being computationally easier, yields a formula for dy/dx that applies at any point on the circle (except, of course, $(\pm 5, 0)$, where slope is undefined). The explicit solution derived from the formula for y_2 applies only to the lower half of the circle. ***Now try Exercise 11.***

To calculate the derivatives of other implicitly defined functions, we proceed as in Examples 1 and 2. We treat y as a differentiable function of x and apply the usual rules to differentiate both sides of the defining equation.

EXAMPLE 3 Solving for *dy/dx*

Show that the slope dy/dx is defined at every point on the graph of $2y = x^2 + \sin y$.

SOLUTION

First we need to know dy/dx, which we find by implicit differentiation:

$$2y = x^2 + \sin y$$

$$\frac{d}{dx}(2y) = \frac{d}{dx}(x^2 + \sin y) \quad \text{Differentiate both sides with respect to } x \ldots$$

$$= \frac{d}{dx}(x^2) + \frac{d}{dx}(\sin y)$$

$$2\frac{dy}{dx} = 2x + \cos y\frac{dy}{dx} \quad \ldots \text{ treating } y \text{ as a function of } x \text{ and using the Chain Rule.}$$

$$2\frac{dy}{dx} - (\cos y)\frac{dy}{dx} = 2x \quad \text{Collect terms with } dy/dx \ldots$$

$$(2 - \cos y)\frac{dy}{dx} = 2x \quad \text{and factor out } dy/dx.$$

$$\frac{dy}{dx} = \frac{2x}{2 - \cos y}. \quad \text{Solve for } dy/dx \text{ by dividing.}$$

The formula for dy/dx is defined at every point (x, y), except for those points at which $\cos y = 2$. Since $\cos y$ cannot be greater than 1, this never happens.

Now try Exercise 13.

Ellen Ochoa (1958–)

After earning a doctorate degree in electrical engineering from Stanford University, Ellen Ochoa became a research engineer and, within a few years, received three patents in the field of optics. In 1990, Ochoa joined the NASA astronaut program, and, three years later, became the first Hispanic female to travel in space. Ochoa's message to young people is: "If you stay in school you have the potential to achieve what you want in the future."

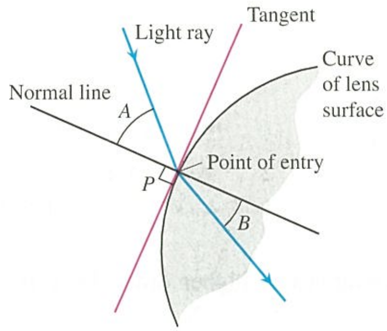


Figure 3.50 The profile of a lens, showing the bending (refraction) of a ray of light as it passes through the lens surface.

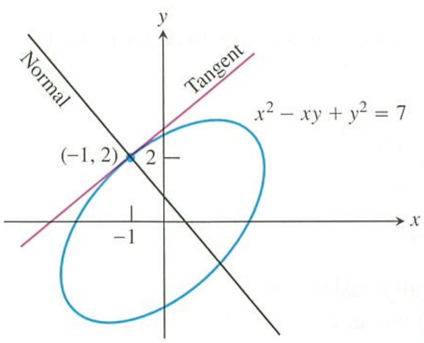


Figure 3.51 Tangent and normal lines to the ellipse $x^2 - xy + y^2 = 7$ at the point $(-1, 2)$. (Example 4)

Implicit Differentiation Process

1. Differentiate both sides of the equation with respect to x.
2. Collect the terms with dy/dx on one side of the equation.
3. Factor out dy/dx.
4. Solve for dy/dx.

Lenses, Tangents, and Normal Lines

In the law that describes how light changes direction as it enters a lens, the important angles are the angles the light makes with the line perpendicular to the surface of the lens at the point of entry (angles A and B in Figure 3.50). This line is called the *normal to the surface* at the point of entry. In a profile view of a lens like the one in Figure 3.50, the normal is a line perpendicular to the tangent to the profile curve at the point of entry.

Profiles of lenses are often described by quadratic curves (see Figure 3.51). When they are, we can use implicit differentiation to find the tangents and normals.

EXAMPLE 4 Tangent and normal to an ellipse

Find the tangent and normal to the ellipse $x^2 - xy + y^2 = 7$ at the point $(-1, 2)$. (See Figure 3.51.)

SOLUTION

We first use implicit differentiation to find dy/dx:

$$x^2 - xy + y^2 = 7$$

$$\frac{d}{dx}(x^2) - \frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = \frac{d}{dx}(7) \qquad \text{Differentiate both sides with respect to } x \ldots$$

$$2x - \left(x\frac{dy}{dx} + y\frac{dx}{dx}\right) + 2y\frac{dy}{dx} = 0 \qquad \ldots \text{treating } xy \text{ as a product and } y \text{ as a function of } x.$$

$$(2y - x)\frac{dy}{dx} = y - 2x \qquad \text{Collect terms.}$$

$$\frac{dy}{dx} = \frac{y - 2x}{2y - x}. \qquad \text{Solve for } dy/dx.$$

We then evaluate the derivative at $x = -1$, $y = 2$ to obtain

$$\left.\frac{dy}{dx}\right|_{(-1,2)} = \left.\frac{y - 2x}{2y - x}\right|_{(-1,2)}$$

$$= \frac{2 - 2(-1)}{2(2) - (-1)}$$

$$= \frac{4}{5}.$$

The tangent to the curve at $(-1, 2)$ is

$$y - 2 = \frac{4}{5}(x - (-1))$$

$$y = \frac{4}{5}x + \frac{14}{5}.$$

continued

The normal to the curve at $(-1, 2)$ is

$$y - 2 = -\frac{5}{4}(x + 1)$$
$$y = -\frac{5}{4}x + \frac{3}{4}.$$

Now try Exercise 17.

Derivatives of Higher Order

Implicit differentiation can also be used to find derivatives of higher order. Here is an example.

EXAMPLE 5 Finding a Second Derivative Implicitly

Find d^2y/dx^2 if $2x^3 - 3y^2 = 8$.

SOLUTION

To start, we differentiate both sides of the equation with respect to x in order to find $y' = dy/dx$.

$$\frac{d}{dx}(2x^3 - 3y^2) = \frac{d}{dx}(8)$$
$$6x^2 - 6yy' = 0$$
$$x^2 - yy' = 0$$
$$y' = \frac{x^2}{y}, \text{ when } y \neq 0$$

We now apply the Quotient Rule to find y''.

$$y'' = \frac{d}{dx}\left(\frac{x^2}{y}\right) = \frac{2xy - x^2y'}{y^2} = \frac{2x}{y} - \frac{x^2}{y^2} \cdot y'$$

Finally, we substitute $y' = x^2/y$ to express y'' in terms of x and y.

$$y'' = \frac{2x}{y} - \frac{x^2}{y^2}\left(\frac{x^2}{y}\right) = \frac{2x}{y} - \frac{x^4}{y^3}, \text{ when } y \neq 0$$

Now try Exercise 29.

EXPLORATION 1 An Unexpected Derivative

Consider the set of all points (x, y) satisfying the equation $x^2 - 2xy + y^2 = 4$. What does the graph of the equation look like? You can find out in two ways in this Exploration.

1. Use implicit differentiation to find dy/dx. Are you surprised by this derivative?
2. Knowing the derivative, what do you conjecture about the graph?
3. What are the possible values of y when $x = 0$? Does this information enable you to refine your conjecture about the graph?
4. The original equation can be written as $(x - y)^2 - 4 = 0$. By factoring the expression on the left, write two equations whose graphs combine to give the graph of the original equation. Then sketch the graph.
5. Explain why your graph is consistent with the derivative found in part 1.

Rational Powers of Differentiable Functions

We know that the Power Rule

$$\frac{d}{dx}x^n = nx^{n-1}$$

holds for any integer n (Rules 2 and 7 of this chapter). We can now prove that it holds when n is any rational number.

> **RULE 9 Power Rule for Rational Powers of x**
>
> If n is any rational number, then
>
> $$\frac{d}{dx}x^n = nx^{n-1}.$$
>
> If $n < 1$, then the derivative does not exist at $x = 0$.

Proof Let p and q be integers with $q > 0$ and suppose that $y = \sqrt[q]{x^p} = x^{p/q}$. Then

$$y^q = x^p.$$

Since p and q are integers (for which we already have the Power Rule), we can differentiate both sides of the equation with respect to x and obtain

$$qy^{q-1}\frac{dy}{dx} = px^{p-1}.$$

If $y \neq 0$, we can divide both sides of the equation by qy^{q-1} to solve for dy/dx, obtaining

$$\begin{aligned} \frac{dy}{dx} &= \frac{px^{p-1}}{qy^{q-1}} \\ &= \frac{p}{q} \cdot \frac{x^{p-1}}{(x^{p/q})^{q-1}} && y = x^{p/q} \\ &= \frac{p}{q} \cdot \frac{x^{p-1}}{x^{p-p/q}} && \frac{p}{q}(q-1) = p - \frac{p}{q} \\ &= \frac{p}{q} \cdot x^{(p-1)-(p-p/q)} && \text{A law of exponents} \\ &= \frac{p}{q} \cdot x^{(p/q)-1}. \end{aligned}$$

This proves the rule. ■

By combining this result with the Chain Rule, we get an extension of the Power Chain Rule to rational powers of u:

If n is a rational number and u is a differentiable function of x, then u^n is a differentiable function of x and

$$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx},$$

provided that $u \neq 0$ if $n < 1$.

The restriction that $u \neq 0$ when $n < 1$ is necessary because 0 might be in the domain of u^n but not in the domain of u^{n-1}, as we see in the first two parts of Example 6.

EXAMPLE 6 Using the Rational Power Rule

(a) $\frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}(x^{1/2}) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

Notice that $\sqrt{x}$ is defined at $x = 0$, but $1/(2\sqrt{x})$ is not.

(b) $\frac{d}{dx}(x^{2/3}) = \frac{2}{3}(x^{-1/3}) = \frac{2}{3x^{1/3}}$

The original function is defined for all real numbers, but the derivative is undefined at $x = 0$. Recall Figure 3.12, which showed that this function's graph has a *cusp* at $x = 0$.

(c)
$$\begin{aligned}\frac{d}{dx}(\cos x)^{-1/5} &= -\frac{1}{5}(\cos x)^{-6/5} \cdot \frac{d}{dx}(\cos x)\\ &= -\frac{1}{5}(\cos x)^{-6/5}(-\sin x)\\ &= \frac{1}{5}\sin x(\cos x)^{-6/5}\end{aligned}$$

Now try Exercise 33.

Quick Review 3.7 *(For help, go to Section 1.2 and Appendix A.5.)*

In Exercises 1–5, sketch the curve defined by the equation and find two functions y_1 and y_2 whose graphs will combine to give the curve.

1. $x - y^2 = 0$ **2.** $4x^2 + 9y^2 = 36$

3. $x^2 - 4y^2 = 0$ **4.** $x^2 + y^2 = 9$

5. $x^2 + y^2 = 2x + 3$

In Exercises 6–8, solve for y' in terms of y and x.

6. $x^2y' - 2xy = 4x - y$

7. $y'\sin x - x\cos x = xy' + y$

8. $x(y^2 - y') = y'(x^2 - y)$

In Exercises 9 and 10, find an expression for the function using rational powers rather than radicals.

9. $\sqrt{x}(x - \sqrt[3]{x})$ **10.** $\frac{x + \sqrt[3]{x^2}}{\sqrt{x^3}}$

Section 3.7 Exercises

In Exercises 1–8, find dy/dx.

1. $x^2y + xy^2 = 6$ **2.** $x^3 + y^3 = 18xy$

3. $y^2 = \frac{x - 1}{x + 1}$ **4.** $x^2 = \frac{x - y}{x + y}$

5. $x = \tan y$ **6.** $x = \sin y$

7. $x + \tan(xy) = 0$ **8.** $x + \sin y = xy$

In Exercises 9–12, find dy/dx and find the slope of the curve at the indicated point.

9. $x^2 + y^2 = 13, \quad (-2, 3)$

10. $x^2 + y^2 = 9, \quad (0, 3)$

11. $(x - 1)^2 + (y - 1)^2 = 13, \quad (3, 4)$

12. $(x + 2)^2 + (y + 3)^2 = 25, \quad (1, -7)$

In Exercises 13–16, find where the slope of the curve is defined.

13. $x^2y - xy^2 = 4$ **14.** $x = \cos y$

15. $x^3 + y^3 = xy$ **16.** $x^2 + 4xy + 4y^2 - 3x = 6$

In Exercises 17–26, find the lines that are **(a)** tangent and **(b)** normal to the curve at the given point.

17. $x^2 + xy - y^2 = 1, \quad (2, 3)$

18. $x^2 + y^2 = 25, \quad (3, -4)$

19. $x^2y^2 = 9, \quad (-1, 3)$

20. $y^2 - 2x - 4y - 1 = 0, \quad (-2, 1)$

21. $6x^2 + 3xy + 2y^2 + 17y - 6 = 0, \quad (-1, 0)$

22. $x^2 - \sqrt{3}xy + 2y^2 = 5, \quad (\sqrt{3}, 2)$

23. $2xy + \pi \sin y = 2\pi, \quad (1, \pi/2)$

24. $x \sin 2y = y \cos 2x, \quad (\pi/4, \pi/2)$

25. $y = 2\sin(\pi x - y), \quad (1, 0)$

26. $x^2\cos^2 y - \sin y = 0, \quad (0, \pi)$

In Exercises 27–30, use implicit differentiation to find dy/dx and then d^2y/dx^2.

27. $x^2 + y^2 = 1$ **28.** $x^{2/3} + y^{2/3} = 1$

29. $y^2 = x^2 + 2x$ **30.** $y^2 + 2y = 2x + 1$

In Exercises 31–42, find dy/dx.

31. $y = x^{9/4}$ **32.** $y = x^{-3/5}$

33. $y = \sqrt[3]{x}$ **34.** $y = \sqrt[4]{x}$

35. $y = (2x + 5)^{-1/2}$ **36.** $y = (1 - 6x)^{2/3}$

37. $y = x\sqrt{x^2 + 1}$ **38.** $y = \frac{x}{\sqrt{x^2 + 1}}$

39. $y = \sqrt{1 - \sqrt{x}}$ **40.** $y = 3(2x^{-1/2} + 1)^{-1/3}$

41. $y = 3(\csc x)^{3/2}$ **42.** $y = [\sin(x + 5)]^{5/4}$

43. Which of the following could be true if $f''(x) = x^{-1/3}$?

(a) $f(x) = \frac{3}{2}x^{2/3} - 3$ **(b)** $f(x) = \frac{9}{10}x^{5/3} - 7$

(c) $f'''(x) = -\frac{1}{3}x^{-4/3}$ **(d)** $f'(x) = \frac{3}{2}x^{2/3} + 6$

44. Which of the following could be true if $g''(t) = 1/t^{3/4}$?

(a) $g'(t) = 4\sqrt[4]{t} - 4$ **(b)** $g'''(t) = -4/\sqrt[4]{t}$

(c) $g(t) = t - 7 + (16/5)t^{5/4}$ **(d)** $g'(t) = (1/4)t^{1/4}$

45. ***The Eight Curve*** **(a)** Find the slopes of the figure-eight-shaped curve

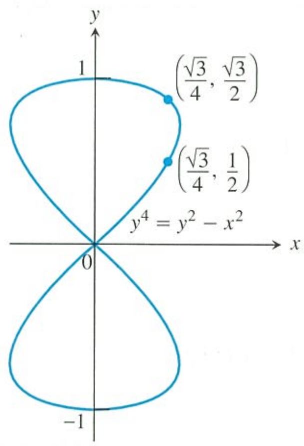
$$y^4 = y^2 - x^2$$

at the two points shown on the graph that follows.

(b) Use parametric mode and the two pairs of parametric equations

$$x_1(t) = \sqrt{t^2 - t^4}, \quad y_1(t) = t,$$
$$x_2(t) = -\sqrt{t^2 - t^4}, \quad y_2(t) = t,$$

to graph the curve. Specify a window and a parameter interval.



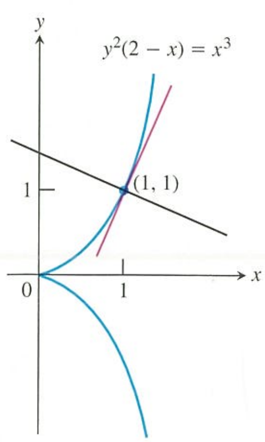
46. ***The Cissoid of Diocles (dates from about 200 B.C.)***

(a) Find equations for the tangent and normal to the cissoid of Diocles,

$$y^2(2 - x) = x^3,$$

at the point (1, 1) as pictured below.

(b) Explain how to reproduce the graph on a grapher.



47. **(a)** Confirm that $(-1, 1)$ is on the curve defined by $x^3y^2 = \cos(\pi y)$.

(b) Use part (a) to find the slope of the line tangent to the curve at $(-1, 1)$.

48. **Grouping Activity**

(a) Show that the relation

$$y^3 - xy = -1$$

cannot be a function of x by showing that there is more than one possible y-value when $x = 2$.

(b) On a small enough square with center (2, 1), the part of the graph of the relation within the square will define a function $y = f(x)$. For this function, find $f'(2)$ and $f''(2)$.

49. Find the two points where the curve $x^2 + xy + y^2 = 7$ crosses the x-axis, and show that the tangents to the curve at these points are parallel. What is the common slope of these tangents?

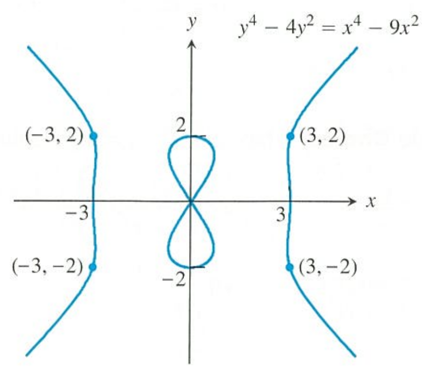
50. Find points on the curve $x^2 + xy + y^2 = 7$ **(a)** where the tangent is parallel to the x-axis and **(b)** where the tangent is parallel to the y-axis. (In the latter case, dy/dx is not defined, but dx/dy is. What value does dx/dy have at these points?)

51. ***Orthogonal Curves*** Two curves are *orthogonal* at a point of intersection if their tangents at that point cross at right angles. Show that the curves $2x^2 + 3y^2 = 5$ and $y^2 = x^3$ are orthogonal at (1, 1) and (1, −1). Use parametric mode to draw the curves and to show the tangent lines.

52. The position of a body moving along a coordinate line at time t is $s = (4 + 6t)^{3/2}$, with s in meters and t in seconds. Find the body's velocity and acceleration when $t = 2$ sec.

53. The velocity of a falling body is $v = 8\sqrt{s - t} + 1$ feet per second at the instant t(sec) the body has fallen s feet from its starting point. Show that the body's acceleration is 32 ft/sec^2.

54. ***The Devil's Curve (Gabriel Cramer [the Cramer of Cramer's Rule], 1750)*** Find the slopes of the devil's curve $y^4 - 4y^2 = x^4 - 9x^2$ at the four indicated points.



55. ***The Folium of Descartes*** (See Figure 3.47 on page 157)

(a) Find the slope of the folium of Descartes, $x^3 + y^3 - 9xy = 0$ at the points (4, 2) and (2, 4).

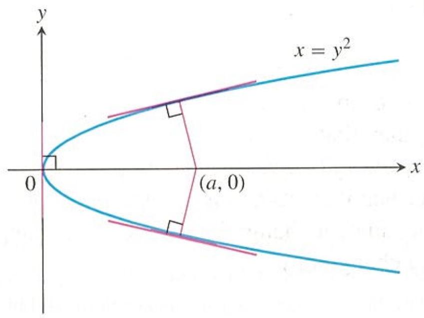
(b) At what point other than the origin does the folium have a horizontal tangent?

(c) Find the coordinates of the point A in Figure 3.47, where the folium has a vertical tangent.

56. The line that is normal to the curve $x^2 + 2xy - 3y^2 = 0$ at $(1, 1)$ intersects the curve at what other point?

57. Find the normals to the curve $xy + 2x - y = 0$ that are parallel to the line $2x + y = 0$.

58. Show that if it is possible to draw these three normals from the point $(a, 0)$ to the parabola $x = y^2$ shown here, then a must be greater than $1/2$. One of the normals is the x-axis. For what value of a are the other two normals perpendicular?



Standardized Test Questions

You should solve the following problems without using a graphing calculator.

59. True or False The slope of $xy^2 + x = 1$ at $(1/2, 1)$ is 2. Justify your answer.

60. True or False The derivative of $y = \sqrt[3]{x}$ is $\frac{1}{3x^{2/3}}$. Justify your answer.

In Exercises 61 and 62, use the curve $x^2 - xy + y^2 = 1$.

61. Multiple Choice Which of the following is equal to dy/dx?

(A) $\dfrac{y - 2x}{2y - x}$ (B) $\dfrac{y + 2x}{2y - x}$

(C) $\dfrac{2x}{x - 2y}$ (D) $\dfrac{2x + y}{x - 2y}$

(E) $\dfrac{y + 2x}{x}$

62. Multiple Choice Which of the following is equal to $\dfrac{d^2y}{dx^2}$?

(A) $-\dfrac{6}{(2y - x)^3}$ (B) $\dfrac{10y^2 - 10x^2 - 10xy}{(2y - x)^3}$

(C) $\dfrac{8x^2 - 4xy + 8y^2}{(x - 2y)^3}$ (D) $\dfrac{10x^2 + 10y^2}{(x - 2y)^3}$

(E) $\dfrac{2}{x}$

63. Multiple Choice Which of the following is equal to dy/dx if $y = x^{3/4}$?

(A) $\dfrac{3x^{1/3}}{4}$ (B) $\dfrac{4x^{1/4}}{3}$ (C) $\dfrac{3x^{1/4}}{4}$ (D) $\dfrac{4}{3x^{1/4}}$ (E) $\dfrac{3}{4x^{1/4}}$

64. Multiple Choice Which of the following is equal to the slope of the tangent to $y^2 - x^2 = 1$ at $(1, \sqrt{2})$?

(A) $-\dfrac{1}{\sqrt{2}}$ (B) $-\sqrt{2}$ (C) $\dfrac{1}{\sqrt{2}}$ (D) $\sqrt{2}$ (E) 0

Extending the Ideas

65. *Finding Tangents*

(a) Show that the tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at the point (x_1, y_1) has equation

$$\frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 1.$$

(b) Find an equation for the tangent to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

at the point (x_1, y_1).

66. *End Behavior Model* Consider the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Show that

(a) $y = \pm\dfrac{b}{a}\sqrt{x^2 - a^2}$.

(b) $g(x) = (b/a)|x|$ is an end behavior model for

$$f(x) = (b/a)\sqrt{x^2 - a^2}.$$

(c) $g(x) = -(b/a)|x|$ is an end behavior model for

$$f(x) = -(b/a)\sqrt{x^2 - a^2}.$$

3.8 Derivatives of Inverse Trigonometric Functions

What you'll learn about

- Derivatives of Inverse Functions
- Derivative of the Arcsine
- Derivative of the Arctangent
- Derivative of the Arcsecant
- Derivatives of the Other Three

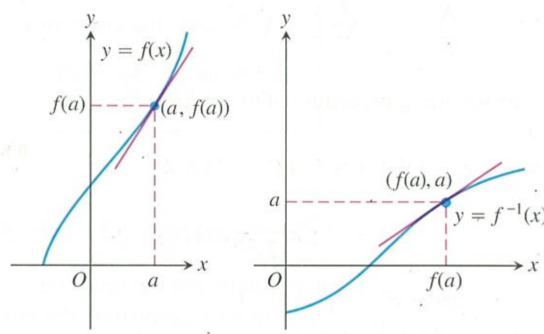
. . . and why

The relationship between the graph of a function and its inverse allows us to see the relationship between their derivatives.

Derivatives of Inverse Functions

In Section 1.5 we learned that the graph of the inverse of a function f can be obtained by reflecting the graph of f across the line $y = x$. If we combine that with our understanding of what makes a function differentiable, we can gain some quick insights into the differentiability of inverse functions.

As Figure 3.52 suggests, the reflection of a continuous curve with no cusps or corners will be another continuous curve with no cusps or corners. Indeed, if there is a tangent line to the graph of f at the point $(a, f(a))$, then that line will reflect across $y = x$ to become a tangent line to the graph of f^{-1} at the point $(f(a), a)$. We can even see geometrically that the *slope* of the reflected tangent line (when it exists and is not zero) will be the *reciprocal* of the slope of the original tangent line, since a change in y becomes a change in x in the reflection, and a change in x becomes a change in y.



The slopes are reciprocal: $\left.\dfrac{df^{-1}}{dx}\right|_{f(a)} = \dfrac{1}{\left.\dfrac{df}{dx}\right|_{a}}$

Figure 3.52 The graphs of a function and its inverse. Notice that the tangent lines have reciprocal slopes.

All of this serves as an introduction to the following theorem, which we will assume as we proceed to find derivatives of inverse functions. Although the essentials of the proof are illustrated in the geometry of Figure 3.52, a careful analytic proof is more appropriate for an advanced calculus text and will be omitted here.

THEOREM 3 Derivatives of Inverse Functions

If f is differentiable at every point of an interval I and df/dx is never zero on I, then f has an inverse and f^{-1} is differentiable at every point of the interval $f(I)$.

EXPLORATION 1 **Finding a Derivative on an Inverse Graph Geometrically**

Let $f(x) = x^5 + 2x - 1$. Since the point (1, 2) is on the graph of f, it follows that the point (2, 1) is on the graph of f^{-1}. Can you find

$$\frac{df^{-1}}{dx}(2),$$

the value of df^{-1}/dx at 2, without knowing a formula for f^{-1}?

1. Graph $f(x) = x^5 + 2x - 1$. A function must be one-to-one to have an inverse function. Is this function one-to-one?
2. Find $f'(x)$. How could this derivative help you to conclude that f has an inverse?
3. Reflect the graph of f across the line $y = x$ to obtain a graph of f^{-1}.
4. Sketch the tangent line to the graph of f^{-1} at the point (2, 1). Call it L.
5. Reflect the line L across the line $y = x$. At what point is the reflection of L tangent to the graph of f?
6. What is the slope of the reflection of L?
7. What is the slope of L?
8. What is $\dfrac{df^{-1}}{dx}(2)$?

Derivative of the Arcsine

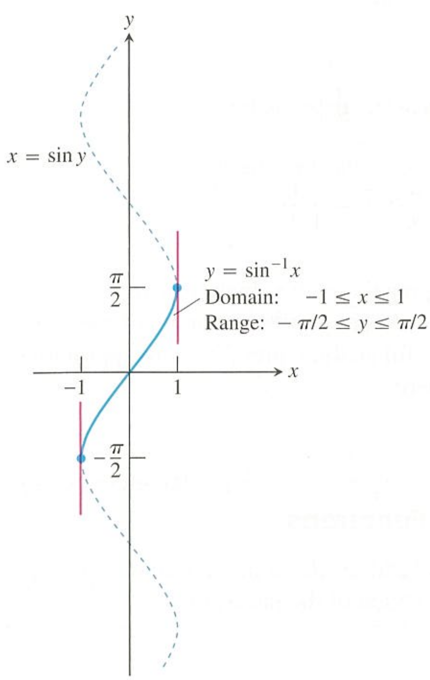


Figure 3.53 The graph of $y = \sin^{-1} x$ has vertical tangents $x = -1$ and $x = 1$.

We know that the function $x = \sin y$ is differentiable in the open interval $-\pi/2 < y < \pi/2$ and that its derivative, the cosine, is positive there. Theorem 3 therefore assures us that the inverse function $y = \sin^{-1}(x)$ (the *arcsine* of x) is differentiable throughout the interval $-1 < x < 1$. We cannot expect it to be differentiable at $x = -1$ or $x = 1$, however, because the tangents to the graph are vertical at these points (Figure 3.53).

We find the derivative of $y = \sin^{-1}(x)$ as follows:

$$y = \sin^{-1} x$$

$$\sin y = x \qquad \text{Inverse function relationship}$$

$$\frac{d}{dx}(\sin y) = \frac{d}{dx}x \qquad \text{Differentiate both sides.}$$

$$\cos y \frac{dy}{dx} = 1 \qquad \text{Implicit differentiation}$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

The division in the last step is safe because $\cos y \neq 0$ for $-\pi/2 < y < \pi/2$. In fact, $\cos y$ is *positive* for $-\pi/2 < y < \pi/2$, so we can replace $\cos y$ with $\sqrt{1 - (\sin y)^2}$, which is $\sqrt{1 - x^2}$. Thus

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}.$$

If u is a differentiable function of x with $|u| < 1$, we apply the Chain Rule to get

$$\frac{d}{dx}\sin^{-1} u = \frac{1}{\sqrt{1 - u^2}}\frac{du}{dx}, \qquad |u| < 1.$$

EXAMPLE 1 Applying the Formula

$$\frac{d}{dx}(\sin^{-1} x^2) = \frac{1}{\sqrt{1-(x^2)^2}} \cdot \frac{d}{dx}(x^2) = \frac{2x}{\sqrt{1-x^4}}$$

Now try Exercise 3.

Derivative of the Arctangent

Although the function $y = \sin^{-1}(x)$ has a rather narrow domain of $[-1, 1]$, the function $y = \tan^{-1} x$ is defined for all real numbers, and is differentiable for all real numbers, as we will now see. The differentiation proceeds exactly as with the arcsine function.

$$y = \tan^{-1} x$$

$$\tan y = x \qquad \text{Inverse function relationship}$$

$$\frac{d}{dx}(\tan y) = \frac{d}{dx}x$$

$$\sec^2 y \frac{dy}{dx} = 1 \qquad \text{Implicit differentiation}$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y}$$

$$= \frac{1}{1+(\tan y)^2} \qquad \text{Trig identity: } \sec^2 y = 1 + \tan^2 y$$

$$= \frac{1}{1+x^2}$$

The derivative is defined for all real numbers. If u is a differentiable function of x, we get the Chain Rule form:

$$\frac{d}{dx}\tan^{-1} u = \frac{1}{1+u^2}\frac{du}{dx}.$$

EXAMPLE 2 A Moving Particle

A particle moves along the x-axis so that its position at any time $t \geq 0$ is $x(t) = \tan^{-1}\sqrt{t}$. What is the velocity of the particle when $t = 16$?

SOLUTION

$$v(t) = \frac{d}{dt}\tan^{-1}\sqrt{t} = \frac{1}{1+(\sqrt{t})^2} \cdot \frac{d}{dt}\sqrt{t} = \frac{1}{1+t} \cdot \frac{1}{2\sqrt{t}}$$

When $t = 16$, the velocity is $v(16) = \frac{1}{1+16} \cdot \frac{1}{2\sqrt{16}} = \frac{1}{136}$.

Now try Exercise 11.

Derivative of the Arcsecant

We find the derivative of $y = \sec^{-1} x$, $|x| > 1$, beginning as we did with the other inverse trigonometric functions.

$$y = \sec^{-1} x$$

$$\sec y = x \qquad \text{Inverse function relationship}$$

$$\frac{d}{dx}(\sec y) = \frac{d}{dx}x$$

$$\sec y \tan y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sec y \tan y}$$

Since $|x| > 1$, y lies in $(0, \pi/2) \cup (\pi/2, \pi)$ and $\sec y \tan y \neq 0$.

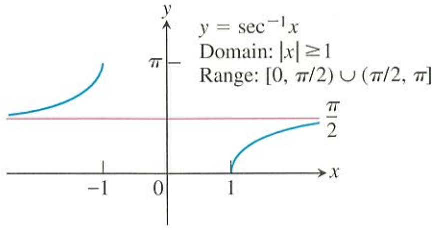


Figure 3.54 The slope of the curve $y = \sec^{-1} x$ is positive for both $x < -1$ and $x > 1$.

To express the result in terms of x, we use the relationships

$$\sec y = x \quad \text{and} \quad \tan y = \pm\sqrt{\sec^2 y - 1} = \pm\sqrt{x^2 - 1}$$

to get

$$\frac{dy}{dx} = \pm\frac{1}{x\sqrt{x^2 - 1}}.$$

Can we do anything about the $\pm$ sign? A glance at Figure 3.54 shows that the slope of the graph $y = \sec^{-1} x$ is always positive. That must mean that

$$\frac{d}{dx}\sec^{-1} x = \begin{cases} +\dfrac{1}{x\sqrt{x^2 - 1}} & \text{if } x > 1 \\ -\dfrac{1}{x\sqrt{x^2 - 1}} & \text{if } x < -1. \end{cases}$$

With the absolute value symbol we can write a single expression that eliminates the "$\pm$" ambiguity:

$$\frac{d}{dx}\sec^{-1} x = \frac{1}{|x|\sqrt{x^2 - 1}}.$$

If u is a differentiable function of x with $|u| > 1$, we have the formula

$$\frac{d}{dx}\sec^{-1} u = \frac{1}{|u|\sqrt{u^2 - 1}}\frac{du}{dx}, \qquad |u| > 1.$$

EXAMPLE 3 Using the Formula

$$\begin{aligned} \frac{d}{dx}\sec^{-1}(5x^4) &= \frac{1}{|5x^4|\sqrt{(5x^4)^2 - 1}}\frac{d}{dx}(5x^4) \\ &= \frac{1}{5x^4\sqrt{25x^8 - 1}}(20x^3) \\ &= \frac{4}{x\sqrt{25x^8 - 1}} \end{aligned}$$

Now try Exercise 17.

Derivatives of the Other Three

We could use the same technique to find the derivatives of the other three inverse trigonometric functions: arccosine, arccotangent, and arccosecant, but there is a much easier way, thanks to the following identities.

Inverse Function–Inverse Cofunction Identities

$$\cos^{-1} x = \pi/2 - \sin^{-1} x$$
$$\cot^{-1} x = \pi/2 - \tan^{-1} x$$
$$\csc^{-1} x = \pi/2 - \sec^{-1} x$$

It follows easily that the derivatives of the inverse cofunctions are the negatives of the derivatives of the corresponding inverse functions (see Exercises 32–34).

You have probably noticed by now that most calculators do not have buttons for $\cot^{-1}$, $\sec^{-1}$, or $\csc^{-1}$. They are not needed because of the following identities:

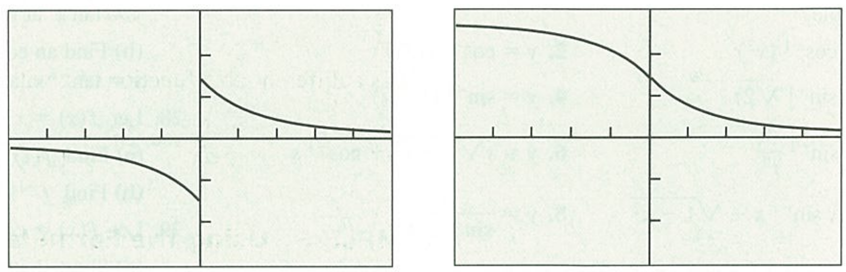
Calculator Conversion Identities

$$\sec^{-1} x = \cos^{-1}(1/x)$$

$$\cot^{-1} x = \pi/2 - \tan^{-1} x$$

$$\csc^{-1} x = \sin^{-1}(1/x)$$

Notice that we do not use $\tan^{-1}(1/x)$ as an identity for $\cot^{-1} x$. A glance at the graphs of $y = \tan^{-1}(1/x)$ and $y = \pi/2 - \tan^{-1} x$ reveals the problem (Figure 3.55).



[–5, 5] by [–3, 3]
(a)

[–5, 5] by [–3, 3]
(b)

Figure 3.55 The graphs of (a) $y = \tan^{-1}(1/x)$ and (b) $y = \pi/2 - \tan^{-1} x$. The graph in (b) is the same as the graph of $y = \cot^{-1} x$.

We cannot replace $\cot^{-1} x$ by the function $y = \tan^{-1}(1/x)$ in the identity for the inverse functions and inverse cofunctions, and so it is not the function we want for $\cot^{-1} x$. The ranges of the inverse trigonometric functions have been chosen in part to make the two sets of identities above hold.

EXAMPLE 4 A Tangent Line to the Arccotangent Curve

Find an equation for the line tangent to the graph of $y = \cot^{-1} x$ at $x = -1$.

SOLUTION

First, we note that

$$\cot^{-1}(-1) = \pi/2 - \tan^{-1}(-1) = \pi/2 - (-\pi/4) = 3\pi/4.$$

The slope of the tangent line is

$$\left.\frac{dy}{dx}\right|_{x=-1} = -\left.\frac{1}{1+x^2}\right|_{x=-1} = -\frac{1}{1+(-1)^2} = -\frac{1}{2}.$$

So the tangent line has equation $y - 3\pi/4 = (-1/2)(x + 1)$.

Now try Exercise 23.

Quick Review 3.8 *(For help, go to Sections 1.2, 1.5, and 1.6.)*

In Exercises 1–5, give the *domain* and *range* of the function, and evaluate the function at $x = 1$.

1. $y = \sin^{-1} x$
2. $y = \cos^{-1} x$
3. $y = \tan^{-1} x$
4. $y = \sec^{-1} x$
5. $y = \tan(\tan^{-1} x)$

In Exercises 6–10, find the inverse of the given function.

6. $y = 3x - 8$
7. $y = \sqrt[3]{x + 5}$
8. $y = \dfrac{8}{x}$
9. $y = \dfrac{3x - 2}{x}$
10. $y = \arctan(x/3)$

Section 3.8 Exercises

In Exercises 1–8, find the derivative of y with respect to the appropriate variable.

1. $y = \cos^{-1}(x^2)$
2. $y = \cos^{-1}(1/x)$
3. $y = \sin^{-1}\sqrt{2}\,t$
4. $y = \sin^{-1}(1 - t)$
5. $y = \sin^{-1}\dfrac{3}{t^2}$
6. $y = s\sqrt{1 - s^2} + \cos^{-1} s$
7. $y = x\sin^{-1} x + \sqrt{1 - x^2}$
8. $y = \dfrac{1}{\sin^{-1}(2x)}$

In Exercises 9–12, a particle moves along the x-axis so that its position at any time $t \geq 0$ is given by $x(t)$. Find the velocity at the indicated value of t.

9. $x(t) = \sin^{-1}\left(\dfrac{t}{4}\right), \quad t = 3$
10. $x(t) = \sin^{-1}\left(\dfrac{\sqrt{t}}{4}\right), \quad t = 4$
11. $x(t) = \tan^{-1} t, \quad t = 2$
12. $x(t) = \tan^{-1}(t^2), \quad t = 1$

In Exercises 13–22, find the derivatives of y with respect to the appropriate variable.

13. $y = \sec^{-1}(2s + 1)$
14. $y = \sec^{-1} 5s$
15. $y = \csc^{-1}(x^2 + 1), \ x > 0$
16. $y = \csc^{-1} x/2$
17. $y = \sec^{-1}\dfrac{1}{t}, \quad 0 < t < 1$
18. $y = \cot^{-1}\sqrt{t}$
19. $y = \cot^{-1}\sqrt{t - 1}$
20. $y = \sqrt{s^2 - 1} - \sec^{-1} s$
21. $y = \tan^{-1}\sqrt{x^2 - 1} + \csc^{-1} x, \quad x > 1$
22. $y = \cot^{-1}\dfrac{1}{x} - \tan^{-1} x$

In Exercises 23–26, find an equation for the tangent to the graph of y at the indicated point.

23. $y = \sec^{-1} x, \quad x = 2$
24. $y = \tan^{-1} x, \quad x = 2$
25. $y = \sin^{-1}\left(\dfrac{x}{4}\right), \quad x = 3$
26. $y = \tan^{-1}(x^2), \quad x = 1$

27. **(a)** Find an equation for the line tangent to the graph of $y = \tan x$ at the point $(\pi/4, 1)$.
 (b) Find an equation for the line tangent to the graph of $y = \tan^{-1} x$ at the point $(1, \pi/4)$.

28. Let $f(x) = x^5 + 2x^3 + x - 1$.
 (a) Find $f(1)$ and $f'(1)$.
 (b) Find $f^{-1}(3)$ and $(f^{-1})'(3)$.

29. Let $f(x) = \cos x + 3x$.
 (a) Show that f has a differentiable inverse.
 (b) Find $f(0)$ and $f'(0)$.
 (c) Find $f^{-1}(1)$ and $(f^{-1})'(1)$.

30. **Group Activity** Graph the function $f(x) = \sin^{-1}(\sin x)$ in the viewing window $[-2\pi, 2\pi]$ by $[-4, 4]$. Then answer the following questions:
 (a) What is the domain of f?
 (b) What is the range of f?
 (c) At which points is f not differentiable?
 (d) Sketch a graph of $y = f'(x)$ without using NDER or computing the derivative.
 (e) Find $f'(x)$ algebraically. Can you reconcile your answer with the graph in part (d)?

31. **Group Activity** A particle moves along the x-axis so that its position at any time $t \geq 0$ is given by $x = \arctan t$.
 (a) Prove that the particle is always moving to the right.
 (b) Prove that the particle is always decelerating.
 (c) What is the limiting position of the particle as t approaches infinity?

In Exercises 32–34, use the inverse function–inverse cofunction identities to derive the formula for the derivative of the function.

32. arccosine
33. arccotangent
34. arccosecant

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

35. True or False The domain of $y = \sin^{-1}x$ is $-1 \le x \le 1$. Justify your answer.

36. True or False The domain of $y = \tan^{-1}x$ is $-1 \le x \le 1$. Justify your answer.

37. Multiple Choice Which of the following is $\frac{d}{dx}\sin^{-1}\left(\frac{x}{2}\right)$?

(A) $-\frac{2}{\sqrt{4-x^2}}$ (B) $-\frac{1}{\sqrt{4-x^2}}$ (C) $\frac{2}{4+x^2}$

(D) $\frac{2}{\sqrt{4-x^2}}$ (E) $\frac{1}{\sqrt{4-x^2}}$

38. Multiple Choice Which of the following is $\frac{d}{dx}\tan^{-1}(3x)$?

(A) $-\frac{3}{1+9x^2}$ (B) $-\frac{1}{1+9x^2}$ (C) $\frac{1}{1+9x^2}$

(D) $\frac{3}{1+9x^2}$ (E) $\frac{3}{\sqrt{1-9x^2}}$

39. Multiple Choice Which of the following is $\frac{d}{dx}\sec^{-1}(x^2)$?

(A) $\frac{2}{x\sqrt{x^4-1}}$ (B) $\frac{2}{x\sqrt{x^2-1}}$ (C) $\frac{2}{x\sqrt{1-x^4}}$

(D) $\frac{2}{x\sqrt{1-x^2}}$ (E) $\frac{2x}{\sqrt{1-x^4}}$

40. Multiple Choice Which of the following is the slope of the tangent line to $y = \tan^{-1}(2x)$ at $x = 1$?

(A) $-2/5$ (B) $1/5$ (C) $2/5$ (D) $5/2$ (E) 5

Explorations

In Exercises 41–46, find **(a)** the right end behavior model, **(b)** the left end behavior model, and **(c)** any horizontal tangents for the function if they exist.

41. $y = \tan^{-1} x$ **42.** $y = \cot^{-1} x$

43. $y = \sec^{-1} x$ **44.** $y = \csc^{-1} x$

45. $y = \sin^{-1} x$ **46.** $y = \cos^{-1} x$

Extending the Ideas

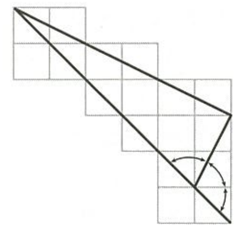
47. *Identities* Confirm the following identities for $x > 0$.

(a) $\cos^{-1} x + \sin^{-1} x = \pi/2$

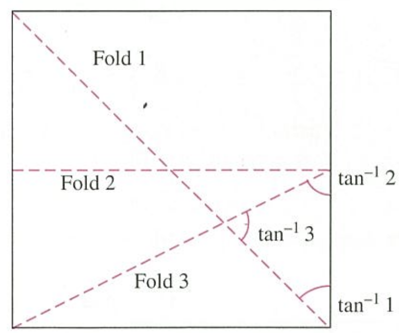
(b) $\tan^{-1} x + \cot^{-1} x = \pi/2$

(c) $\sec^{-1} x + \csc^{-1} x = \pi/2$

48. *Proof Without Words* The figure gives a proof without words that $\tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3 = \pi$. Explain what is going on.



49. *(Continuation of Exercise 48)* Here is a way to construct $\tan^{-1} 1$, $\tan^{-1} 2$, and $\tan^{-1} 3$ by folding a square of paper. Try it and explain what is going on.



3.9 Derivatives of Exponential and Logarithmic Functions

What you'll learn about

- Derivative of e^x
- Derivative of a^x
- Derivative of $\ln x$
- Derivative of $\log_a x$
- Power Rule for Arbitrary Real Powers

. . . and why

The relationship between exponential and logarithmic functions provides a powerful differentiation tool called logarithmic differentiation.

Derivative of e^x

At the end of the brief review of exponential functions in Section 1.3, we mentioned that the function $y = e^x$ was a particularly important function for modeling exponential growth. The number e was defined in that section to be the limit of $(1 + 1/x)^x$ as $x \to \infty$. This intriguing number shows up in other interesting limits as well, but the one with the most interesting implications for the *calculus* of exponential functions is this one:

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1.$$

(The graph and the table in Figure 3.56 provide strong support for this limit being 1. A formal algebraic proof that begins with our limit definition of e would require some rather subtle limit arguments, so we will not include one here.)

The fact that the limit is 1 creates a remarkable relationship between the function e^x and its derivative, as we will now see.

$$\begin{aligned}\frac{d}{dx}(e^x) &= \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} \\ &= \lim_{h \to 0} \frac{e^x \cdot e^h - e^x}{h} \\ &= \lim_{h \to 0} \left(e^x \cdot \frac{e^h - 1}{h}\right) \\ &= e^x \cdot \lim_{h \to 0} \left(\frac{e^h - 1}{h}\right) \\ &= e^x \cdot 1 \\ &= e^x\end{aligned}$$

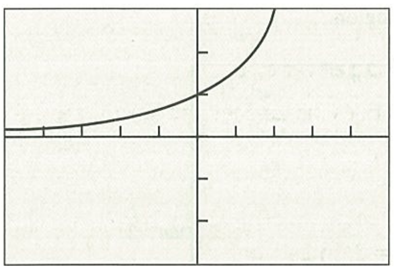
In other words, the derivative of this particular function is itself!

$$\frac{d}{dx}(e^x) = e^x$$

If u is a differentiable function of x, then we have

$$\frac{d}{dx}e^u = e^u \frac{du}{dx}.$$

We will make extensive use of this formula when we study exponential growth and decay in Chapter 6.



[−4.9, 4.9] by [−2.9, 2.9]

(a)

X	Y1	
−.03	.98515	
−.02	.99007	
−.01	.99502	
0	ERROR	
.01	1.005	
.02	1.0101	
.03	1.0152	
X=0		

(b)

Figure 3.56 (a) The graph and (b) the table support the conclusion that

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1.$$

EXAMPLE 1 Using the Formula

Find dy/dx if $y = e^{(x+x^2)}$.

SOLUTION

Let $u = x + x^2$ then $y = e^u$. Then

$$\frac{dy}{dx} = e^u \frac{du}{dx}, \quad \text{and} \quad \frac{du}{dx} = 1 + 2x.$$

Thus, $\dfrac{dy}{dx} = e^u \dfrac{du}{dx} = e^{(x+x^2)}(1 + 2x).$

Now try Exercise 9.

Is any other function its own derivative?

The zero function is also its own derivative, but this hardly seems worth mentioning. (Its value is always 0 and its slope is always 0.) In addition to e^x, however, we can also say that any constant *multiple* of e^x is its own derivative:

$$\frac{d}{dx}(c \cdot e^x) = c \cdot e^x.$$

The next obvious question is whether there are still *other* functions that are their own derivatives, and this time the answer is no. The only functions that satisfy the condition $dy/dx = y$ are functions of the form $y = ke^x$ (and notice that the zero function can be included in this category). We will prove this significant fact in Chapter 6.

Derivative of a^x

What about an exponential function with a base other than e? We will assume that the base is positive and different from 1, since negative numbers to arbitrary real powers are not always real numbers, and $y = 1^x$ is a constant function.

If $a > 0$ and $a \neq 1$, we can use the properties of logarithms to write a^x in terms of e^x. The formula for doing so is

$$a^x = e^{x \ln a}. \qquad e^{x \ln a} = e^{\ln(a^x)} = a^x$$

We can then find the derivative of a^x with the Chain Rule.

$$\frac{d}{dx}a^x = \frac{d}{dx}e^{x \ln a} = e^{x \ln a} \cdot \frac{d}{dx}(x \ln a) = e^{x \ln a} \cdot \ln a = a^x \ln a$$

Thus, if u is a differentiable function of x, we get the following rule.

For $a > 0$ and $a \neq 1$,

$$\frac{d}{dx}(a^u) = a^u \ln a \frac{du}{dx}.$$

EXAMPLE 2 Reviewing the Algebra of Logarithms

At what point on the graph of the function $y = 2^t - 3$ does the tangent line have slope 21?

SOLUTION

The slope is the derivative:

$$\frac{d}{dt}(2^t - 3) = 2^t \cdot \ln 2 - 0 = 2^t \ln 2.$$

We want the value of t for which $2^t \ln 2 = 21$. We could use the solver on the calculator, but we will use logarithms for the sake of review.

$$2^t \ln 2 = 21$$

$$2^t = \frac{21}{\ln 2}$$

$$\ln 2^t = \ln\left(\frac{21}{\ln 2}\right) \qquad \text{Logarithm of both sides}$$

$$t \cdot \ln 2 = \ln 21 - \ln(\ln 2) \qquad \text{Properties of logarithms}$$

$$t = \frac{\ln 21 - \ln(\ln 2)}{\ln 2}$$

$$t \approx 4.921$$

$$y = 2^t - 3 \approx 27.297 \qquad \text{Using the stored value of } t$$

The point is approximately (4.9, 27.3). *Now try Exercise 29.*

EXPLORATION 1 Leaving Milk on the Counter

A glass of cold milk from the refrigerator is left on the counter on a warm summer day. Its temperature y (in degrees Fahrenheit) after sitting on the counter t minutes is

$$y = 72 - 30(0.98)^t.$$

Answer the following questions by interpreting y and dy/dt.

1. What is the temperature of the refrigerator? How can you tell?
2. What is the temperature of the room? How can you tell?
3. When is the milk warming up the fastest? How can you tell?
4. Determine algebraically when the temperature of the milk reaches 55°F.
5. At what rate is the milk warming when its temperature is 55°F? Answer with an appropriate unit of measure.

Derivative of ln *x*

Now that we know the derivative of e^x, it is relatively easy to find the derivative of its inverse function, $\ln x$.

$$y = \ln x$$

$$e^y = x \quad \text{Inverse function relationship}$$

$$\frac{d}{dx}(e^y) = \frac{d}{dx}(x) \quad \text{Differentiate implicitly.}$$

$$e^y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

This equation answers what was once a perplexing problem: Is there a function with derivative x^{-1}? All of the other power functions follow the Power Rule,

$$\frac{d}{dx}x^n = nx^{n-1}.$$

However, this formula is not much help if one is looking for a function with x^{-1} as its derivative! Now we know why: The function we should be looking for is not a power function at all; it is the natural logarithm function.

If u is a differentiable function of x and $u > 0$,

$$\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}.$$

EXAMPLE 3 A Tangent through the Origin

A line with slope m passes through the origin and is tangent to the graph of $y = \ln x$. What is the value of m?

SOLUTION

This problem is a little harder than it looks, since we do not know the point of tangency. However, we do know two important facts about that point:

1. it has coordinates $(a, \ln a)$ for some positive a, and
2. the tangent line there has slope $m = 1/a$ (Figure 3.57).

Since the tangent line passes through the origin, its slope is

$$m = \frac{\ln a - 0}{a - 0} = \frac{\ln a}{a}.$$

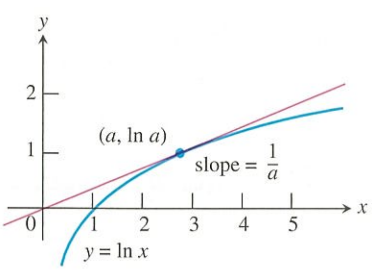


Figure 3.57 The tangent line intersects the curve at some point $(a, \ln a)$, where the slope of the curve is $1/a$. (Example 3)

continued

Setting these two formulas for m equal to each other, we have

$$\frac{\ln a}{a} = \frac{1}{a}$$
$$\ln a = 1$$
$$e^{\ln a} = e^1$$
$$a = e$$
$$m = \frac{1}{e}.$$

Now try Exercise 31.

Derivative of $\log_a x$

To find the derivative of $\log_a x$ for an arbitrary base ($a > 0$, $a \neq 1$), we use the change-of-base formula for logarithms to express $\log_a x$ in terms of natural logarithms, as follows:

$$\log_a x = \frac{\ln x}{\ln a}.$$

The rest is easy:

$$\frac{d}{dx}\log_a x = \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right)$$
$$= \frac{1}{\ln a} \cdot \frac{d}{dx}\ln x \quad \text{Since } \ln a \text{ is a constant}$$
$$= \frac{1}{\ln a} \cdot \frac{1}{x}$$
$$= \frac{1}{x \ln a}.$$

So, if u is a differentiable function of x and $u > 0$, the formula is as follows.

> For $a > 0$ and $a \neq 1$,
> $$\frac{d}{dx}\log_a u = \frac{1}{u \ln a}\frac{du}{dx}.$$

EXAMPLE 4 Going the Long Way with the Chain Rule

Find dy/dx if $y = \log_a a^{\sin x}$.

SOLUTION

Carefully working from the outside in, we apply the Chain Rule to get:

$$\frac{d}{dx}(\log_a a^{\sin x}) = \frac{1}{a^{\sin x}\ln a} \cdot \frac{d}{dx}(a^{\sin x}) \quad \log_a u,\ u = a^{\sin x}$$
$$= \frac{1}{a^{\sin x}\ln a} \cdot a^{\sin x}\ln a \cdot \frac{d}{dx}(\sin x) \quad a^u,\ u = \sin x$$
$$= \frac{a^{\sin x}\ln a}{a^{\sin x}\ln a} \cdot \cos x$$
$$= \cos x.$$

Now try Exercise 23.

We could have saved ourselves a lot of work in Example 4 if we had noticed at the beginning that $\log_a a^{\sin x}$, being the composite of inverse functions, is equal to $\sin x$. It is always a good idea to simplify functions *before* differentiating, wherever possible. On the other hand, it is comforting to know that all these rules do work if applied correctly.

Power Rule for Arbitrary Real Powers

We are now ready to prove the Power Rule in its final form. As long as $x > 0$, we can write any real power of x as a power of e, specifically

$$x^n = e^{n \ln x}.$$

This enables us to differentiate x^n for any real power n, as follows:

$$\begin{aligned}
\frac{d}{dx}(x^n) &= \frac{d}{dx}(e^{n \ln x}) \\
&= e^{n \ln x} \cdot \frac{d}{dx}(n \ln x) \qquad e^u,\ u = n \ln x \\
&= e^{n \ln x} \cdot \frac{n}{x} \\
&= x^n \cdot \frac{n}{x} \\
&= nx^{n-1}.
\end{aligned}$$

The Chain Rule extends this result to the Power Rule's final form.

RULE 10 Power Rule for Arbitrary Real Powers

If u is a positive differentiable function of x and n is any real number, then u^n is a differentiable function of x, and

$$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}.$$

EXAMPLE 5 Using the Power Rule in all its Power

(a) If $y = x^{\sqrt{2}}$, then

$$\frac{dy}{dx} = \sqrt{2}x^{(\sqrt{2}-1)}.$$

(b) If $y = (2 + \sin 3x)^{\pi}$, then

$$\begin{aligned}
\frac{d}{dx}(2 + \sin 3x)^{\pi} &= \pi(2 + \sin 3x)^{\pi-1}(\cos 3x) \cdot 3 \\
&= 3\pi(2 + \sin 3x)^{\pi-1}(\cos 3x).
\end{aligned}$$

Now try Exercise 35.

EXAMPLE 6 Finding Domain

If $f(x) = \ln(x - 3)$, find $f'(x)$. State the domain of f'.

SOLUTION

The domain of f is $(3, \infty)$ and

$$f'(x) = \frac{1}{x-3}.$$

continued

The domain of f' appears to be all $x \neq 3$. However, since f is not defined for $x < 3$, neither is f'. Thus,

$$f'(x) = \frac{1}{x-3}, \qquad x > 3.$$

That is, the domain of f' is $(3, \infty)$. *Now try Exercise 37.*

Sometimes the properties of logarithms can be used to simplify the differentiation process, even if we must introduce the logarithms ourselves as a step in the process. Example 7 shows a clever way to differentiate $y = x^x$ for $x > 0$.

EXAMPLE 7 Logarithmic Differentiation

Find dy/dx for $y = x^x$, $x > 0$.

SOLUTION

$$y = x^x$$

$$\ln y = \ln x^x \qquad \text{Logs of both sides}$$

$$\ln y = x \ln x \qquad \text{Property of logs}$$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}(x \ln x) \qquad \text{Differentiate implicitly.}$$

$$\frac{1}{y}\frac{dy}{dx} = 1 \cdot \ln x + x \cdot \frac{1}{x}$$

$$\frac{dy}{dx} = y(\ln x + 1)$$

$$\frac{dy}{dx} = x^x(\ln x + 1)$$

Now try Exercise 43.

EXAMPLE 8 How Fast does a Flu Spread?

The spread of a flu in a certain school is modeled by the equation

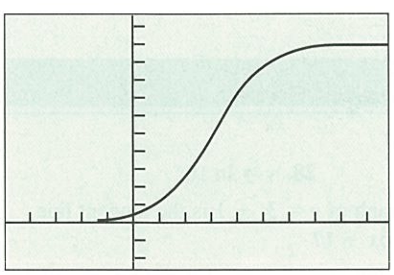
$$P(t) = \frac{100}{1 + e^{3-t}},$$

where $P(t)$ is the total number of students infected t days after the flu was first noticed. Many of them may already be well again at time t.

(a) Estimate the initial number of students infected with the flu.

(b) How fast is the flu spreading after 3 days?

(c) When will the flu spread at its maximum rate? What is this rate?



[–5, 10] by [–25, 120]

Figure 3.58 The graph of

$$P(t) = \frac{100}{1 + e^{3-t}},$$

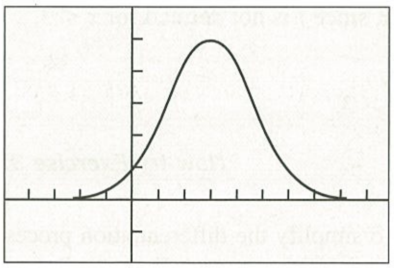
modeling the spread of a flu. (Example 8)

SOLUTION

The graph of P as a function of t is shown in Figure 3.58.

(a) $P(0) = 100/(1 + e^3) = 5$ students (to the nearest whole number).

continued



[–5, 10] by [–10, 30]

Figure 3.59 The graph of dP/dt, the rate of spread of the flu in Example 8. The graph of P is shown in Figure 3.58.

(b) To find the rate at which the flu spreads, we find dP/dt. To find dP/dt, we need to invoke the Chain Rule twice:

$$\frac{dP}{dt} = \frac{d}{dt}(100(1 + e^{3-t})^{-1}) = 100 \cdot (-1)(1 + e^{3-t})^{-2} \cdot \frac{d}{dt}(1 + e^{3-t})$$

$$= -100(1 + e^{3-t})^{-2} \cdot \left(0 + e^{3-t} \cdot \frac{d}{dt}(3 - t)\right)$$

$$= -100(1 + e^{3-t})^{-2}(e^{3-t} \cdot (-1))$$

$$= \frac{100e^{3-t}}{(1 + e^{3-t})^2}$$

At $t = 3$, then, $dP/dt = 100/4 = 25$. The flu is spreading to 25 students per day.

(c) We could estimate when the flu is spreading the fastest by seeing where the graph of $y = P(t)$ has the steepest upward slope, but we can answer both the "when" and the "what" parts of this question most easily by finding the maximum point on the graph of the derivative (Figure 3.59).

We see by tracing on the curve that the maximum rate occurs at about 3 days, when (as we have just calculated) the flu is spreading at a rate of 25 students per day.

Now try Exercise 51.

Quick Review 3.9 *(For help, go to Sections 1.3 and 1.5.)*

1. Write $\log_5 8$ in terms of natural logarithms.
2. Write 7^x as a power of e.

In Exercises 3–7, simplify the expression using properties of exponents and logarithms.

3. $\ln(e^{\tan x})$
4. $\ln(x^2 - 4) - \ln(x + 2)$
5. $\log_2(8^{x-5})$
6. $(\log_4 x^{15})/(\log_4 x^{12})$
7. $3 \ln x - \ln 3x + \ln(12x^2)$

In Exercises 8–10, solve the equation algebraically using logarithms. Give an *exact* answer, such as $(\ln 2)/3$, and also an approximate answer to the nearest hundredth.

8. $3^x = 19$
9. $5^t \ln 5 = 18$
10. $3^{x+1} = 2^x$

Section 3.9 Exercises

In Exercises 1–28, find dy/dx. Remember that you can use NDER to support your computations.

1. $y = 2e^x$
2. $y = e^{2x}$
3. $y = e^{-x}$
4. $y = e^{-5x}$
5. $y = e^{2x/3}$
6. $y = e^{-x/4}$
7. $y = xe^2 - e^x$
8. $y = x^2e^x - xe^x$
9. $y = e^{\sqrt{x}}$
10. $y = e^{(x^2)}$
11. $y = 8^x$
12. $y = 9^{-x}$
13. $y = 3^{\csc x}$
14. $y = 3^{\cot x}$
15. $y = \ln(x^2)$
16. $y = (\ln x)^2$
17. $y = \ln(1/x)$
18. $y = \ln(10/x)$
19. $y = \ln(\ln x)$
20. $y = x \ln x - x$
21. $y = \log_4 x^2$
22. $y = \log_5 \sqrt{x}$
23. $y = \log_2(1/x)$
24. $y = 1/\log_2 x$
25. $y = \ln 2 \cdot \log_2 x$
26. $y = \log_3(1 + x \ln 3)$
27. $y = \log_{10} e^x$
28. $y = \ln 10^x$
29. At what point on the graph of $y = 3^x + 1$ is the tangent line parallel to the line $y = 5x - 1$?
30. At what point on the graph of $y = 2e^x - 1$ is the tangent line perpendicular to the line $y = -3x + 2$?
31. A line with slope m passes through the origin and is tangent to $y = \ln(2x)$. What is the value of m?
32. A line with slope m passes through the origin and is tangent to $y = \ln(x/3)$. What is the value of m?

In Exercises 33–36, find dy/dx.

33. $y = x^{\pi}$
34. $y = x^{1+\sqrt{2}}$
35. $y = x^{-\sqrt{2}}$
36. $y = x^{1-e}$

In Exercises 37–42, find $f'(x)$ and state the domain of f'.

37. $f(x) = \ln(x + 2)$
38. $f(x) = \ln(2x + 2)$

39. $f(x) = \ln(2 - \cos x)$

40. $f(x) = \ln(x^2 + 1)$

41. $f(x) = \log_2(3x + 1)$

42. $f(x) = \log_{10}\sqrt{x + 1}$

Group Activity In Exercises 43–48, use the technique of logarithmic differentiation to find dy/dx.

43. $y = (\sin x)^x, \quad 0 < x < \pi/2$

44. $y = x^{\tan x}, \quad x > 0$

45. $y = \sqrt[5]{\dfrac{(x-3)^4(x^2+1)}{(2x+5)^3}}$

46. $y = \dfrac{x\sqrt{x^2+1}}{(x+1)^{2/3}}$

47. $y = x^{\ln x}$ **48.** $y = x^{(1/\ln x)}$

49. Find an equation for a line that is tangent to the graph of $y = e^x$ and goes through the origin.

50. Find an equation for a line that is normal to the graph of $y = xe^x$ and goes through the origin.

51. ***Spread of a Rumor*** The spread of a rumor in a certain school is modeled by the equation

$$P(t) = \frac{300}{1 + 2^{4-t}},$$

where $P(t)$ is the total number of students who have heard the rumor t days after the rumor first started to spread.

(a) Estimate the initial number of students who first heard the rumor.

(b) How fast is the rumor spreading after 4 days?

(c) When will the rumor spread at its maximum rate? What is that rate?

52. ***Spread of Flu*** The spread of flu in a certain school is modeled by the equation

$$P(t) = \frac{200}{1 + e^{5-t}},$$

where $P(t)$ is the total number of students infected t days after the flu first started to spread.

(a) Estimate the initial number of students infected with this flu.

(b) How fast is the flu spreading after 4 days?

(c) When will the flu spread at its maximum rate? What is that rate?

53. ***Radioactive Decay*** The amount A (in grams) of radioactive plutonium remaining in a 20-gram sample after t days is given by the formula

$$A = 20 \cdot (1/2)^{t/140}.$$

At what rate is the plutonium decaying when $t = 2$ days? Answer in appropriate units.

54. For any positive constant k, the derivative of $\ln(kx)$ is $1/x$. Prove this fact

(a) by using the Chain Rule.

(b) by using a property of logarithms and differentiating.

55. Let $f(x) = 2^x$.

(a) Find $f'(0)$.

(b) Use the definition of the derivative to write $f'(0)$ as a limit.

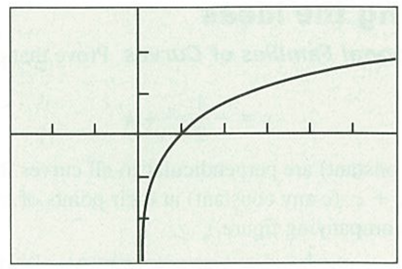
(c) Deduce the exact value of

$$\lim_{h \to 0} \frac{2^h - 1}{h}.$$

(d) What is the exact value of

$$\lim_{h \to 0} \frac{7^h - 1}{h}?$$

56. **Writing to Learn** The graph of $y = \ln x$ looks as though it might be approaching a horizontal asymptote. Write an argument based on the graph of $y = e^x$ to explain why it does not.



[–3, 6] by [–3, 3]

Standardized Test Questions

You should solve the following problems without using a graphing calculator.

57. **True or False** The derivative of $y = 2^x$ is 2^x. Justify your answer.

58. **True or False** The derivative of $y = e^{2x}$ is $2(\ln 2)e^{2x}$. Justify your answer.

59. **Multiple Choice** If a flu is spreading at the rate of

$$P(t) = \frac{150}{1 + e^{4-t}},$$

which of the following is the initial number of persons infected?

(A) 1 **(B)** 3 **(C)** 7 **(D)** 8 **(E)** 75

60. **Multiple Choice** Which of the following is the domain of $f'(x)$ if $f(x) = \log_2(x + 3)$?

(A) $x < -3$ **(B)** $x \le 3$ **(C)** $x \ne -3$ **(D)** $x > -3$ **(E)** $x \ge -3$

61. **Multiple Choice** Which of the following gives dy/dx if $y = \log_{10}(2x - 3)$?

(A) $\dfrac{2}{(2x-3)\ln 10}$ **(B)** $\dfrac{2}{2x-3}$ **(C)** $\dfrac{1}{(2x-3)\ln 10}$

(D) $\dfrac{1}{2x-3}$ **(E)** $\dfrac{1}{2x}$

62. **Multiple Choice** Which of the following gives the slope of the tangent line to the graph of $y = 2^{1-x}$ at $x = 2$?

(A) $-\dfrac{1}{2}$ **(B)** $\dfrac{1}{2}$ **(C)** -2 **(D)** 2 **(E)** $-\dfrac{\ln 2}{2}$

Exploration

63. Let $y_1 = a^x$, $y_2 = \text{NDER } y_1$, $y_3 = y_2/y_1$, and $y_4 = e^{y_3}$.

(a) Describe the graph of y_4 for $a = 2, 3, 4, 5$. Generalize your description to an arbitrary $a > 1$.

(b) Describe the graph of y_3 for $a = 2, 3, 4, 5$. Compare a table of values for y_3 for $a = 2, 3, 4, 5$ with $\ln a$. Generalize your description to an arbitrary $a > 1$.

(c) Explain how parts (a) and (b) support the statement

$$\frac{d}{dx} a^x = a^x \quad \text{if and only if} \quad a = e.$$

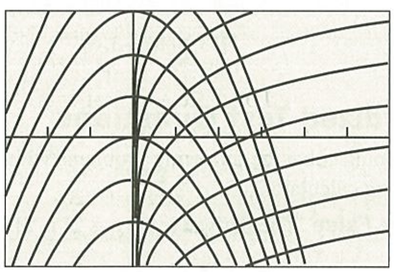
(d) Show algebraically that $y_1 = y_2$ if and only if $a = e$.

Extending the Ideas

64. ***Orthogonal Families of Curves*** Prove that all curves in the family

$$y = -\frac{1}{2}x^2 + k$$

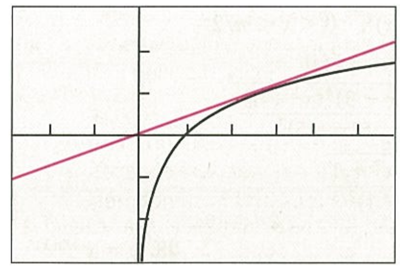
(k any constant) are perpendicular to all curves in the family $y = \ln x + c$ (c any constant) at their points of intersection. (See accompanying figure.)



[–3, 6] by [–3, 3]

65. ***Which is Bigger, π^e or e^π?*** Calculators have taken some of the mystery out of this once-challenging question. (Go ahead and check; you will see that it is a surprisingly close call.) You can answer the question without a calculator, though, by using the result from Example 3 of this section.

Recall from that example that the line through the origin tangent to the graph of $y = \ln x$ has slope $1/e$.



[–3, 6] by [–3, 3]

(a) Find an equation for this tangent line.

(b) Give an argument based on the graphs of $y = \ln x$ and the tangent line to explain why $\ln x < x/e$ for all positive $x \neq e$.

(c) Show that $\ln (x^e) < x$ for all positive $x \neq e$.

(d) Conclude that $x^e < e^x$ for all positive $x \neq e$.

(e) So which is bigger, π^e or e^π?

Quick Quiz for AP* Preparation: Sections 3.7–3.9

You may use a graphing calculator to solve the following problems.

1. Multiple Choice Which of the following gives dy/dx at $x = 1$ if $x^3 + 2xy = 9$?

(A) 11/2 **(B)** 5/2 **(C)** 3/2 **(D)** −5/2 **(E)** −11/2

2. Multiple Choice Which of the following gives dy/dx if $y = \cos^3(3x - 2)$?

(A) $-9 \cos^2 (3x - 2) \sin (3x - 2)$

(B) $-3 \cos^2 (3x - 2) \sin (3x - 2)$

(C) $9 \cos^2 (3x - 2) \sin (3x - 2)$

(D) $-9 \cos^2(3x - 2)$

(E) $-3 \cos^2 (3x - 2)$

3. Multiple Choice Which of the following gives dy/dx if $y = \sin^{-1} (2x)$?

(A) $-\frac{2}{\sqrt{1 - 4x^2}}$ **(B)** $-\frac{1}{\sqrt{1 - 4x^2}}$ **(C)** $\frac{2}{\sqrt{1 - 4x^2}}$

(D) $\frac{1}{\sqrt{1 - 4x^2}}$ **(E)** $\frac{2x}{1 + 4x^2}$

4. Free Response A curve in the xy-plane is defined by $xy^2 - x^3y = 6$.

(a) Find dy/dx.

(b) Find an equation for the tangent line at each point on the curve with x-coordinate 1.

(c) Find the x-coordinate of each point on the curve where the tangent line is vertical.

Calculus at Work

I work at Ramsey County Hospital and other community hospitals in the Minneapolis area, both with patients and in a laboratory. I have wanted to be a physician since I was about 12 years old, and I began attending medical school when I was 30 years old. I am now working in the field of internal medicine.

Cardiac patients are common in my field, especially in the diagnostic stages. One of the machines that is sometimes used in the emergency room to diagnose problems is called a Swan-Ganz catheter, named after its inventors Harold James Swan and William Ganz. The catheter is inserted into the pulmonary artery and then is hooked up to a cardiac monitor. A program measures cardiac output by looking at changes of slope in the curve. This information alerts me to left-sided heart failure.

Lupe Bolding, M.D.
Ramsey County Hospital
Minneapolis, MN

Chapter 3 Key Terms

acceleration (p. 130)
average velocity (p. 128)
Chain Rule (p. 149)
Constant Multiple Rule (p. 117)
Derivative of a Constant Function (p. 116)
derivative of f at a (p. 99)
differentiable function (p. 99)
differentiable on a closed interval (p. 104)
displacement (p. 128)
free-fall constants (p. 130)
implicit differentiation (p. 157)
instantaneous rate of change (p. 127)
instantaneous velocity (p. 128)
Intermediate Value Theorem for Derivatives (p. 113)
inverse function–inverse cofunction identities (p. 168)
jerk (p. 144)
left-hand derivative (p. 104)
local linearity (p. 110)
logarithmic differentiation (p. 177)
marginal cost (p. 134)
marginal revenue (p. 134)
nth derivative (p. 122)
normal to the surface (p. 159)
numerical derivative (NDER) (p. 111)
orthogonal curves (p. 154)
orthogonal families (p. 180)
Power Chain Rule (p. 151)
Power Rule for Arbitrary Real Powers (p. 176)
Power Rule for Negative Integer Powers of x (p. 121)
Power Rule for Positive Integer Powers of x (p. 116)
Power Rule for Rational Powers of x (p. 161)
Product Rule (p. 119)
Quotient Rule (p. 120)
right-hand derivative (p. 104)
sensitivity to change (p. 133)
simple harmonic motion (p. 143)
speed (p. 129)
Sum and Difference Rule (p. 117)
symmetric difference quotient (p. 111)
velocity (p. 128)

Chapter 3 Review Exercises

The collection of exercises marked in red could be used as a chapter test.

In Exercises 1–30, find the derivative of the function.

1. $y = x^5 - \frac{1}{8}x^2 + \frac{1}{4}x$
2. $y = 3 - 7x^3 + 3x^7$
3. $y = 2 \sin x \cos x$
4. $y = \frac{2x + 1}{2x - 1}$
5. $s = \cos(1 - 2t)$
6. $s = \cot \frac{2}{t}$
7. $y = \sqrt{x} + 1 + \frac{1}{\sqrt{x}}$
8. $y = x\sqrt{2x + 1}$
9. $r = \sec(1 + 3\theta)$
10. $r = \tan^2(3 - \theta^2)$
11. $y = x^2 \csc 5x$
12. $y = \ln \sqrt{x}$
13. $y = \ln(1 + e^x)$
14. $y = xe^{-x}$
15. $y = e^{(1 + \ln x)}$
16. $y = \ln(\sin x)$
17. $r = \ln(\cos^{-1} x)$
18. $r = \log_2(\theta^2)$
19. $s = \log_5(t - 7)$
20. $s = 8^{-t}$
21. $y = x^{\ln x}$
22. $y = \frac{(2x)2^x}{\sqrt{x^2 + 1}}$
23. $y = e^{\tan^{-1} x}$
24. $y = \sin^{-1}\sqrt{1 - u^2}$
25. $y = t \sec^{-1} t - \frac{1}{2}\ln t$
26. $y = (1 + t^2)\cot^{-1} 2t$
27. $y = z \cos^{-1} z - \sqrt{1 - z^2}$
28. $y = 2\sqrt{x - 1}\csc^{-1}\sqrt{x}$

29. $y = \csc^{-1}(\sec x),\ 0 \le x \le 2\pi$

30. $r = \left(\dfrac{1 + \sin\theta}{1 - \cos\theta}\right)^2$

In Exercises 31–34, find all values of x for which the function is differentiable.

31. $y = \ln x^2$

32. $y = \sin x - x\cos x$

33. $y = \sqrt{\dfrac{1 - x}{1 + x^2}}$

34. $y = (2x - 7)^{-1}(x + 5)$

In Exercises 35–38, find dy/dx.

35. $xy + 2x + 3y = 1$

36. $5x^{4/5} + 10y^{6/5} = 15$

37. $\sqrt{xy} = 1$

38. $y^2 = \dfrac{x}{x + 1}$

In Exercises 39–42, find d^2y/dx^2 by implicit differentiation.

39. $x^3 + y^3 = 1$

40. $y^2 = 1 - \dfrac{2}{x}$

41. $y^3 + y = 2\cos x$

42. $x^{1/3} + y^{1/3} = 4$

In Exercises 43 and 44, find all derivatives of the function.

43. $y = \dfrac{x^4}{2} - \dfrac{3}{2}x^2 - x$

44. $y = \dfrac{x^5}{120}$

In Exercises 45–48, find an equation for the **(a)** tangent and **(b)** normal to the curve at the indicated point.

45. $y = \sqrt{x^2 - 2x}, \quad x = 3$

46. $y = 4 + \cot x - 2\csc x, \quad x = \pi/2$

47. $x^2 + 2y^2 = 9, \quad (1, 2)$

48. $x + \sqrt{xy} = 6, \quad (4, 1)$

In Exercises 49–52, find an equation for the line tangent to the curve at the point defined by the given value of t.

49. $x = 2\sin t, \quad y = 2\cos t, \quad t = 3\pi/4$

50. $x = 3\cos t, \quad y = 4\sin t, \quad t = 3\pi/4$

51. $x = 3\sec t, \quad y = 5\tan t, \quad t = \pi/6$

52. $x = \cos t, \quad y = t + \sin t, \quad t = -\pi/4$

53. Writing to Learn

(a) Graph the function

$$f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2 - x, & 1 < x \le 2. \end{cases}$$

(b) Is f continuous at $x = 1$? Explain.

(c) Is f differentiable at $x = 1$? Explain.

54. Writing to Learn For what values of the constant m is

$$f(x) = \begin{cases} \sin 2x, & x \le 0 \\ mx, & x > 0 \end{cases}$$

(a) continuous at $x = 0$? Explain.

(b) differentiable at $x = 0$? Explain.

In Exercises 55–58, determine where the function is **(a)** differentiable, **(b)** continuous but not differentiable, and **(c)** neither continuous nor differentiable.

55. $f(x) = x^{4/5}$

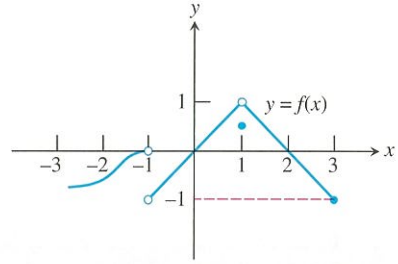
56. $g(x) = \sin(x^2 + 1)$

57. $f(x) = \begin{cases} 2x - 3, & -1 \le x < 0 \\ x - 3, & 0 \le x \le 4 \end{cases}$

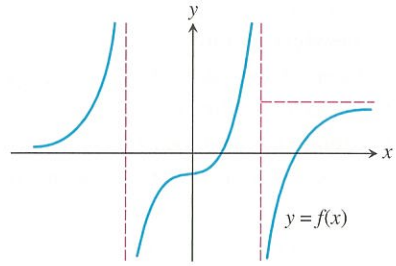
58. $g(x) = \begin{cases} \dfrac{x - 1}{x}, & -2 \le x < 0 \\ \dfrac{x + 1}{x}, & 0 \le x \le 2 \end{cases}$

In Exercises 59 and 60, use the graph of f to sketch the graph of f'.

59. *Sketching f′ from f*

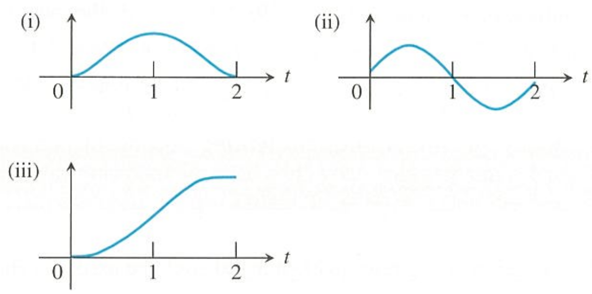


60. *Sketching f′ from f*



61. *Recognizing Graphs* The following graphs show the distance traveled, velocity, and acceleration for each second of a 2-minute automobile trip. Which graph shows

(a) distance ? **(b)** velocity? **(c)** acceleration?



62. *Sketching f from f′* Sketch the graph of a continuous function f with $f(0) = 5$ and

$$f'(x) = \begin{cases} -2, & x < 2 \\ -0.5, & x > 2. \end{cases}$$

63. *Sketching f from f′* Sketch the graph of a continuous function f with $f(-1) = 2$ and

$$f'(x) = \begin{cases} -2, & x < 1 \\ 1, & 1 < x < 4 \\ -1, & 4 < x < 6. \end{cases}$$

64. Which of the following statements could be true if $f''(x) = x^{1/3}$?

i. $f(x) = \frac{9}{28}x^{7/3} + 9$ **ii.** $f'(x) = \frac{9}{28}x^{7/3} - 2$

iii. $f'(x) = \frac{3}{4}x^{4/3} + 6$ **iv.** $f(x) = \frac{3}{4}x^{4/3} - 4$

A. i only **B.** iii only

C. ii and iv only **D.** i and iii only

65. ***Derivative from Data*** The following data give the coordinates of a moving body for various values of t.

t (sec)	0	0.5	1	1.5	2	2.5	3	3.5	4
s (ft)	10	38	58	70	74	70	58	38	10

(a) Make a scatter plot of the (t, s) data and sketch a smooth curve through the points.

(b) Compute the average velocity between consecutive points of the table.

(c) Make a scatter plot of the data in part (b) using the midpoints of the t values to represent the data. Then sketch a smooth curve through the points.

(d) Writing to Learn Why does the curve in part (c) approximate the graph of ds/dt?

66. ***Working with Numerical Values*** Suppose that a function f and its first derivative have the following values at $x = 0$ and $x = 1$.

x	$f(x)$	$f'(x)$
0	9	−2
1	−3	1/5

Find the first derivative of the following combinations at the given value of x.

(a) $\sqrt{x}f(x)$, $x = 1$ **(b)** $\sqrt{f(x)}$, $x = 0$

(c) $f(\sqrt{x})$, $x = 1$ **(d)** $f(1 - 5\tan x)$, $x = 0$

(e) $\frac{f(x)}{2 + \cos x}$, $x = 0$ **(f)** $10\sin\left(\frac{\pi x}{2}\right)f^2(x)$, $x = 1$

67. ***Working with Numerical Values*** Suppose that functions f and g and their first derivatives have the following values at $x = -1$ and $x = 0$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
−1	0	−1	2	1
0	−1	−3	−2	4

Find the first derivative of the following combinations at the given value of x.

(a) $3f(x) - g(x)$, $x = -1$ **(b)** $f^2(x)g^3(x)$, $x = 0$

(c) $g(f(x))$, $x = -1$ **(d)** $f(g(x))$, $x = -1$

(e) $\frac{f(x)}{g(x) + 2}$, $x = 0$ **(f)** $g(x + f(x))$, $x = 0$

68. Find the value of dw/ds at $s = 0$ if $w = \sin(\sqrt{r} - 2)$ and $r = 8\sin(s + \pi/6)$.

69. Find the value of dr/dt at $t = 0$ if $r = (\theta^2 + 7)^{1/3}$ and $\theta^2 t + \theta = 1$.

70. ***Particle Motion*** The position at time $t \geq 0$ of a particle moving along the s-axis is

$$s(t) = 10\cos(t + \pi/4).$$

(a) Give parametric equations that can be used to simulate the motion of the particle.

(b) What is the particle's initial position $(t = 0)$?

(c) What points reached by the particle are farthest to the left and right of the origin?

(d) When does the particle first reach the origin? What are its velocity, speed, and acceleration then?

71. ***Vertical Motion*** On Earth, if you shoot a paper clip 64 ft straight up into the air with a rubber band, the paper clip will be $s(t) = 64t - 16t^2$ feet above your hand at t sec after firing.

(a) Find ds/dt and d^2s/dt^2.

(b) How long does it take the paper clip to reach its maximum height?

(c) With what velocity does it leave your hand?

(d) On the moon, the same force will send the paper clip to a height of $s(t) = 64t - 2.6t^2$ ft in t sec. About how long will it take the paper clip to reach its maximum height, and how high will it go?

72. ***Free Fall*** Suppose two balls are falling from rest at a certain height in centimeters above the ground. Use the equation $s = 490t^2$ to answer the following questions.

(a) How long does it take the balls to fall the first 160 cm? What is their average velocity for the period?

(b) How fast are the balls falling when they reach the 160-cm mark? What is their acceleration then?

73. ***Filling a Bowl*** If a hemispherical bowl of radius 10 in. is filled with water to a depth of x in., the volume of water is given by $V = \pi[10 - (x/3)]x^2$. Find the rate of increase of the volume per inch increase of depth.

74. ***Marginal Revenue*** A bus will hold 60 people. The fare charged (p dollars) is related to the number x of people who use the bus by the formula $p = [3 - (x/40)]^2$.

(a) Write a formula for the total revenue per trip received by the bus company.

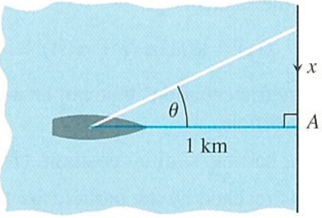
(b) What number of people per trip will make the marginal revenue equal to zero? What is the corresponding fare?

(c) Writing to Learn Do you think the bus company's fare policy is good for its business?

75. ***Searchlight*** The figure shows a boat 1 km offshore sweeping the shore with a searchlight. The light turns at a constant rate, $d\theta/dt = -0.6$ rad/sec.

(a) How fast is the light moving along the shore when it reaches point *A*?

(b) How many revolutions per minute is 0.6 rad/sec?



76. ***Horizontal Tangents*** The graph of $y = \sin(x - \sin x)$ appears to have horizontal tangents at the x-axis. Does it?

77. ***Fundamental Frequency of a Vibrating Piano String*** We measure the frequencies at which wires vibrate in cycles (trips back and forth) per sec. The unit of measure is a *hertz:* 1 cycle per sec. Middle A on a piano has a frequency 440 hertz. For any given wire, the fundamental frequency y is a function of four variables:

r: the radius of the wire;

l: the length;

d: the density of the wire;

T: the tension (force) holding the wire taut.

With r and l in centimeters, d in grams per cubic centimeter, and T in dynes (it takes about 100,000 dynes to lift an apple), the fundamental frequency of the wire is

$$y = \frac{1}{2rl}\sqrt{\frac{T}{\pi d}}.$$

If we keep all the variables fixed except one, then y can be alternatively thought of as four different functions of one variable, $y(r)$, $y(l)$, $y(d)$, and $y(T)$. How would changing each variable affect the string's fundamental frequency? To find out, calculate $y'(r)$, $y'(l)$, $y'(d)$, and $y'(T)$.

78. ***Spread of Measles*** The spread of measles in a certain school is given by

$$P(t) = \frac{200}{1 + e^{5-t}},$$

where t is the number of days since the measles first appeared, and $P(t)$ is the total number of students who have caught the measles to date.

(a) Estimate the initial number of students infected with measles.

(b) About how many students in all will get the measles?

(c) When will the rate of spread of measles be greatest? What is this rate?

79. Graph the function $f(x) = \tan^{-1}(\tan 2x)$ in the window $[-\pi, \pi]$ by $[-4, 4]$. Then answer the following questions.

(a) What is the domain of f?

(b) What is the range of f?

(c) At which points is f not differentiable?

(d) Describe the graph of f'.

80. If $x^2 - y^2 = 1$, find d^2y/dx^2 at the point $(2, \sqrt{3})$.

AP* Examination Preparation

You may use a graphing calculator to solve the following problems.

81. A particle moves along the x-axis so that at any time $t \geq 0$ its position is given by $x(t) = t^3 - 12t + 5$.

(a) Find the velocity of the particle at any time t.

(b) Find the acceleration of the particle at any time t.

(c) Find all values of t for which the particle is at rest.

(d) Find the speed of the particle when its acceleration is zero.

(e) Is the particle moving toward the origin or away from the origin when $t = 3$? Justify your answer.

82. Let $y = \dfrac{e^x + e^{-x}}{2}$.

(a) Find $\dfrac{dy}{dx}$.

(b) Find $\dfrac{d^2y}{dx^2}$.

(c) Find an equation of the line tangent to the curve at $x = 1$.

(d) Find an equation of the line normal to the curve at $x = 1$.

(e) Find any points where the tangent line is horizontal.

83. Let $f(x) = \ln(1 - x^2)$.

(a) State the domain of f.

(b) Find $f'(x)$.

(c) State the domain of f'.

(d) Prove that $f''(x) < 0$ for all x in the domain of f.