Chapter 10

Parametric, Vector, and Polar Functions

In 1935, air traffic control was conducted with a system of teletype machines, wall-sized blackboards, large table maps, and movable markers representing airplanes. Today's radar data processing includes an automatic display of aircraft identification, speed, altitude, and velocity vectors.

A DC-10 plane flying due west at 600 mph enters a region with a steady air current coming from the southwest at 100 mph. How should the pilot adjust the airplane's course and speed to maintain its original velocity vector? This type of problem is covered in Section 10.2.

Chapter 10 Overview

The material in this book is generally described as the calculus of a single variable, since it deals with functions of one independent variable (usually x or t). In this chapter you will apply your understanding of single-variable calculus in three kinds of two-variable contexts, enabling you to analyze some new kinds of curves (parametrically defined and polar) and to analyze motion in the plane that does not proceed along a straight line. Interestingly enough, this will not require the tools of multi-variable calculus, which you will probably learn in your next calculus course. We will simply use single-variable calculus in some new and interesting ways.

10.1 Parametric Functions

What you'll learn about

- Parametric Curves in the Plane
- Slope and Concavity
- Arc Length
- Cycloids

... and why

Parametric equations enable us to define some interesting and important curves that would be difficult or impossible to define in the form $y = f(x)$.

Parametric Curves in the Plane

We reviewed parametrically defined functions in Section 1.4. Instead of defining the points (x, y) on a planar curve by relating y directly to x, we can define both coordinates as functions of a parameter t. The resulting set of points may or may not define y as a function of x (that is, the parametric curve might fail the vertical line test).

EXAMPLE 1   Reviewing Some Parametric Curves

Sketch the parametric curves and identify those which define y as a function of x. In each case, eliminate the parameter to find an equation that relates x and y directly.

(a) $x = \cos t$ and $y = \sin t$ for t in the interval $[0, 2\pi)$

(b) $x = 3 \cos t$ and $y = 2 \sin t$ for t in the interval $[0, 4\pi]$

(c) $x = \sqrt{t}$ and $y = t - 2$ for t in the interval $[0, 4]$

SOLUTION

(a) This is probably the best-known parametrization of all. The curve is the unit circle (Figure 10.1a), and it does not define y as a function of x. To eliminate the parameter, we use the identity $(\cos t)^2 + (\sin t)^2 = 1$ to write $x^2 + y^2 = 1$.

(b) This parametrization stretches the unit circle by a factor of 3 horizontally and by a factor of 2 vertically. The result is an ellipse (Figure 10.1b), which is traced twice as t covers the interval $[0, 4\pi]$. (In fact, the point $(3, 0)$ is visited three times.) It does not define y as a function of x. We use the same identity as in part (a) to write $\left(\dfrac{x}{3}\right)^2 + \left(\dfrac{y}{2}\right)^2 = 1$.

(c) This parametrization produces a segment of a parabola (Figure 10.1c). It does define y as a function of x. Since $t = x^2$, we write $y = x^2 - 2$.   ***Now try Exercise 1.***
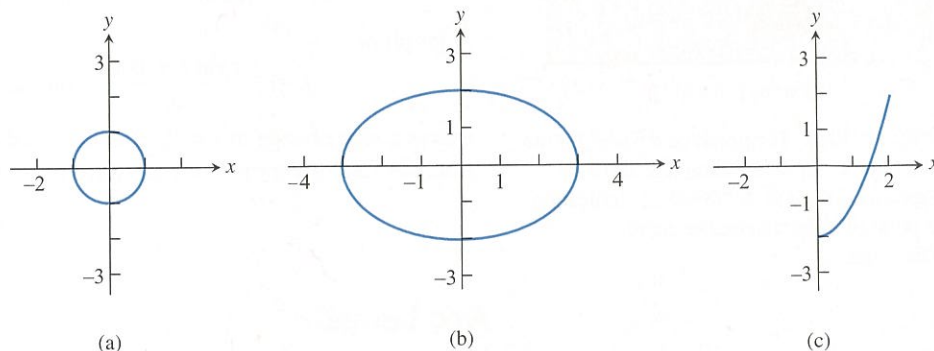

(a)	(b)	(c)

Figure 10.1 A collection of parametric curves (Example 1). Each point (x, y) is determined by parametric functions of t, but only the parametrization in graph (c) determines y as a function of x.

Slope and Concavity

We can analyze the slope and concavity of parametric curves just as we can with explicitly-defined curves. The slope of the curve is still dy/dx, and the concavity still depends on d^2y/dx^2, so all that is needed is a way of differentiating with respect to x when everything is given in terms of t. The required parametric differentiation formulas are straightforward applications of the Chain Rule.

Parametric Differentiation Formulas

If x and y are both differentiable functions of t and if $dx/dt \neq 0$, then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

If $y' = dy/dx$ is also a differentiable function of t, then

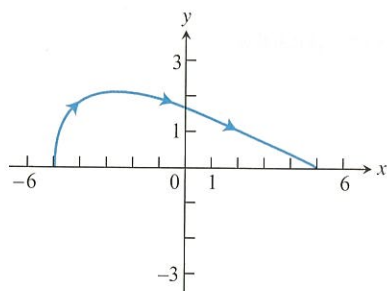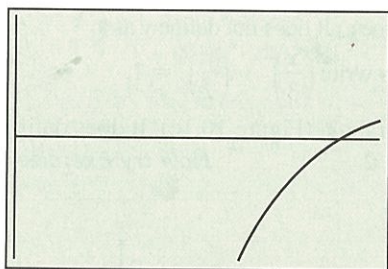$$\frac{d^2y}{dx^2} = \frac{d}{dx}(y') = \frac{dy'/dt}{dx/dt}.$$

Figure 10.2 The parametric curve defined in Example 2.

EXAMPLE 2   Analyzing a Parametric Curve

Consider the curve defined parametrically by $x = t^2 - 5$ and $y = 2 \sin t$ for $0 \leq t \leq \pi$.

(a) Sketch a graph of the curve in the viewing window $[-7, 7]$ by $[-4, 4]$. Indicate the direction in which it is traced.

(b) Find the highest point on the curve. Justify your answer.

(c) Find all points of inflection on the curve. Justify your answer.

SOLUTION

(a) The curve is shown in Figure 10.2.

(b) We seek to maximize y as a function of t, so we compute $dy/dt = 2 \cos t$. Since dy/dt is positive for $0 \leq t < \pi/2$ and negative for $\pi/2 < t \leq \pi$, the maximum occurs when $t = \pi/2$. Substituting this t value into the parametrization, we find the highest point to be approximately $(-2.533, 2)$.

(c) First we compute d^2y/dx^2.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2 \cos t}{2t} = \frac{\cos t}{t}$$

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{\dfrac{(-\sin t)(t) - (1)(\cos t)}{t^2}}{2t} = -\frac{t \sin t + \cos t}{2t^3}$$

A graph of

$$y = -\frac{t \sin t + \cos t}{2t^3} \quad \text{on the interval } [0, \pi] \text{ (Figure 10.3)}$$

$[0, \pi]$ by $[-0.1, 0.1]$

Figure 10.3 The graph of d^2y/dx^2 for the parametric curve in Example 2 shows a sign change at $t = 2.798386\ldots$, indicating a point of inflection on the curve. (Example 2)

shows a sign change at $t = 2.798386\ldots$. Substituting this t value into the parametrization, we find the point of inflection to be approximately $(2.831, 0.673)$.

Now try Exercise 19.

Arc Length

In Section 7.4 we derived two different formulas for arc length, each of them based on an approximation of the curve by tiny straight line segments with length $\sqrt{\Delta x_k^2 + \Delta y_k^2}$. (See Figure 10.4.)
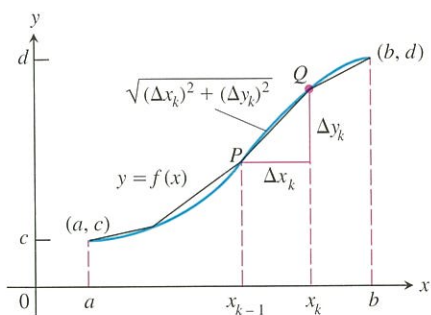
Figure 10.4 The graph of f, approximated by line segments.

Here is a third formula based on the same approximation.

> ## Arc Length of a Parametrized Curve
>
> Let L be the length of a parametric curve that is traversed exactly once as t increases from t_1 to t_2.
>
> If dx/dt and dy/dt are continuous functions of t, then
>
> $$L = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

$x = \cos^3 t,\ y = \sin^3 t,\ 0 \le t \le 2\pi$
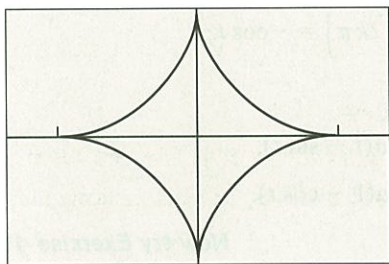
Figure 10.5 The astroid in Example 3.

EXAMPLE 3   Measuring a Parametric Curve

Find the length of the astroid (Figure 10.5)

$$x = \cos^3 t, \qquad y = \sin^3 t, \qquad 0 \le t \le 2\pi.$$

SOLUTION

Solve Analytically The curve is traced once as t goes from 0 to 2π. Because of the curve's symmetry with respect to the coordinate axes, its length is four times the length of the first quadrant portion. We have

$$\left(\frac{dx}{dt}\right)^2 = \left((3\cos^2 t)(-\sin t)\right)^2 = 9\cos^4 t \sin^2 t$$

$$\left(\frac{dy}{dt}\right)^2 = \left((3\sin^2 t)(\cos t)\right)^2 = 9\sin^4 t \cos^2 t$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{9\cos^2 t \sin^2 t \underbrace{(\cos^2 t + \sin^2 t)}_{1}}$$

$$= \sqrt{9\cos^2 t \sin^2 t}$$

$$= 3|\cos t \sin t|.$$

Thus, the length of the first quadrant portion of the curve is

$$\int_0^{\pi/2} 3|\cos t \sin t|\, dt = 3\int_0^{\pi/2} \cos t \sin t\, dt \qquad \cos t \sin t \ge 0,\ 0 \le t \le \pi/2$$

$$= \frac{3}{2}\sin^2 t \Big]_0^{\pi/2} \qquad u = \sin t,\ du = \cos t\, dt$$

$$= \frac{3}{2}.$$

The length of the astroid is $4(3/2) = 6$.

Support Numerically NINT $(3|\cos t \sin t|,\, t,\, 0,\, 2\pi) = 6$.   ***Now try Exercise 29.***

Figure 10.6 The position of $P(x, y)$ on the edge of the wheel when the wheel has turned t radians. (Example 4)
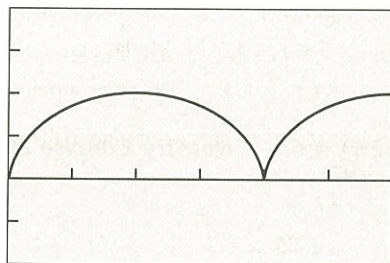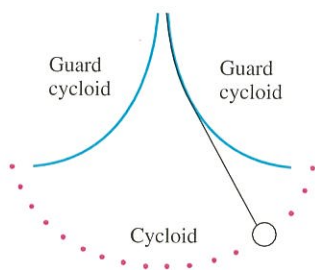
Cycloids

Suppose that a wheel of radius a rolls along a horizontal line without slipping (see Figure 10.6). The path traced by a point P on the wheel's edge is a **cycloid**, where P is originally at the origin.

Huygens's Clock

The problem with a pendulum clock whose bob swings in a circular arc is that the frequency of the swing depends on the amplitude of the swing. The wider the swing, the longer it takes the bob to return to center.

This does not happen if the bob can be made to swing in a cycloid. In 1673, Christiaan Huygens (1629–1695), the Dutch mathematician, physicist, and astronomer who discovered the rings of Saturn, designed a pendulum clock whose bob would swing in a cycloid. Driven by a need to make accurate determinations of longitude at sea, he hung the bob from a fine wire constrained by guards that caused it to draw up as it swung away from center. How were the guards shaped? They were cycloids, too.

Guard cycloid   Guard cycloid

Cycloid

EXAMPLE 4 Finding Parametric Equations for a Cycloid

Find parametric equations for the path of the point P in Figure 10.6.

SOLUTION

We suppose that the wheel rolls to the right, P being at the origin when the turn angle t equals 0. Figure 10.6 shows the wheel after it has turned t radians. The base of the wheel is at distance at from the origin. The wheel's center is at (at, a), and the coordinates of P are

$$x = at + a \cos \theta, \quad y = a + a \sin \theta.$$

To express θ in terms of t, we observe that $t + \theta = 3\pi/2 + 2k\pi$ for some integer k, so

$$\theta = \frac{3\pi}{2} - t + 2k\pi.$$

Thus,

$$\cos \theta = \cos \left(\frac{3\pi}{2} - t + 2k\pi \right) = -\sin t,$$

$$\sin \theta = \sin \left(\frac{3\pi}{2} - t + 2k\pi \right) = -\cos t.$$

Therefore,

$$x = at - a \sin t = a(t - \sin t),$$

$$y = a - a \cos t = a(1 - \cos t).$$

Now try Exercise 41.

EXPLORATION 1 **Investigating Cycloids**

Consider the cycloids with parametric equations

$$x = a(t - \sin t), \quad y = a(1 - \cos t), \quad a > 0.$$

1. Graph the equations for $a = 1$, 2, and 3.
2. Find the x-intercepts.
3. Show that $y \geq 0$ for all t.
4. Explain why the arches of a cycloid are congruent.
5. What is the maximum value of y? Where is it attained?
6. Describe the graph of a cycloid.

EXAMPLE 5 Finding Length

Find the length of one arch of the cycloid

$$x = a(t - \sin t), \quad y = a(1 - \cos t), \quad a > 0.$$

SOLUTION

Figure 10.7 shows the first arch of the cycloid and part of the next for $a = 1$. In Exploration 1 you found that the x-intercepts occur at t equal to multiples of 2π and that the arches are congruent.

The length of the first arch is

$$\int_0^{2\pi} \sqrt{\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2} \, dt.$$

$[0, 3\pi]$ by $[-2, 4]$

Figure 10.7 The graph of the cycloid $x = t - \sin t$, $y = 1 - \cos t$, $t \geq 0$. (Example 5)

continued

We have

$$\left(\frac{dx}{dt}\right)^2 = [a(1 - \cos t)]^2 = a^2(1 - 2\cos t + \cos^2 t)$$

$$\left(\frac{dy}{dt}\right)^2 = [a \sin t]^2 = a^2 \sin^2 t$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = a\sqrt{2 - 2\cos t}. \quad a > 0, \sin^2 t + \cos^2 t = 1$$

Therefore,

$$\int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = a \int_0^{2\pi} \sqrt{2 - 2\cos t}\, dt = 8a. \quad \text{Using NINT}$$

The length of one arch of the cycloid is $8a$.

Now try Exercise 43.

Quick Review 10.1 *(For help, go to Appendix A.1.)*

Use algebra or a trig identity to write an equation relating x and y.

1. $x = t + 1$ and $y = 2t + 3$

2. $x = 3t$ and $y = 54t^3 - 3$

3. $x = \sin t$ and $y = \cos t$

4. $x = \sin t \cos t$ and $y = \sin(2t)$

5. $x = \tan \theta$ and $y = \sec \theta$

6. $x = \csc \theta$ and $y = \cot \theta$

7. $x = \cos \theta$ and $y = \cos(2\theta)$

8. $x = \sin \theta$ and $y = \cos(2\theta)$

9. $x = \cos \theta$ and $y = \sin \theta$ $\quad (0 \le \theta \le \pi)$

10. $x = \cos \theta$ and $y = \sin \theta$ $\quad (\pi \le \theta \le 2\pi)$

Section 10.1 Exercises

In Exercises 1–6, sketch the parametric curves and identify those which define y as a function of x. In each case, eliminate the parameter to find an equation that relates x and y directly.

1. $x = 2t + 3$ and $y = 4t - 3$ for t in the interval $[0, 3]$

2. $x = \sqrt{t - 2}$ and $y = \dfrac{t + 5}{4}$ for t in the interval $[3, 11]$

3. $x = \tan t$ and $y = \sec t$ for t in the interval $[0, \pi/4]$

4. $x = \sin t$ and $y = 2 \cos t$ for t in the interval $[0, \pi]$

5. $x = \sin t$ and $y = \cos(2t)$ for t in the interval $[0, 2\pi]$

6. $x = \sin 6t$ and $y = 2t$ for t in the interval $[0, \pi/2]$

In Exercises 7–16, find **(a)** dy/dx and **(b)** d^2y/dx^2 in terms of t.

7. $x = 4 \sin t, \quad y = 2 \cos t$

8. $x = \cos t, \quad y = \sqrt{3} \cos t$

9. $x = -\sqrt{t + 1}, \quad y = \sqrt{3t}$

10. $x = 1/t, \quad y = -2 + \ln t$

11. $x = t^2 - 3t, \quad y = t^3$

12. $x = t^2 + t, \quad y = t^2 - t$

13. $x = \tan t, \quad y = \sec t$

14. $x = 2\cos t, \quad y = \cos(2t)$

15. $x = \ln(2t), \quad y = \ln(3t)^4$

16. $x = \ln(5t), \quad y = e^{5t}$

In Exercises 17–22,

(a) sketch the curve over the given t-interval, indicating the direction in which it is traced,

(b) identify the requested point, and

(c) justify that you have found the requested point by analyzing an appropriate derivative.

17. $x = t + 1, \quad y = t^2 + t, \quad -2 \le t \le 2$ $\qquad$ Lowest point

18. $x = t^2 + 2t, \quad y = t^2 - 2t + 3, \quad -2 \le t \le 3$ $\quad$ Leftmost point

19. $x = 2 \sin t, \quad y = \cos t, \quad 0 \le t \le \pi$ $\qquad$ Rightmost point

20. $x = \tan t, \quad y = 2 \sec t, \quad -1 \le t \le 1$ $\qquad$ Lowest point

21. $x = 2 \sin t, \quad y = \cos(2t), \quad 1.5 \le t \le 4.5$ $\qquad$ Highest point

22. $x = \ln(5t), \quad y = \ln(4t^2), \quad 0 < t \le 10$ $\qquad$ Rightmost point

In Exercises 23–26, find the points at which the tangent line to the curve is **(a)** horizontal or **(b)** vertical.

23. $x = 2 + \cos t, \quad y = -1 + \sin t$

24. $x = \sec t, \quad y = \tan t$ $\qquad$ **25.** $x = 2 - t, \quad y = t^3 - 4t$

26. $x = -2 + 3 \cos t, \quad y = 1 + 3 \sin t$

In Exercises 27–34, find the length of the curve. (For an algebraic challenge, try evaluating the integrals without a calculator.)

27. $x = \cos t, \quad y = \sin t, \quad 0 \le t \le 2\pi$

28. $x = 3 \sin t, \quad y = 3 \cos t, \quad 0 \le t \le \pi$

29. $x = 8 \cos t + 8t \sin t, \quad y = 8 \sin t - 8t \cos t, \quad 0 \le t \le \pi/2$

30. $x = 2 \cos^3 t, \ y = 2 \sin^3 t, \ 0 \le t \le 2\pi$

31. $x = \dfrac{(2t + 3)^{3/2}}{3}, \quad y = t + \dfrac{t^2}{2}, \quad 0 \le t \le 3$

32. $x = \dfrac{(8t + 8)^{3/2}}{12}, \quad y = t^2 + t, \quad 0 \le t \le 2$

33. $x = \dfrac{1}{3}t^3, \quad y = \dfrac{1}{2}t^2, \quad 0 \le t \le 1$

34. $x = \ln(\sec t + \tan t) - \sin t, \quad y = \cos t, \quad 0 \le t \le \pi/3$

35. *Length is Independent of Parametrization* To illustrate the fact that the numbers we get for length do not usually depend on the way we parametrize our curves, calculate the length of the semicircle $y = \sqrt{1 - x^2}$ with these two different parametrizations.

(a) $x = \cos 2t, \quad y = \sin 2t, \quad 0 \le t \le \pi/2$

(b) $x = \sin \pi t, \quad y = \cos \pi t, \quad -1/2 \le t \le 1/2$

36. *Perimeter of an Ellipse* Find the length of the ellipse

$$x = 3 \cos t, \quad y = 4 \sin t, \quad 0 \le t \le 2\pi.$$

37. *Cartesian Length Formula* The graph of a function $y = f(x)$ over an interval $[a, b]$ automatically has the parametrization

$$x = x, \quad y = f(x), \quad a \le x \le b.$$

The parameter in this case is x itself. Show that for this parametrization, the length formula

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

reduces to the Cartesian formula

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

derived in Section 7.4.

38. (*Continuation of Exercise 37*) Show that the Cartesian formula

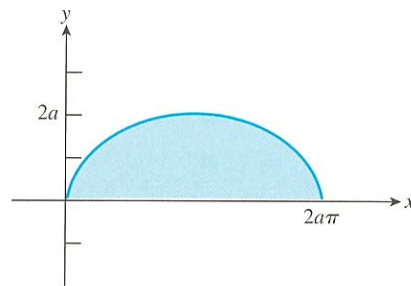$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

for the length of the curve $x = g(y), \ c \le y \le d$, from Section 7.4 is a special case of the parametric length formula

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Exercises 39 and 40 refer to the region bounded by the x-axis and one arch of the cycloid

$$x = a(t - \sin t), \quad y = a(1 - \cos t)$$

that is shaded in the figure shown at the top of the next column.

39. Find the area of the shaded region. (*Hint:* $dx = (dx/dt)\, dt$)

40. Find the volume swept out by revolving the region about the x-axis. (*Hint:* $dV = \pi y^2\, dx = \pi y^2 (dx/dt)\, dt$)

41. *Curtate Cycloid* Modify Example 4 slightly to find the parametric equations for the motion of a point in the *interior* of a wheel of radius a as the wheel rolls along the horizontal line without slipping. Assume that the point is at distance b from the center of the wheel, where $0 < b < a$. This curve, known as a *curtate cycloid,* has been used by artisans in designing the arches of violins (*Source:* mathworld.wolfram.com).

42. *Prolate Cycloid* Modify Example 4 slightly to find the parametric equations for the motion of a point on the *exterior* of a wheel of radius a as the wheel rolls along the horizontal line without slipping. Assume that the point is at distance b from the center of the wheel, where $a < b < 2a$. This curve, known as a *prolate cycloid,* is traced out by a point on the outer edge of a train's flanged wheel as the train moves along a track. (If you graph a prolate cycloid, you can see why they say that there is always part of a forward-moving train that is moving backwards!)

43. *Arc Length* Find the length of one arch (that is, the curve over one period) of the curtate cycloid defined parametrically by $x = 3t - 2 \sin t$ and $y = 3 - 2 \cos t$.

44. *Arc Length* Find the length of one arch (that is, the curve over one period) of the prolate cycloid defined parametrically by $x = 2t - 3 \sin t$ and $y = 2 - 3 \cos t$.

Standardized Test Questions

You should solve the following problems without using a graphing calculator.

45. True or False In a parametrization, if x is a continuous function of t and y is a continuous function of t, then y is a continuous function of x. Justify your answer.

46. True or False If f is a function with domain all real numbers, then the graph of f can be defined parametrically by $x = t$ and $y = f(t)$ for $-\infty < t < \infty$. Justify your answer.

47. Multiple Choice For which of the following parametrizations of the unit circle will the circle be traversed clockwise?

(A) $x = \cos t, \quad y = \sin t, \qquad 0 \le t \le 2\pi$

(B) $x = \sin t, \quad y = \cos t, \qquad 0 \le t \le 2\pi$

(C) $x = -\cos t, \quad y = -\sin t, \quad 0 \le t \le 2\pi$

(D) $x = -\sin t, \quad y = \cos t, \qquad 0 \le t \le 2\pi$

(E) $x = \sin t, \quad y = -\cos t, \qquad 0 \le t \le 2\pi$

48. Multiple Choice A parametric curve is defined by $x = \sin t$ and $y = \csc t$ for $0 < t < \pi/2$. This curve is

(A) increasing and concave up.

(B) increasing and concave down.

(C) decreasing and concave up.

(D) decreasing and concave down.

(E) decreasing with a point of inflection.

49. Multiple Choice The parametric curve defined by $x = \ln(t)$, $y = t$ for $t > 0$ is identical to the graph of the function
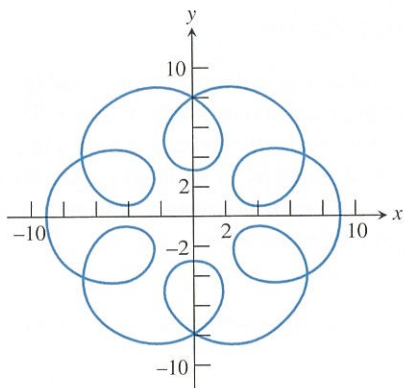
(A) $y = \ln x$ for all real x.

(B) $y = \ln^x$ for $x > 0$.

(C) $y = e^x$ for all real x.

(D) $y = e^x$ for $x > 0$.

(E) $y = \ln(e^x)$ for $x > 0$.

50. Multiple Choice The curve parametrized by

$$x = 6 \sin t - 3 \sin(7t) \text{ and } y = 6 \cos t - 3 \cos(7t),$$

as shown in the diagram below, is traversed exactly once as t increases from 0 to 2π. The total length of the curve is given by

(A) $\int_0^{2\pi} \sqrt{(6 \sin t - 3 \sin(7t))^2 + (6 \cos t - 3 \cos(7t))^2}\, dt$

(B) $\int_0^{2\pi} \sqrt{(6 \cos t - 3 \cos(7t))^2 + (6 \sin t - 3 \sin(7t))^2}\, dt$

(C) $\int_0^{2\pi} \sqrt{(6 \cos t - 21 \cos(7t))^2 - (6 \sin t - 21 \sin(7t))^2}\, dt$

(D) $\int_0^{2\pi} \sqrt{(6 \cos t - 21 \cos(7t))^2 + (-6 \sin t + 21 \sin(7t))^2}\, dt$

(E) $\int_0^{2\pi} 7\sqrt{(6 \cos t - 3 \cos(7t))^2 + (6 \sin t + 3 \sin(3t))^2}\, dt$

Explorations

51. Group Activity *Involute of a Circle* If a string wound around a fixed circle is unwound while being held taut in the plane of the circle, its end P traces an *involute* of the circle as suggested by the diagram below. In the diagram, the circle is the unit circle in the xy-plane, and the initial position of the tracing point is the point $(1, 0)$ on the x-axis. The unwound portion of

the string is tangent to the circle at Q, and t is the radian measure of the angle from the positive x-axis to the segment OQ.

(a) Derive parametric equations for the involute by expressing the coordinates x and y of P in terms of t for $t \geq 0$.

(b) Find the length of the involute for $0 \leq t \leq 2\pi$.
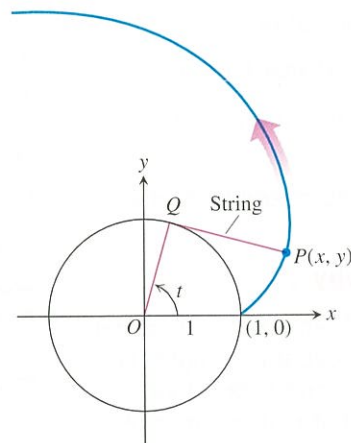
52. *(Continuation of Exercise 51)* Repeat Exercise 51 using the circle of radius a centered at the origin, $x^2 + y^2 = a^2$.

In Exercises 53–56, a projectile is launched over horizontal ground at an angle θ with the horizontal and with initial velocity v_0 ft/sec. Its path is given by the parametric equations

$$x = (v_0 \cos \theta)t, \quad y = (v_0 \sin \theta)t - 16t^2.$$

(a) Find the length of the path traveled by the projectile.

(b) Estimate the maximum height of the projectile.

53. $\theta = 20°, \quad v_0 = 150$ **54.** $\theta = 30°, \quad v_0 = 150$

55. $\theta = 60°, \quad v_0 = 150$ **56.** $\theta = 90°, \quad v_0 = 150$

Extending the Ideas

If dx/dt and dy/dt are continuous, the parametric curve defined by $(x(t), y(t))$ for $a \leq t \leq b$ is called *smooth*. If the curve is traversed exactly once as t increases from a to b, and if y is a positive function of x, then the curve can be revolved about the x-axis to form a solid of revolution (see Section 7.3). The *surface area* of such a solid is given by

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Apply this formula in Exercises 57–60 to find the surface area when the parametric curve is revolved about the x-axis.

57. $x = \cos t, \quad y = 2 + \sin t, \quad 0 \leq t \leq 2\pi$

58. $x = 2\sqrt{t}, \quad y = (2/3)t^{3/2}, \quad 0 \leq t \leq 2$

59. $x = t^2 + 2, \quad y = t + 1, \quad 0 \leq t \leq 3$

60. $x = \ln(\sec t + \tan t) - \sin t, \quad y = \cos t, \quad 0 \leq t \leq \pi/3$

Two-Dimensional Vectors

When an object moves *along a straight line*, its velocity can be determined by a single number that represents both magnitude and direction (forward if the number is positive, backward if it is negative). The speed of an object moving on a path *in a plane* can still be represented by a number, but how can we represent its direction when there are an infinite number of directions possible? Fortunately, we can represent both magnitude and direction with just two numbers, just as we can represent any point in the plane with just two coordinates (which is possible essentially for the same reason). This representation is what two-dimensional vectors were designed to do.

While the pair (a, b) determines a point in the plane, it also determines a **directed line segment** (or **arrow**) with its tail at the origin and its head at (a, b) (Figure 10.8). The length of this arrow represents magnitude, while the direction in which it points represents direction. In this way, the ordered pair (a, b) represents a mathematical object with both magnitude and direction, called the **position vector of (a, b)**.
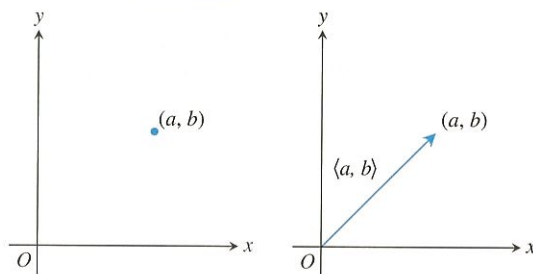

Figure 10.8 The point represents the ordered pair (a, b). The arrow (directed line segment) represents the vector $\langle a, b \rangle$.

DEFINITION Two-Dimensional Vector

A **two-dimensional vector v** is an ordered pair of real numbers, denoted in **component form** as $\langle a, b \rangle$. The numbers a and b are the **components** of the vector **v**. The **standard representation** of the vector $\langle a, b \rangle$ is the arrow from the origin to the point (a, b). The **magnitude** (or **absolute value**) of **v**, denoted $|\mathbf{v}|$, is the length of the arrow, and the **direction of v** is the direction in which the arrow is pointing. The vector $\mathbf{0} = \langle 0, 0 \rangle$, called the **zero vector**, has zero length and no direction.

The distance formula in the plane gives a simple computational formula for magnitude.

Magnitude of a Vector

The **magnitude** or **absolute value** of the vector $\langle a, b \rangle$ is the nonnegative real number $|\langle a, b \rangle| = \sqrt{a^2 + b^2}$.

Direction can be quantified in several ways; for example, navigators use bearings from compass points. The simplest choice for us is to measure direction as we do with the trigonometric functions, using the usual position angle formed with the positive x-axis as the initial ray and the vector as the terminal ray. In this way, every nonzero vector determines a unique **direction angle** θ satisfying (in degrees) $0 \leq \theta < 360$ or (in radians) $0 \leq \theta < 2\pi$. (See Figure 10.10 for an example.)
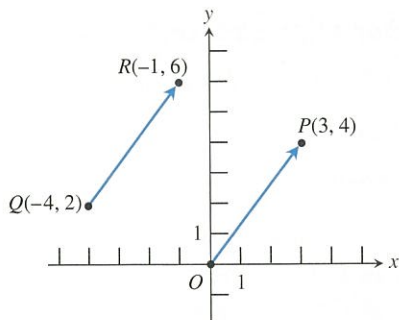
Figure 10.9 The arrows $\overrightarrow{QR}$ and $\overrightarrow{OP}$ both represent the vector $\langle 3, 4 \rangle$, as would any arrow with the same length pointing in the same direction. Such arrows are called *equivalent*.
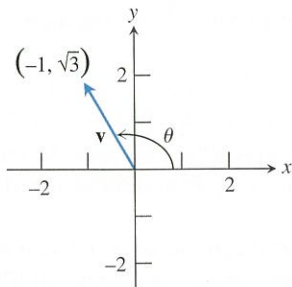

Figure 10.10 The vector **v** in Example 1 is represented by an arrow from the origin to the point $(-1, \sqrt{3})$.
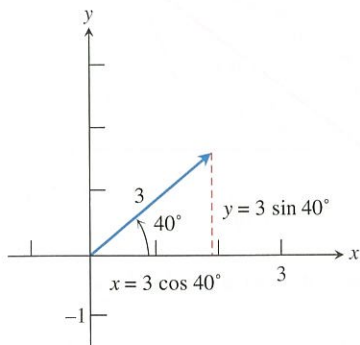

Figure 10.11 The vector in Example 2 is represented by an arrow from the origin to the point $(3 \cos 40°, 3 \sin 40°)$.

Why Not Use Slope for Direction?

Notice that *slope* is inadequate for determining the direction of a vector, since two vectors with the same slope could be pointing in opposite directions. Moreover, vectors are still useful in dimensions higher than 2, while slope is not.

Direction Angle of a Vector

The **direction angle** of a nonzero vector **v** is the smallest nonnegative angle θ formed with the positive x-axis as the initial ray and the standard representation of **v** as the terminal ray.

This textbook uses boldface variables to represent vectors (for example, **u** and **v**) to distinguish them from numbers. In handwritten form it is customary to distinguish vector variables by arrows (for example, $\vec{u}$ and $\vec{v}$). We also use angled brackets to distinguish a vector $\langle x, y \rangle$ from a point (x, y) in the plane, although it is not uncommon to see (x, y) used for both, especially in handwritten form.

It is often convenient in applications to represent vectors with arrows that begin at points other than the origin. The important thing to remember is that *any two arrows with the same length and pointing in the same direction represent the same vector*. In Figure 10.9, for example, the vector $\langle 3, 4 \rangle$ is shown represented by $\overrightarrow{QR}$, an arrow with **initial point** Q and **terminal point** R, as well as by its standard representation $\overrightarrow{OP}$. Two arrows that represent the same vector are said to be **equivalent**.

The quick way to associate arrows with the vectors they represent is to use the following rule.

Head Minus Tail (HMT) Rule

If an arrow has initial point (x_1, y_1) and terminal point (x_2, y_2), it represents the vector $\langle x_2 - x_1, y_2 - y_1 \rangle$.

EXAMPLE 1 Finding Magnitude and Direction

Find the magnitude and the direction angle θ of the vector $\mathbf{v} = \langle -1, \sqrt{3} \rangle$ (Figure 10.10).

SOLUTION

The magnitude of **v** is $|\mathbf{v}| = \sqrt{(-1 - 0)^2 + (\sqrt{3} - 0)^2} = 2$. Using triangle ratios, we see that the direction angle θ satisfies $\cos \theta = -1/2$ and $\sin \theta = \sqrt{3}/2$, so $\theta = 120°$ or $2\pi/3$ radians.

Now try Exercise 5.

EXAMPLE 2 Finding Component Form

Find the component form of a vector with magnitude 3 and direction angle 40°.

SOLUTION

The components of the vector, found trigonometrically, are $x = 3 \cos 40°$ and $y = 3 \sin 40°$ (Figure 10.11).

The vector is $\langle 3 \cos 40°, 3 \sin 40° \rangle \approx \langle 2.298, 1.928 \rangle$.

Now try Exercise 13.

Vector Operations

The algebra of vectors sometimes involves working with vectors and numbers at the same time. In this context, we refer to the numbers as **scalars**. The two most basic algebraic operations involving vectors are *vector addition* (adding a vector to a vector) and *scalar multiplication* (multiplying a vector by a number). Both operations are easily represented geometrically.

> ### DEFINITION   Vector Addition and Scalar Multiplication
>
> Let $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ be vectors and let k be a real number (scalar).
>
> The **sum** (or **resultant**) **of the vectors u and v** is the vector
> $$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2 \rangle.$$
>
> The **product of the scalar k and the vector u** is
> $$k\mathbf{u} = k\langle u_1, u_2 \rangle = \langle ku_1, ku_2 \rangle.$$
>
> The **opposite of a vector v** is $-\mathbf{v} = (-1)\mathbf{v}$. We define vector subtraction by
> $$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}).$$

The vector $\dfrac{\mathbf{v}}{|\mathbf{v}|}$ is a vector of magnitude 1, called a **unit vector**. Its component form is $\langle \cos\theta, \sin\theta \rangle$, where θ is the direction angle of $\mathbf{v}$. For this reason, $\dfrac{\mathbf{v}}{|\mathbf{v}|}$ is sometimes called the **direction vector** of $\mathbf{v}$.

The sum of two vectors $\mathbf{u}$ and $\mathbf{v}$ can be represented geometrically by arrows in two ways. In the **tail-to-head representation**, the arrow from the origin to (u_1, u_2) is the standard representation of $\mathbf{u}$, the arrow from (u_1, u_2) to $(u_1 + v_1, u_2 + v_2,)$ represents $\mathbf{v}$ (as you can verify by the HMT Rule), and the arrow from the origin to $(u_1 + v_1, u_2 + v_2)$ then is the standard representation of $\mathbf{u} + \mathbf{v}$ (Figure 10.12a).

In the **parallelogram representation**, the standard representations of $\mathbf{u}$ and $\mathbf{v}$ determine a parallelogram whose diagonal is the standard representation of $\mathbf{u} + \mathbf{v}$ (Figure 10.12b).
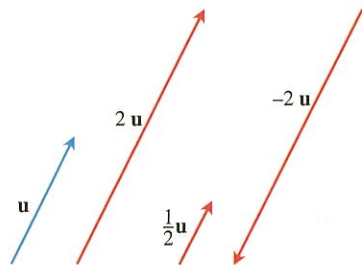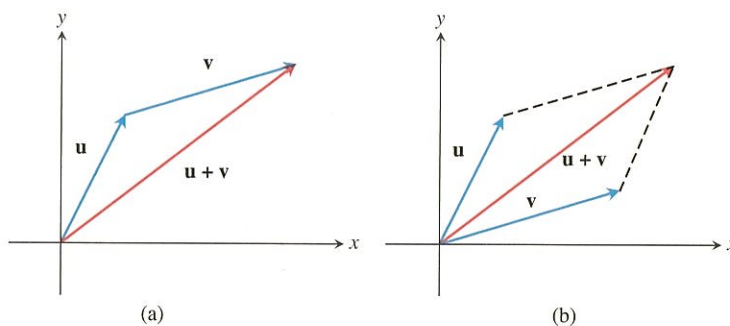

(a)   (b)

Figure 10.12 Two ways to represent vector addition geometrically: (a) tail-to-head and (b) parallelogram.

The product $k\mathbf{u}$ of the scalar k and the vector $\mathbf{u}$ can be represented by a stretch (or shrink) of $\mathbf{u}$ by a factor of k. If $k > 0$, then $k\mathbf{u}$ points in the same direction as $\mathbf{u}$; if $k < 0$, then $k\mathbf{u}$ points in the opposite direction (Figure 10.13).

Figure 10.13 Representations of $\mathbf{u}$ and several scalar multiples of $\mathbf{u}$.

EXAMPLE 3   Performing Operations on Vectors

Let $\mathbf{u} = \langle -1, 3 \rangle$ and $\mathbf{v} = \langle 4, 7 \rangle$. Find the following.

(a) $2\mathbf{u} + 3\mathbf{v}$ **(b)** $\mathbf{u} - \mathbf{v}$ **(c)** $\left| \dfrac{1}{2}\mathbf{u} \right|$

SOLUTION

(a) $2\mathbf{u} + 3\mathbf{v} = 2\langle -1, 3 \rangle + 3\langle 4, 7 \rangle$
$$= \langle 2(-1) + 3(4), 2(3) + 3(7) \rangle = \langle 10, 27 \rangle$$

continued

(b) $\mathbf{u} - \mathbf{v} = \langle -1, 3 \rangle - \langle 4, 7 \rangle$

$\qquad = \langle -1 - 4, 3 - 7 \rangle = \langle -5, -4 \rangle$

(c) $\left| \dfrac{1}{2}\mathbf{u} \right| = \left| \left\langle -\dfrac{1}{2}, \dfrac{3}{2} \right\rangle \right| = \sqrt{\left(-\dfrac{1}{2} \right)^2 + \left(\dfrac{3}{2} \right)^2} = \dfrac{1}{2}\sqrt{10}$

Now try Exercise 21.

Vector operations have many of the properties of their real-number counterparts.

Properties of Vector Operations

Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be vectors and a, b be scalars.

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

3. $\mathbf{u} + \mathbf{0} = \mathbf{u}$

4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

5. $0\mathbf{u} = \mathbf{0}$

6. $1\mathbf{u} = \mathbf{u}$

7. $a(b\mathbf{u}) = (ab)\mathbf{u}$

8. $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$

9. $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$

Modeling Planar Motion

Although vectors are used in many other physical applications, our primary reason for introducing them into this course is to model the motion of objects moving in a coordinate plane. You may have seen vector problems of the following type in a physics or mechanics course.

EXAMPLE 4 Finding Ground Speed and Direction

A Boeing® 727® airplane, flying due east at 500 mph in still air, encounters a 70-mph tail wind acting in the direction 60° north of east. The airplane holds its compass heading due east but, because of the wind, acquires a new ground speed and direction. What are they?

SOLUTION

If $\mathbf{u}$ = the velocity of the airplane alone and $\mathbf{v}$ = the velocity of the tail wind, then $|\mathbf{u}| = 500$ and $|\mathbf{v}| = 70$ (Figure 10.14).

We need to find the magnitude and direction of the *resultant vector* $\mathbf{u} + \mathbf{v}$. If we let the positive x-axis represent east and the positive y-axis represent north, then the component forms of $\mathbf{u}$ and $\mathbf{v}$ are

$$\mathbf{u} = \langle 500, 0 \rangle \quad \text{and} \quad \mathbf{v} = \langle 70 \cos 60°, 70 \sin 60° \rangle = \langle 35, 35\sqrt{3} \rangle.$$

Therefore,

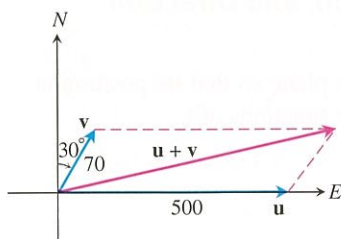$$\mathbf{u} + \mathbf{v} = \langle 535, 35\sqrt{3} \rangle,$$

$$|\mathbf{u} + \mathbf{v}| = \sqrt{535^2 + (35\sqrt{3})^2} \approx 538.4,$$

and

$$\theta = \tan^{-1} \frac{35\sqrt{3}}{535} \approx 6.5°.$$

Interpret The new ground speed of the airplane is about 538.4 mph, and its new direction is about 6.5° north of east.

Now try Exercise 25.

Figure 10.14 Vectors representing the velocities of the airplane and tail wind in Example 4.

Recall that if the position x of an object moving along a line is given as a function of time t, then the velocity of the object is dx/dt and the acceleration of the object is d^2x/dt^2. It is almost as simple to relate position, velocity, and acceleration for an object moving in the plane, because we can model those functions with vectors and treat *the components of the vectors as separate linear models*. Example 5 shows how simple this modeling actually is.

EXAMPLE 5 Doing Calculus Componentwise

A particle moves in the plane so that its position at any time $t \geq 0$ is given by $(\sin t, t^2/2)$.

(a) Find the position vector of the particle at time t.

(b) Find the velocity vector of the particle at time t.

(c) Find the acceleration of the particle at time t.

(d) Describe the position and motion of the particle at time $t = 6$.

SOLUTION

(a) The position vector, which has the same components as the position point, is $\langle \sin t, t^2/2 \rangle$. In fact, it could also be represented as $(\sin t, t^2/2)$, since the context would identify it as a vector.

(b) Differentiate each component of the position vector to get $\langle \cos t, t \rangle$.

(c) Differentiate each component of the velocity vector to get $\langle -\sin t, 1 \rangle$.

(d) The particle is at the point $(\sin 6, 18)$, with velocity $\langle \cos 6, 6 \rangle$ and acceleration $\langle -\sin 6, 1 \rangle$.

You can graph the path of this particle parametrically, letting $x = \sin(t)$ and $y = t^2/2$. In Figure 10.15 we show the path of the particle from $t = 0$ to $t = 6$. The red arrow at the point $(\sin 6, 18)$ represents the velocity vector $(\cos 6, 6)$. It shows both the magnitude and direction of the velocity at that moment in time. *Now try Exercise 31.*
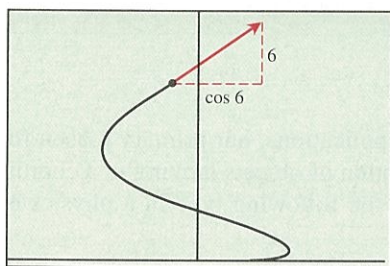
[−2, 2] by [0, 25]

$0 \leq t \leq 6$

Figure 10.15 The path of the particle in Example 5 from $t = 0$ to $t = 6$. The red arrow shows the velocity vector at $t = 6$.

Velocity, Acceleration, and Speed

We are now ready to give some definitions.

A Word About Differentiability

Our definitions can be expanded to a **calculus of vectors**, in which (for example) $d\mathbf{v}/dt = \mathbf{a}(t)$, but it is not our intention to get into that here. We have therefore finessed the fine point of vector differentiability by requiring the path of our particle to be "smooth." The path can have vertical tangents, fail the vertical line test, and loop back on itself, but corners and cusps are still problematic.

**DEFINITIONS Velocity, Speed, Acceleration, and Direction of Motion**

Suppose a particle moves along a smooth curve in the plane so that its position at any time t is $(x(t), y(t))$, where x and y are differentiable functions of t.

1. The particle's **position vector** is $\mathbf{r}(t) = \langle x(t), y(t) \rangle$.

2. The particle's **velocity vector** is $\mathbf{v}(t) = \left\langle \dfrac{dx}{dt}, \dfrac{dy}{dt} \right\rangle$.

3. The particle's **speed** is the magnitude of $\mathbf{v}$, denoted $|\mathbf{v}|$. Speed is a *scalar*, not a vector.

4. The particle's **acceleration vector** is $\mathbf{a}(t) = \left\langle \dfrac{d^2x}{dt^2}, \dfrac{d^2y}{dt^2} \right\rangle$.

5. The particle's **direction of motion** is the **direction vector** $\dfrac{\mathbf{v}}{|\mathbf{v}|}$.

EXAMPLE 6 Studying Planar Motion

A particle moves in the plane with position vector $\mathbf{r}(t) = \langle \sin(3t), \cos(5t) \rangle$. Find the velocity and acceleration vectors and determine the path of the particle.

continued
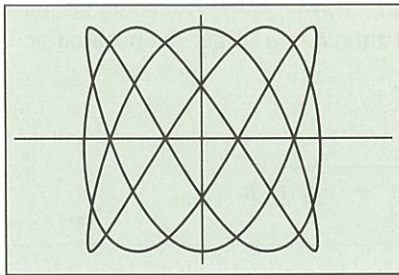
[−1.6, 1.6] by [−1.1, 1.1]

$0 \le t \le 6.3$

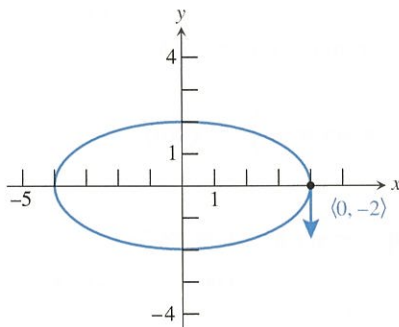**Figure 10.16** The path of the busy particle in Example 6.

SOLUTION

Velocity $\mathbf{v}(t) = \left\langle \dfrac{d}{dt}(\sin(3t)), \dfrac{d}{dt}(\cos(5t)) \right\rangle = \langle 3 \cos(3t), -5 \sin(5t) \rangle$.

Acceleration $\mathbf{a}(t) = \left\langle \dfrac{d}{dt}(3 \cos(3t)), \dfrac{d}{dt}(-5 \sin(5t)) \right\rangle = \langle -9 \sin(3t), -25 \cos(5t) \rangle$.

The path of the particle is found by graphing (in parametric mode) the curve defined by $x = \sin(3t)$ and $y = \cos(5t)$ (Figure 10.16).

Now try Exercise 33.

EXAMPLE 7   Studying Planar Motion

A particle moves in an elliptical path so that its position at any time $t \ge 0$ is given by $(4 \sin t, 2 \cos t)$.

(a) Find the velocity and acceleration vectors.

(b) Find the velocity, acceleration, speed, and direction of motion at $t = \pi/4$.

(c) Sketch the path of the particle and show the velocity vector at the point $(4, 0)$.

(d) Does the particle travel clockwise or counterclockwise around the origin?

SOLUTION

(a) Velocity $\mathbf{v}(t) = \left\langle \dfrac{d}{dt}(4 \sin t), \dfrac{d}{dt}(2 \cos t) \right\rangle = \langle 4 \cos t, -2 \sin t \rangle$

Acceleration $\mathbf{a}(t) = \left\langle \dfrac{d}{dt}(4 \cos t), \dfrac{d}{dt}(-2 \sin t) \right\rangle = \langle -4 \sin t, -2 \cos t \rangle$

(b) Velocity $\mathbf{v}(\pi/4) = \langle 4 \cos(\pi/4), -2 \sin(\pi/4) \rangle = \langle 2\sqrt{2}, -\sqrt{2} \rangle$

Acceleration $\mathbf{a}(\pi/4) = \langle -4 \sin(\pi/4), -2 \cos(\pi/4) \rangle = \langle -2\sqrt{2}, -\sqrt{2} \rangle$

Speed $= |\mathbf{v}(\pi/4)| = |\langle 2\sqrt{2}, -\sqrt{2} \rangle| = \sqrt{(2\sqrt{2})^2 + (-\sqrt{2})^2} = \sqrt{10}$

Figure 10.17 The ellipse on which the particle travels in Example 7. The velocity vector at the point $(4, 0)$ is $\langle 0, -2 \rangle$, represented by an arrow tangent to the ellipse at $(4, 0)$ and pointing down. The direction of the velocity at that point indicates that the particle travels clockwise around the origin.

(c) The ellipse defined parametrically by $x = 4 \sin t$ and $y = 2 \cos t$ is shown in Figure 10.17. At the point $(4, 0)$, $\sin t = 1$ and $\cos t = 0$, so $\mathbf{v}(t) = \langle 4 \cos t, -2 \sin t \rangle = \langle 0, -2 \rangle$. The vector $\langle 0, -2 \rangle$ is drawn tangent to the curve at $(4, 0)$.

(d) As the vector in Figure 10.17 shows, the particle travels clockwise around the origin.

Now try Exercise 35.

Displacement and Distance Traveled

Recall that when a particle moves along a line with velocity $v(t)$, the displacement (or net distance traveled) from time $t = a$ to time $t = b$ is given by $\int_a^b v(t)\, dt$, while the (total) distance traveled in that time interval is given by $\int_a^b |v(t)|\, dt$. When a particle moves in the plane with velocity vector $\mathbf{v}(t)$, displacement and distance traveled can be found by applying the same integrals to the vector $\mathbf{v}$, although in slightly different ways.

DEFINITIONS   Displacement and Distance Traveled

Suppose a particle moves along a path in the plane so that its velocity at any time t is $\mathbf{v}(t) = (v_1(t), v_2(t))$, where v_1 and v_2 are integrable functions of t.

The **displacement** from $t = a$ to $t = b$ is given by the vector

$$\left\langle \int_a^b v_1(t)\, dt, \int_a^b v_2(t)\, dt \right\rangle.$$

continued

The preceding vector is added to the position at time $t = a$ to get the position at time $t = b$.

The **distance traveled** from $t = a$ to $t = b$ is

$$\int_a^b |\mathbf{v}(t)|\,dt = \int_a^b \sqrt{(v_1(t))^2 + (v_2(t))^2}\,dt.$$

There are two things worth noting about the formula for distance traveled. First of all, it is a nice example of the integral as an accumulator, since we are summing up bits of speed multiplied by bits of time, which equals bits of positive distance. Secondly, it is actually a new look at an old formula. Substitute dx/dt for $v_1(t)$ and dy/dt for $v_2(t)$ and you get the arc length formula for a curve defined parametrically (Section 10.1). This formula makes sense, since the distance the particle travels is precisely the length of the path along which it moves.

EXAMPLE 8 Finding Displacement and Distance Traveled

A particle moves in the plane with velocity vector $\mathbf{v}(t) = (t - 3\pi \cos \pi t, 2t - \pi \sin \pi t)$. At $t = 0$, the particle is at the point $(1, 5)$.

(a) Find the position of the particle at $t = 4$.

(b) What is the total distance traveled by the particle from $t = 0$ to $t = 4$?

SOLUTION

(a) Displacement $= \left\langle \int_0^4 (t - 3\pi \cos \pi t)\,dt, \int_0^4 (2t - \pi \sin \pi t)\,dt \right\rangle = \langle 8, 16 \rangle$.

The particle is at the point $(1 + 8, 5 + 16) = (9, 21)$.

(b) Distance traveled $= \int_0^4 \sqrt{(t - 3\pi \cos \pi t)^2 + (2t - \pi \sin \pi t)^2}\,dt \approx 33.533$.

Now try Exercise 37.

EXAMPLE 9 Finding the Path of the Particle

Determine the path that the particle in Example 8 travels going from $(1, 5)$ to $(9, 21)$.

SOLUTION

The velocity vector and the position at $t = 0$ combine to give us the vector equivalent of an initial value problem. We simply find the components of the position vector separately.

$$\frac{dx}{dt} = t - 3\pi \cos \pi t$$

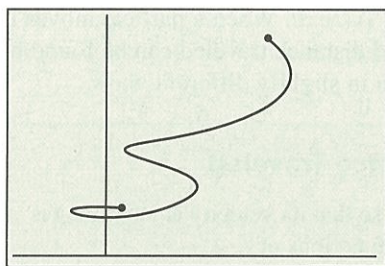$$x = \frac{t^2}{2} - 3 \sin \pi t + C \qquad \text{Antidifferentiate.}$$

$$x = \frac{t^2}{2} - 3 \sin \pi t + 1 \qquad x = 1 \text{ when } t = 0.$$

$$\frac{dy}{dt} = 2t - \pi \sin \pi t$$

$$y = t^2 + \cos \pi t + C \qquad \text{Antidifferentiate.}$$

$$y = t^2 + \cos \pi t + 4 \qquad y = 5 \text{ when } t = 0.$$

We then graph the position $\langle t^2/2 - 3 \sin \pi t + 1, t^2 + \cos \pi t + 4 \rangle$ parametrically from $t = 0$ to $t = 4$. The path is shown in Figure 10.18.

Now try Exercise 41.

[−5, 15] by [0, 23]

$0 \le t \le 4$

Figure 10.18 The path traveled by the particle in Example 8 as it goes from $(1, 5)$ to $(9, 21)$ in four seconds (Example 9).

Quick Review 10.2   *(For help, go to Sections 1.1, 4.3, and 10.1.)*

In Exercises 1–4, let $P = (1, 2)$ and $Q = (5, 3)$.

1. Find the distance between the points P and Q.

2. Find the slope of the line segment PQ.

3. If $R = (3, b)$, determine b so that segments PQ and RQ are collinear.

4. If $R = (3, b)$, determine b so that segments PQ and RQ are perpendicular.

In Exercises 5 and 6, determine the missing coordinate so that the four points form a parallelogram $ABCD$.

5. $A = (0, 0)$, $B = (1, 3)$, $C = (5, 3)$, $D = (a, 0)$

6. $A = (1, 1)$, $B = (3, 5)$, $C = (8, b)$, $D = (6, 2)$

7. Find the velocity and acceleration of a particle moving along a line if its position at time t is given by $x(t) = t \sin t$.

8. A particle moves along the x-axis with velocity $v(t) = 3t^2 - 12t$ for $t \geq 0$. If its position is $x = 40$ when $t = 0$, where is the particle when $t = 4$?

9. A particle moves along the x-axis with velocity $v(t) = 3t^2 - 12t$ for $t \geq 0$. What is the total distance traveled by the particle from $t = 0$ to $t = 4$?

10. Find the length of the curve defined parametrically by $x = \sin(2t)$ and $y = \cos(3t)$ for $0 \leq t \leq 2\pi$.

Section 10.2 Exercises

In Exercises 1–4, find the component form of the vector.

1. the vector from the origin to the point $A = (2, 3)$

2. the vector from the point $A = (2, 3)$ to the origin

3. the vector $\overrightarrow{PQ}$, where $P = (1, 3)$ and $Q = (2, -1)$

4. the vector $\overrightarrow{OP}$, where O is the origin and P is the midpoint of the segment RS connecting $R = (2, -1)$ and $S = (-4, 3)$.

In Exercises 5–10, find the magnitude of the vector and the direction angle θ it forms with the positive x-axis ($0 \leq \theta < 360°$).

5. $\langle 2, 2 \rangle$

6. $\langle -\sqrt{2}, \sqrt{2} \rangle$

7. $\langle \sqrt{3}, 1 \rangle$

8. $\langle -2, -2\sqrt{3} \rangle$

9. $\langle -5, 0 \rangle$

10. $\langle 0, 4 \rangle$

In Exercises 11–16, find the component form of the vector with the given magnitude that forms the given directional angle with the positive x-axis.

11. 4, 180°

12. 6, 270°

13. 5, 100°

14. 13, 200°

15. $3\sqrt{2}$, $\pi/4$ radians

16. $2\sqrt{3}$, $\pi/6$ radians

In Exercises 17–24, let $\mathbf{u} = \langle 3, -2 \rangle$ and $\mathbf{v} = \langle -2, 5 \rangle$. Find the (a) component form and (b) magnitude of the vector.

17. $3\mathbf{u}$

18. $-2\mathbf{v}$

19. $\mathbf{u} + \mathbf{v}$

20. $\mathbf{u} - \mathbf{v}$

21. $2\mathbf{u} - 3\mathbf{v}$

22. $-2\mathbf{u} + 5\mathbf{v}$

23. $\dfrac{3}{5}\mathbf{u} + \dfrac{4}{5}\mathbf{v}$

24. $-\dfrac{5}{13}\mathbf{u} + \dfrac{12}{13}\mathbf{v}$

25. *Navigation* An airplane, flying in the direction 20° east of north at 325 mph in still air, encounters a 40-mph tail wind acting in the direction 40° west of north. The airplane maintains its compass heading but, because of the wind, acquires a new ground speed and direction. What are they?

26. A river is flowing due east at 2 mph. A canoeist paddles across the river at 4 mph with his bow aimed directly northwest (a direction angle of 135°). What is the true direction angle of the canoeist's path, and how fast is the canoe going?

In Exercises 27–32, a particle travels in the plane with position vector $\mathbf{r}(t)$. Find (a) the velocity vector $\mathbf{v}(t)$ and (b) the acceleration vector $\mathbf{a}(t)$.

27. $\mathbf{r}(t) = \langle 3t^2, 2t^3 \rangle$

28. $\mathbf{r}(t) = \langle \sin 2t, 2 \cos t \rangle$

29. $\mathbf{r}(t) = \langle te^{-t}, e^{-t} \rangle$

30. $\mathbf{r}(t) = \langle 2 \cos 3t, 2 \sin 4t \rangle$

31. $\mathbf{r}(t) = \langle t^2 + \sin 2t, t^2 - \cos 2t \rangle$

32. $\mathbf{r}(t) = \langle t \sin t, t \cos t \rangle$

33. A particle moves in the plane with position vector $\langle \cos 3t, \sin 2t \rangle$. Find the velocity and acceleration vectors and determine the path of the particle.

34. A particle moves in the plane with position vector $\langle \sin 4t, \cos 3t \rangle$. Find the velocity and acceleration vectors and determine the path of the particle.

35. A particle moves in the plane so that its position at any time $t \geq 0$ is given by $x = \sin 4t \cos t$ and $y = \sin 2t$.

(a) Find the velocity and speed of the particle when $t = 5\pi/4$.

(b) Draw the path of the particle and show the velocity vector at $t = 5\pi/4$.

(c) Is the particle moving to the left or to the right when $t = 5\pi/4$?

36. A particle moves in the plane so that its position at any time $t \geq 0$ is given by $x = e^t + e^{-t}$ and $y = e^t - e^{-t}$.

(a) Find the velocity vector.

(b) Find $\lim\limits_{t \to \infty} \dfrac{dy/dt}{dx/dt}$.

(c) Show algebraically that the particle moves on the hyperbola $x^2 - y^2 = 4$.

(d) Sketch the path of the particle, showing the velocity vector at $t = 0$.

In Exercises 37–40, the velocity $\mathbf{v}(t)$ of a particle moving in the plane is given, along with the position of the particle at time $t = 0$. Find (a) the position of the particle at time $t = 3$, and (b) the distance the particle travels from $t = 0$ to $t = 3$.

37. $\mathbf{v}(t) = \langle 3t^2 - 2t, 1 + \cos \pi t \rangle$;   (2, 6)

38. $\mathbf{v}(t) = \langle 2\pi \cos 4\pi t, 4\pi \sin 2\pi t \rangle$;   (7, 2)

39. $\mathbf{v}(t) = \langle (t + 1)^{-1}, (t + 2)^{-2} \rangle$;   (3, −2)

40. $\mathbf{v}(t) = \langle e^t - t, e^t + t \rangle$;   (1, 1)

41. Sketch the path that the particle travels in Exercise 37.

42. Sketch the path that the particle travels in Exercise 38.

43. A point moves in the plane so that $x = 5 \cos(\pi t/6)$ and $y = 3 \sin(\pi t/6)$.

   (a) Find the speed of the point at $t = 2$.

   (b) Find the acceleration vector at $t = 2$.

   (c) Eliminate the parameter and find an equation in x and y that defines the curve on which the point moves.

44. A particle moves with position vector $\langle \sec \pi t, \tan \pi t \rangle$ for $0 \le t < 1/2$.

   (a) Find the velocity and speed of the particle at $t = 1/4$.

   (b) The particle moves along a hyperbola. Eliminate the parameter to find an equation of the hyperbola in terms of x and y.

   (c) Sketch the path of the particle over the time interval $0 \le t < 1/2$.

45. A particle moves on the circle $x^2 + y^2 = 1$ so that its position vector at any time $t \ge 0$ is $\left\langle \dfrac{1 - t^2}{1 + t^2}, \dfrac{2t}{1 + t^2} \right\rangle$.

   (a) Find the velocity vector.

   (b) Is the particle ever at rest? Justify your answer.

   (c) Give the coordinates of the point that the particle approaches as t increases without bound.

46. A particle moves in the plane so that its position at any time t, $0 \le t \le 2\pi$, is given parametrically by $x = \sin t$ and $y = \cos(2t)$.

   (a) Find the velocity vector for the particle.

   (b) For what values of t is the particle at rest?
   (c) Write an equation for the path of the particle in terms of x and y that does not involve trigonometric functions.

   (d) Sketch the path of the particle.

47. A particle moves in the plane so that its position at any time t, $0 \le t \le 2\pi$, is given parametrically by $x = e^t \sin t$ and $y = e^t \cos t$.

   (a) Find the slope of the path of the particle at time $t = \pi/2$.

   (b) Find the speed of the particle when $t = 1$.

   (c) Find the distance traveled by the particle along the path from $t = 0$ to $t = 1$.

48. The position of a particle at any time $t \ge 0$ is given by

   $x(t) = t^2 - 3$ and $y(t) = \frac{2}{3}t^3$.

   (a) Find the magnitude of the velocity vector at $t = 4$.

   (b) Find the total distance traveled by the particle from $t = 0$ to $t = 4$.

   (c) Find dy/dx as a function of x.

49. An object moving along a curve in the xy-plane has position $(x(t), y(t))$ at time $t \ge 0$ with $dx/dt = 2 + \sin(t^2)$. The derivative dy/dt is not explicitly given. At time $t = 2$, the object is at position (3, 5).

   (a) Find the x-coordinate of the position of the object at time $t = 4$.

   (b) At time $t = 2$, the value of dy/dt is -6. Write an equation for the line tangent to the curve at the point $(x(2), y(2))$.

   (c) Find the speed of the object at time $t = 2$.

   (d) For $t \ge 3$, the line tangent to the curve at $(x(t), y(t))$ has a slope of $2t - 1$. Find the acceleration vector of the object at time $t = 4$.

50. For $0 \le t \le 3$, an object moving along a curve in the xy-plane has position $(x(t), y(t))$ with $dx/dt = \sin(t^3)$ and $dy/dt = 3 \cos(t^2)$. At time $t = 2$, the object is at position (4, 5).

   (a) Write an equation for the line tangent to the curve at (4, 5).

   (b) Find the speed of the object at time $t = 2$.

   (c) Find the total distance traveled by the object over the time interval $0 \le t \le 1$.

   (d) Find the position of the object at time $t = 3$.

Standardized Test Questions

 You may use a graphing calculator to solve the following problems.

51. True or False A scalar multiple of a vector $\mathbf{v}$ has the same direction as $\mathbf{v}$. Justify your answer.

52. True or False If a vector with direction angle 0° is added to a vector with direction angle 90°, the result is a vector with direction angle 45°. Justify your answer.

53. Multiple Choice The position of a particle in the xy-plane is given by $x = t^2 + 1$ and $y = \ln(2t + 3)$ for all $t \ge 0$. The acceleration vector of the particle is

   (A) $\left(2t, \dfrac{2}{2t + 3} \right)$.   (B) $\left(2t, -\dfrac{4}{(2t + 3)^2} \right)$.   (C) $\left(2, \dfrac{4}{(2t + 3)^2} \right)$.

   (D) $\left(2, \dfrac{2}{(2t + 3)^2} \right)$.   (E) $\left(2, -\dfrac{4}{(2t + 3)^2} \right)$.

54. Multiple Choice An object moving along a curve in the xy-plane has position $(x(t), y(t))$ with $dx/dt = \cos(t^2)$ and $dy/dt = \sin(t^3)$. At time $t = 0$, the object is at position (4, 7). Where is the particle when $t = 2$?

   (A) $\langle -0.654, 0.989 \rangle$ (B) $\langle 0.461, 0.452 \rangle$ (C) $\langle 3.346, 7.989 \rangle$
   (D) $\langle 4.461, 7.452 \rangle$ (E) $\langle 5.962, 8.962 \rangle$

55. Multiple Choice A vector with magnitude 7 and direction angle 40° is added to a vector with magnitude 4 and direction angle 140°. The result is a vector with magnitude

   (A) 4.684.   (B) 7.435.   (C) 8.062.   (D) 9.369.   (E) 11.

56. Multiple Choice The path of a particle moving in the plane is defined parametrically as a function of time t by $x = \sin 2t$ and $y = \cos 5t$. What is the speed of the particle when $t = 2$?

   (A) 1.130   (B) 3.018   (C) $\langle -1.307, 2.720 \rangle$
   (D) $\langle 0.757, 0.839 \rangle$ (E) $\langle 1.307, 2.720 \rangle$

Explorations

Two nonzero vectors are said to be *orthogonal* if they are perpendicular to each other. The zero vector is considered to be orthogonal to every vector.

57. **Orthogonal vectors** A particle with coordinates (x, y) moves along a curve in the first quadrant in such a way that $dx/dt = -x$ and $dy/dt = \sqrt{1 - x^2}$ for every $t \geq 0$. Find the acceleration vector in terms of x and show that it is orthogonal to the corresponding velocity vector.

58. **Orthogonal vectors** A particle moves around the unit circle with position vector $\langle \cos t, \sin t \rangle$. Use vectors to show that the particle's velocity is always orthogonal to both its position and its acceleration.

59. **Colliding particles** The paths of two particles for $t \geq 0$ are given by the position vectors

$$\mathbf{r}_1(t) = \langle t - 3, (t - 3)^2 \rangle$$

$$\mathbf{r}_2(t) = \left\langle \frac{3t}{2} - 4, \frac{3t}{2} - 2 \right\rangle.$$

(a) Determine the exact time(s) at which the particles collide.

(b) Find the direction of motion of each particle at the time(s) of collision.

60. **A Satellite in Circular Orbit** A satellite of mass m is moving at a constant speed v around a planet of mass M in a circular orbit of radius r_0, as measured from the planet's center of mass. Determine the satellite's orbital period T (the time to complete one full orbit), as follows:

(a) Coordinatize the orbital plane by placing the origin at the planet's center of mass, with the satellite on the x-axis at $t = 0$ and moving counterclockwise, as in the accompanying figure.
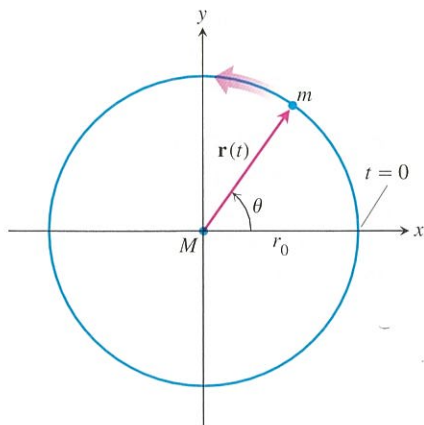
Let $\mathbf{r}(t)$ be the satellite's position vector at time t. Show that $\theta = vt/r_0$ and hence that

$$\mathbf{r}(t) = \left\langle r_0 \cos \frac{vt}{r_0}, r_0 \sin \frac{vt}{r_0} \right\rangle.$$

(b) Find the acceleration of the satellite.

(c) According to Newton's law of gravitation, the gravitational force exerted on the satellite by the planet is directed toward the origin and is given by

$$\mathbf{F} = \left(-\frac{GmM}{r_0^2} \right) \frac{\mathbf{r}}{r_0},$$

where G is the universal constant of gravitation. Using Newton's second law, $\mathbf{F} = m\mathbf{a}$, show that $v^2 = GM/r_0$.

(d) Show that the orbital period T satisfies $vT = 2\pi r_0$.

(e) From parts (c) and (d), deduce that

$$T^2 = \frac{4\pi^2}{GM} r_0^3;$$

that is, the square of the period of a satellite in circular orbit is proportional to the cube of the radius from the orbital center.

Extending the Ideas

Let $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ be vectors in the plane. The **dot product** or **inner product** $\mathbf{u} \cdot \mathbf{v}$ is a scalar defined by

$$\mathbf{u} \cdot \mathbf{v} = \langle u_1, u_2 \rangle \cdot \langle v_1, v_2 \rangle = u_1 v_1 + u_2 v_2.$$

61. **Using the Dot Product** Show that the dot product of two perpendicular vectors is zero.

62. **An Alternate Formula for Dot Product** Let $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ be vectors in the plane, and let $\mathbf{w} = \mathbf{u} - \mathbf{v}$.

(a) Explain why $\mathbf{w}$ can be represented by the arrow in the accompanying diagram.

(b) Explain why $|\mathbf{w}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}||\mathbf{v}| \cos \theta$, where θ is the angle between vectors $\mathbf{u}$ and $\mathbf{v}$.

(c) Find the component form of $\mathbf{w}$ and use it to prove that

$$|\mathbf{u}|^2 + |\mathbf{v}|^2 - |\mathbf{w}|^2 = 2(u_1 v_1 + u_2 v_2).$$

(d) Finally, prove that $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \theta$, where θ is the angle between vectors $\mathbf{u}$ and $\mathbf{v}$.
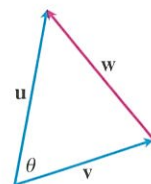
10.3 Polar Functions

. . . and why

Polar equations enable us to define some interesting and important curves that would be difficult or impossible to define in the form $y = f(x)$.

Polar Coordinates

If you graph the two functions $y = \sin 3x$ and $y = \cos 5x$ on the same pair of axes, you will get two sinusoids. But if you graph the curve defined *parametrically* by $x = \sin 3t$ and $y = \cos 5t$, you will get the figure shown. Parametric graphing opens up a whole new world of curves that can be defined using our familiar basic functions.
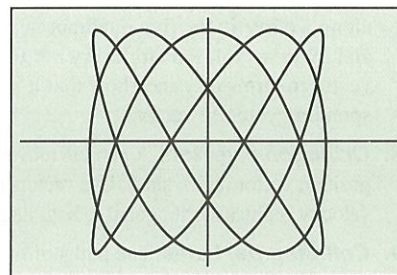
Another way to enter that world is to use a different coordinate system. In **polar coordinates** we identify the origin O as the **pole** and the positive x-axis as the **initial ray** of angles measured in the usual trigonometric way. We can then identify each point P in the plane by polar coordinates (r, θ), where r gives the directed distance from O to P and θ gives the directed angle from the initial ray to the ray $\overrightarrow{OP}$. In Figure 10.19 we see that the point P with rectangular (Cartesian) coordinates (2, 2) has polar coordinates $(2\sqrt{2}, \pi/4)$.
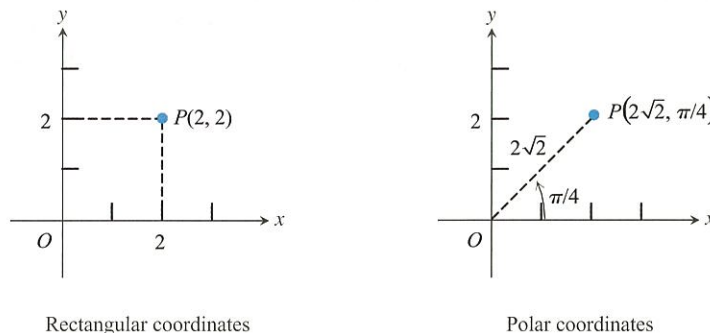

Rectangular coordinates   Polar coordinates

Figure 10.19 Point P has rectangular coordinates (2, 2) and polar coordinates $(2\sqrt{2}, \pi/4)$.

As you would expect, we can also coordinatize point P with the polar coordinates $(2\sqrt{2}, 9\pi/4)$ or $(2\sqrt{2}, -7\pi/4)$, since those angles determine the same ray $\overrightarrow{OP}$. Less obviously, we can also coordinatize P with polar coordinates $(-2\sqrt{2}, -3\pi/4)$, since the *directed* distance $-2\sqrt{2}$ in the $-3\pi/4$ direction is the same as the directed distance $2\sqrt{2}$ in the $\pi/4$ direction (Figure 10.20). So, although each pair (r, θ) determines a unique point in the plane, each point in the plane can be coordinatized by an infinite number of polar ordered pairs.
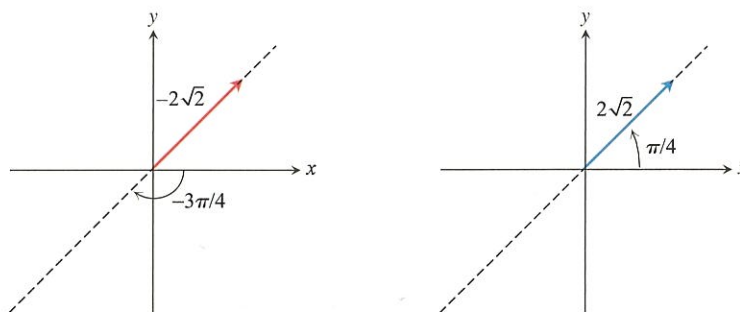

Figure 10.20 The directed *negative* distance $-2\sqrt{2}$ in the $-3\pi/4$ direction is the same as the directed *positive* distance $2\sqrt{2}$ in the $\pi/4$ direction. Thus the polar coordinates $(-2\sqrt{2}, -3\pi/4)$ and $(2\sqrt{2}, \pi/4)$ determine the same point.

EXAMPLE 1   Rectangular and Polar Coordinates

(a) Find rectangular coordinates for the points with given polar coordinates.

(i) $(4, \pi/2)$ (ii) $(-3, \pi)$ (iii) $(16, 5\pi/6)$ (iv) $(-\sqrt{2}, -\pi/4)$

(b) Find two different sets of polar coordinates for the points with given rectangular coordinates.

(i) $(1, 0)$ (ii) $(-3, 3)$ (iii) $(0, -4)$ (iv) $(1, \sqrt{3})$

SOLUTION

(a) (i) $(0, 4)$ (ii) $(3, 0)$ (iii) $(-8\sqrt{3}, 8)$ (iv) $(-1, 1)$

(b) A point has infinitely many sets of polar coordinates, so here we list just two typical examples for each given point.

(i) $(1, 0), (1, 2\pi)$ (ii) $(3\sqrt{2}, 3\pi/4), (-3\sqrt{2}, -\pi/4)$

(iii) $(4, -\pi/2), (4, 3\pi/2)$ (iv) $(2, \pi/3), (-2, 4\pi/3)$

Now try Exercises 1 and 3.

EXAMPLE 2   Graphing with Polar Coordinates

Graph all points in the plane that satisfy the given polar equation

(a) $r = 2$ (b) $r = -2$ (c) $\theta = \pi/6$

SOLUTION

First, note that we do *not* label our axes r and θ. We are graphing *polar* equations in the usual *xy*-plane, not renaming our rectangular variables!

(a) The set of all points with directed distance 2 units from the pole is a circle of radius 2 centered at the origin (Figure 10.21a).

(b) The set of all points with directed distance -2 units from the pole is also a circle of radius 2 centered at the origin (Figure 10.21b).

(c) The set of all points of positive or negative directed distance from the pole in the $\pi/6$ direction is a line through the origin with slope $\tan(\pi/6)$ (Figure 10.21c).
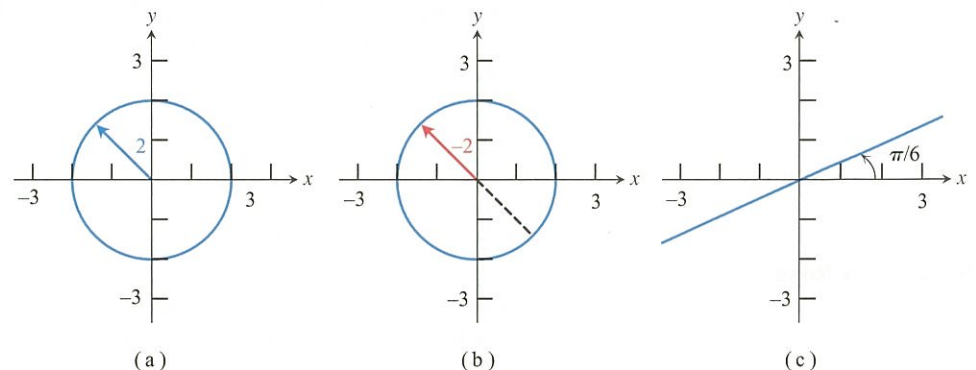
Now try Exercise 7.

(a)   (b)   (c)

Figure 10.21 Polar graphs of (a) $r = 2$, (b) $r = -2$, and (c) $\theta = \pi/6$. (Example 2)

Polar Curves

The curves in Example 2 are a start, but we would not introduce a new coordinate system just to graph circles and lines; there are far more interesting polar curves to study. In the past it was hard work to produce reasonable polar graphs by hand, but today, thanks to graphing technology, it is

just a matter of finding the right window and pushing the right buttons. Our intent in this section is to use the technology to produce the graphs and then concentrate on how calculus can be used to give us further information.

EXAMPLE 3 Polar Graphing with Technology

Find an appropriate graphing window and produce a graph of the polar curve.

(a) $r = \sin 6\theta$ (b) $r = 1 - 2 \cos \theta$ (c) $r = 4 \sin \theta$

SOLUTION

For all these graphs, set your calculator to POLAR mode.

(a) First we find the window. Notice that $|r| = |\sin 6\theta| \leq 1$ for all θ, so points on the graph are all within 1 unit from the pole. We want a window at least as large as $[-1, 1]$ by $[-1, 1]$, but we choose the window $[-1.5, 1.5]$ by $[-1, 1]$ in order to keep the *aspect ratio* close to the screen dimensions, which have a ratio of 3:2. We choose a θ-range of $0 \leq \theta \leq 2\pi$ to get a full rotation around the graph, after which we know that $\sin 6\theta$ will repeat the same graph periodically. Choose θ step = 0.05. The result is shown in Figure 10.22a.

(b) In this graph we notice that $|r| = |1 - 2 \cos \theta| \leq 3$, so we choose $[-3, 3]$ for our y-range and, to get the right aspect ratio, $[-4.5, 4.5]$ for our x-range. Due to the cosine's period, $0 \leq \theta \leq 2\pi$ again suffices for our θ-range. The graph is shown in Figure 10.22b.

(c) Since $|r| = |4 \sin \theta| \leq 4$, we choose $[-4, 4]$ for our y-range and $[-6, 6]$ for our x-range. Due to the sine's period, $0 \leq \theta \leq 2\pi$ again suffices for our θ-range. The graph is shown in Figure 10.22c.

Now try Exercise 13.

$[-1.5, 1.5]$ by $[-1, 1]$
$0 \leq \theta \leq 2\pi$

(a)

$[-4.5, 4.5]$ by $[-3, 3]$
$0 \leq \theta \leq 2\pi$

(b)

$[-6, 6]$ by $[-4, 4]$
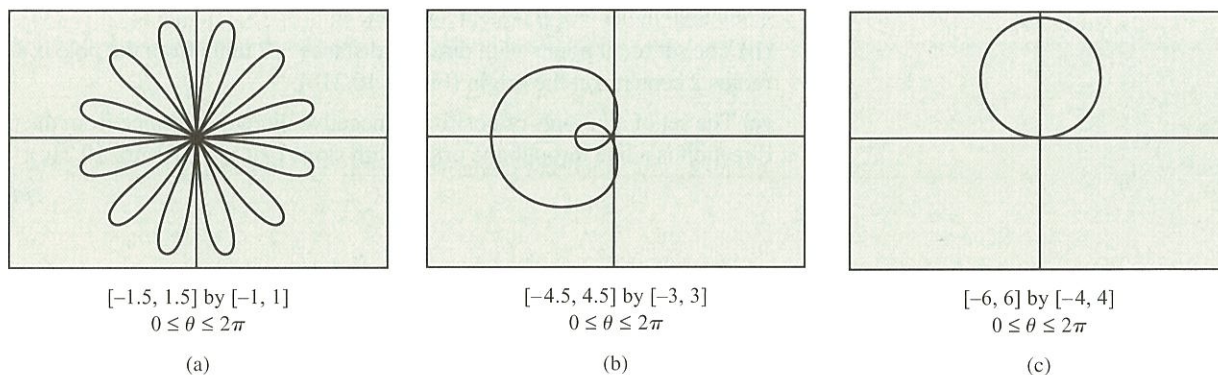$0 \leq \theta \leq 2\pi$

(c)

Figure 10.22 The graphs of the three polar curves in Example 3. The curves are (a) a 12-petaled rose, (b) a limaçon, and (c) a circle.

A Rose is a Rose

The graph in Figure 10.22a is called a 12-petaled rose, because it looks like a flower and some flowers are roses. The graph in Figure 10.22b is called a limaçon (LEE-ma-sohn) from an old French word for *snail.* We will have more names for you at the end of the section.

With a little experimentation, it is possible to improve on the "safe" windows we chose in Example 3 (at least in parts (b) and (c)), but it is always a good idea to keep a 3:2 ratio of the x-range to the y-range so that shapes do not become distorted. Also, an astute observer may have noticed that the graph in part (c) was traversed *twice* as θ went from 0 to 2π, so a range of $0 \leq \theta \leq \pi$ would have sufficed to produce the entire graph. From 0 to π, the circle is swept out by positive r values; then from π to 2π, the same circle is swept out by negative r values.

Although the graph in Figure 10.22c certainly looks like a circle, how can we tell for sure that it really is? One way is to convert the polar equation to a Cartesian equation and verify that it is the equation of a circle. Trigonometry gives us a simple way to convert polar equations to rectangular equations and vice versa.

Polar–Rectangular Conversion Formulas

$$x = r \cos \theta \qquad r^2 = x^2 + y^2$$

$$y = r \sin \theta \qquad \tan \theta = \frac{y}{x}$$

EXAMPLE 4 Converting Polar to Rectangular

Use the polar–rectangular conversion formulas to show that the polar graph of $r = 4 \sin \theta$ is a circle.

SOLUTION

To facilitate the substitutions, multiply both sides of the original equation by r. (This could introduce extraneous solutions with $r = 0$, but the pole is the only such point, and we notice that it is already on the graph.)

$$r = 4 \sin \theta$$

$$r^2 = 4r \sin \theta \qquad \text{Multiply by } r.$$

$$x^2 + y^2 = 4y \qquad \text{Polar–rectangular conversion}$$

$$x^2 + y^2 - 4y = 0$$

$$x^2 + y^2 - 4y + 4 = 4 \qquad \text{Completing the square}$$

$$x^2 + (y - 2)^2 = 2^2 \qquad \text{Circle in standard form}$$

Sure enough, the graph is a circle centered at $(0, 2)$ with radius 2.   ***Now try Exercise 25.***

The polar–rectangular conversion formulas also reveal the calculator's secret to polar graphing: It is really just parametric graphing with θ as the parameter.

Parametric Equations of Polar Curves

The polar graph of $r = f(\theta)$ is the curve defined parametrically by:

$$x = r \cos \theta = f(\theta) \cos \theta$$

$$y = r \sin \theta = f(\theta) \sin \theta$$

EXPLORATION 1 Graphing Polar Curves Parametrically

Switch your grapher to parametric mode and enter the equations

$$x = \sin (6t) \cos t$$

$$y = \sin (6t) \sin t.$$

1. Set an appropriate window and see if you can reproduce the polar graph in Figure 10.22a.

2. Then produce the graphs in Figures 10.22b and 10.22c in the same way.

Slopes of Polar Curves

Since polar curves are drawn in the *xy*-plane, the *slope* of a polar curve is still the slope of the tangent line, which is dy/dx. The polar–rectangular conversion formulas enable us to write *x* and *y* as functions of θ, so we can find dy/dx as we did with parametrically defined functions:

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}.$$
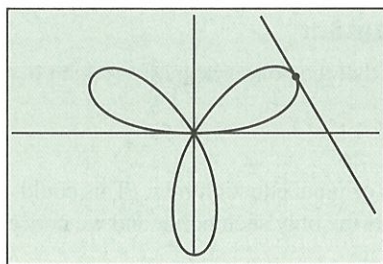
[−3, 3] by [−2, 2]

$0 \le \theta \le \pi$

Figure 10.23 The 3-petaled rose curve $r = 2 \sin 3\theta$. Example 5 shows how to find the tangent line to the curve at $\theta = \pi/6$.

EXAMPLE 5 Finding Slope of a Polar Curve

Find the slope of the rose curve $r = 2 \sin 3\theta$ at the point where $\theta = \pi/6$ and use it to find the equation of the tangent line (Figure 10.23).

SOLUTION

The slope is

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{6}} = \left.\frac{dy/d\theta}{dx/d\theta}\right|_{\theta=\frac{\pi}{6}} = \left.\frac{\dfrac{d}{d\theta}(2 \sin 3\theta \sin\theta)}{\dfrac{d}{d\theta}(2 \sin 3\theta \cos\theta)}\right|_{\theta=\frac{\pi}{6}}.$$

This expression can be computed by hand, but it is an excellent candidate for your calculator's numerical derivative functionality (Section 3.2). NDERIV quickly gives an answer of -1.732050808, which you might recognize as $-\sqrt{3}$.

When $\theta = \pi/6$,

$$x = 2 \sin(\pi/2) \cos(\pi/6) = \sqrt{3} \quad \text{and} \quad y = 2 \sin(\pi/2) \sin(\pi/6) = 1.$$

So the tangent line has equation $y - 1 = -\sqrt{3}(x - \sqrt{3})$.   ***Now try Exercise 39.***

Areas Enclosed by Polar Curves

We would like to be able to use numerical integration to find areas enclosed by polar curves just as we did with curves defined by their rectangular coordinates. Converting the equations to rectangular coordinates is not a reasonable option for most polar curves, so we would like to have a formula involving small changes in θ rather than small changes in *x*. While a small change Δx produces a thin *rectangular* strip of area, a small change $\Delta\theta$ produces a thin *circular sector* of area (Figure 10.24).

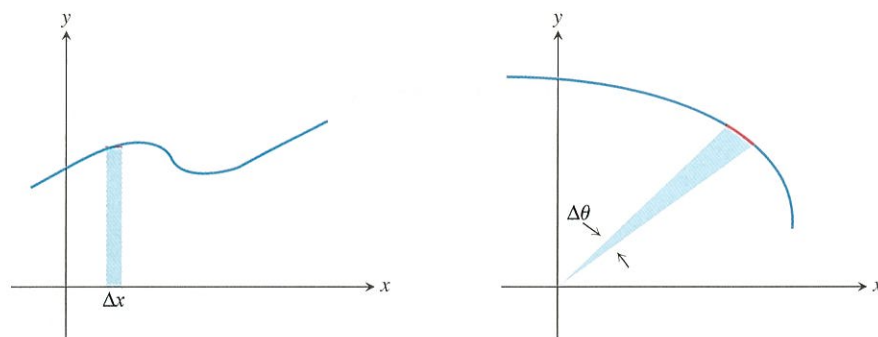**Figure 10.24** A small change in *x* produces a rectangular strip of area, while a small change in θ produces a thin *sector* of area.

Recall from geometry that the area of a sector of a circle is $\frac{1}{2}r^2\theta$, where r is the radius and θ is the central angle measured in radians. If we replace θ by the differential $d\theta$, we get **the area differential** $dA = \frac{1}{2}r^2 d\theta$ (Figure 10.25), which is exactly the quantity that we need to integrate to get an area in polar coordinates.
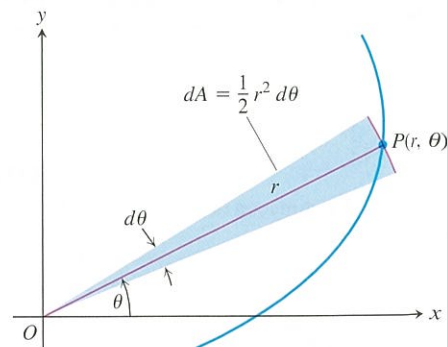

Figure 10.25 The area differential dA.

Area in Polar Coordinates

The area of the region between the origin and the curve $r = f(\theta)$ for $\alpha \leq \theta \leq \beta$ is

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta = \int_{\alpha}^{\beta} \frac{1}{2}\big(f(\theta)\big)^2 d\theta.$$

EXAMPLE 6 Finding Area

Find the area of the region in the plane enclosed by the cardioid $r = 2(1 + \cos\theta)$.

SOLUTION

Figure 10.26 The cardioid in Example 6.

We graph the cardioid (Figure 10.26) and determine that the *radius OP* sweeps out the region exactly once as θ runs from 0 to 2π.

Solve Analytically The area is therefore

$$\int_{\theta=0}^{\theta=2\pi} \frac{1}{2} r^2 \, d\theta = \int_0^{2\pi} \frac{1}{2} \cdot 4(1 + \cos\theta)^2 \, d\theta$$

$$= \int_0^{2\pi} 2(1 + 2\cos\theta + \cos^2\theta) \, d\theta$$

$$= \int_0^{2\pi} \left(2 + 4\cos\theta + 2\frac{1 + \cos 2\theta}{2}\right) d\theta$$

$$= \int_0^{2\pi} (3 + 4\cos\theta + \cos 2\theta) \, d\theta$$

$$= \left[3\theta + 4\sin\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi} = 6\pi - 0 = 6\pi.$$

Support Numerically NINT $(2(1 + \cos\theta)^2, \theta, 0, 2\pi) = 18.84955592$, which agrees with 6π to eight decimal places.
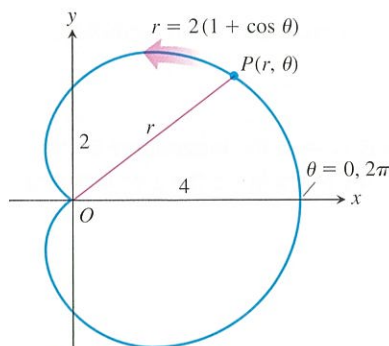
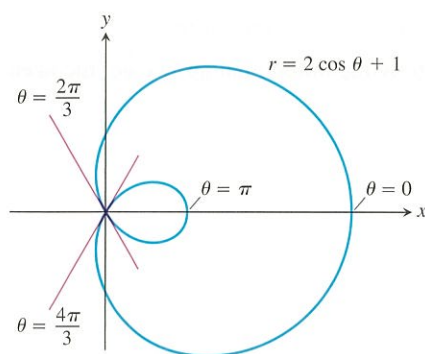*Now try Exercise 43.*

Figure 10.27 The limaçon in Example 7.

EXAMPLE 7 Finding Area
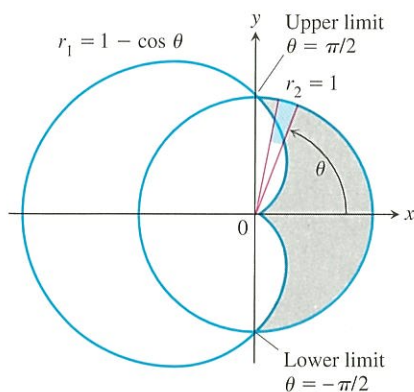
Find the area inside the smaller loop of the limaçon $r = 2 \cos \theta + 1$.

SOLUTION

After watching the grapher generate the curve over the interval $0 \le \theta \le 2\pi$ (Figure 10.27), we see that the smaller loop is traced by the point (r, θ) as θ increases from $\theta = 2\pi/3$ to $\theta = 4\pi/3$ (the values for which $r = 2 \cos \theta + 1 = 0$). The area we seek is

$$A = \int_{2\pi/3}^{4\pi/3} \frac{1}{2} r^2 \, d\theta = \frac{1}{2} \int_{2\pi/3}^{4\pi/3} (2 \cos \theta + 1)^2 \, d\theta.$$

Solve Numerically

$$\frac{1}{2} \text{ NINT } ((2 \cos \theta + 1)^2, \, \theta, \, 2\pi/3, \, 4\pi/3) \approx 0.544.$$

Now try Exercise 47.

To find the area of a region like the one in Figure 10.28, which lies between two polar curves $r_1 = r_1(\theta)$ and $r_2 = r_2(\theta)$ from $\theta = \alpha$ to $\theta = \beta$, we subtract the integral of $(1/2)r_1^2$ from the integral of $(1/2)r_2^2$. This leads to the following formula.

Figure 10.28 The area of the shaded region is calculated by subtracting the area of the region between r_1 and the origin from the area of the region between r_2 and the origin.

Area Between Polar Curves

The area of the region between $r_1(\theta)$ and $r_2(\theta)$ for $\alpha \le \theta \le \beta$ is

$$A = \int_\alpha^\beta \frac{1}{2} r_2^2 \, d\theta - \int_\alpha^\beta \frac{1}{2} r_1^2 \, d\theta = \int_\alpha^\beta \frac{1}{2} (r_2^2 - r_1^2) \, d\theta.$$

EXAMPLE 8 Finding Area Between Curves

Find the area of the region that lies inside the circle $r = 1$ and outside the cardioid $r = 1 - \cos \theta$.

SOLUTION

The region is shown in Figure 10.29. The outer curve is $r_2 = 1$, the inner curve is $r_1 = 1 - \cos \theta$, and θ runs from $-\pi/2$ to $\pi/2$. Using the formula for the area between polar curves, the area is
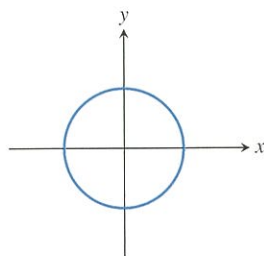
$$A = \int_{-\pi/2}^{\pi/2} \frac{1}{2} (r_2^2 - r_1^2) \, d\theta$$

$$= 2 \int_0^{\pi/2} \frac{1}{2} (r_2^2 - r_1^2) \, d\theta \qquad \text{Symmetry}$$

$$= \int_0^{\pi/2} (1 - (1 - 2 \cos \theta + \cos^2 \theta)) \, d\theta$$

$$= \int_0^{\pi/2} (2 \cos \theta - \cos^2 \theta) \, d\theta \approx 1.215. \qquad \text{Using NINT}$$

In case you are interested, the exact value is $2 - \pi/4$.

Now try Exercise 53.

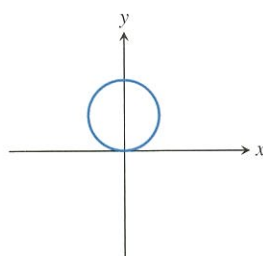**Figure 10.29** The region and limits of integration in Example 8.

A SMALL POLAR GALLERY

Here are a few of the more common polar graphs and the θ-intervals that can be used to produce them.

CIRCLES
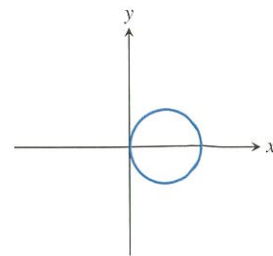
$r = \text{constant}$
$0 \le \theta \le 2\pi$
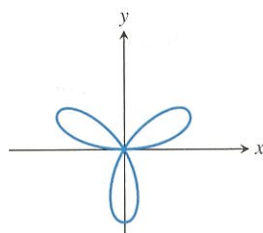
$r = a \sin \theta$
$0 \le \theta \le \pi$

$r = a \cos \theta$
$0 \le \theta \le \pi$

ROSE CURVES
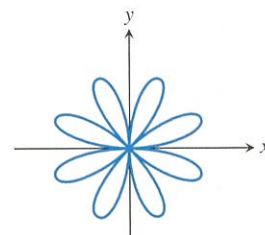
$r = a \sin n\theta,\ n \text{ odd}$
$0 \le \theta \le \pi$
n petals
y-axis symmetry

$r = a \sin n\theta,\ n \text{ even}$
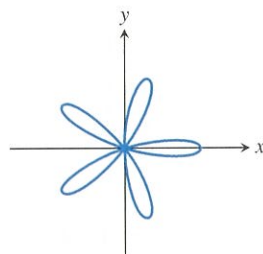$0 \le \theta \le 2\pi$
$2n$ petals
y-axis symmetry and
x-axis symmetry
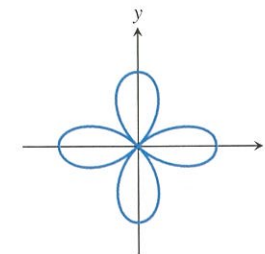
$r = a \cos n\theta,\ n \text{ odd}$
$0 \le \theta \le \pi$
n petals
x-axis symmetry

$r = a \cos n\theta,\ n \text{ even}$
$0 \le \theta \le 2\pi$
$2n$ petals
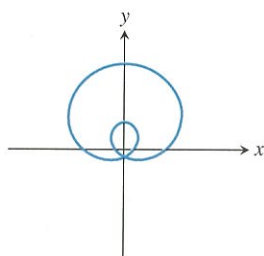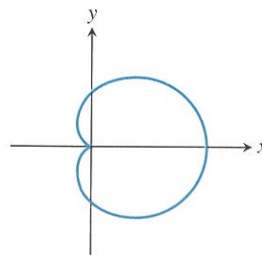y-axis symmetry and
x-axis symmetry

LIMAÇON CURVES

$r = a \pm b \sin \theta$ or $r = a \pm b \cos \theta$ with $a > 0$ and $b > 0$

($r = a \pm b \sin \theta$ has y-axis symmetry; $r = a \pm b \cos \theta$ has x-axis symmetry.)

$$\frac{a}{b} < 1$$
$$0 \le \theta \le 2\pi$$
Limaçon with loop

$$\frac{a}{b} = 1$$
$$0 \le \theta \le 2\pi$$
Cardioid
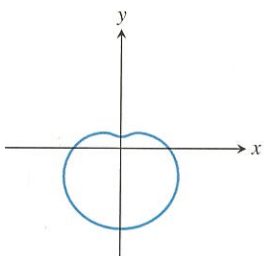
$$1 < \frac{a}{b} < 2$$
$$0 \le \theta \le \pi$$
Dimpled limaçon

$$\frac{a}{b} \ge 2$$
$$0 \le \theta \le 2\pi$$
Convex limaçon

LEMNISCATE CURVES

$$r^2 = a^2 \sin 2\theta$$
$$0 \le \theta \le \pi$$

$$r^2 = a^2 \cos 2\theta$$
$$0 \le \theta \le \pi$$

SPIRAL OF ARCHIMEDES

$$r = \theta > 0$$

Quick Review 10.3 *(For help, go to Sections 10.1 and 10.2.)*
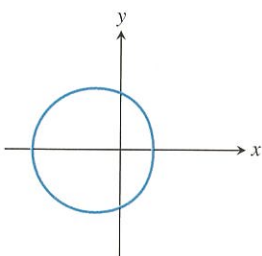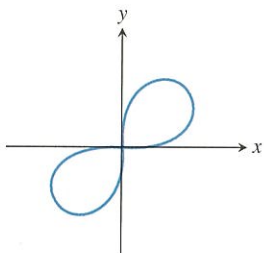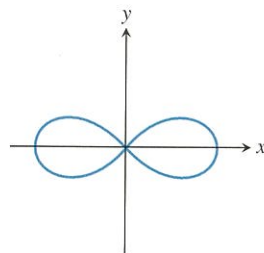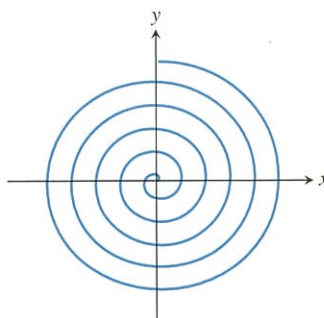
1. Find the component form of a vector with magnitude 4 and direction angle 30°.

2. Find the area of a 30° sector of a circle of radius 6.

3. Find the area of a sector of a circle of radius 8 that has a central angle of $\pi/8$ radians.

4. Find the rectangular equation of a circle of radius 5 centered at the origin.

5. Explain how to use your calculator in function mode to graph the curve $x^2 + 3y^2 = 4$.

Exercises 6–10 refer to the parametrized curve
$$x = 3\cos t, \; y = 5\sin t, \; 0 \le t \le 2\pi.$$

6. Find dy/dx.

7. Find the slope of the curve at $t = 2$.

8. Find the points on the curve where the slope is zero.

9. Find the points on the curve where the slope is undefined.

10. Find the length of the curve from $t = 0$ to $t = \pi$.

Section 10.3 Exercises

In Exercises 1 and 2, plot each point with the given polar coordinates and find the corresponding rectangular coordinates.

1. (a) $(\sqrt{2}, \pi/4)$ (b) $(1, 0)$

    (c) $(0, \pi/2)$ (d) $(-\sqrt{2}, \pi/4)$

2. (a) $(-3, 5\pi/6)$ (b) $(5, \tan^{-1}(4/3))$

    (c) $(-1, 7\pi)$ (d) $(2\sqrt{3}, 2\pi/3)$

In Exercises 3 and 4, plot each point with the given rectangular coordinates and find two sets of corresponding polar coordinates.

3. (a) $(-1, 1)$ (b) $(1, -\sqrt{3})$

    (c) $(0, 3)$ (d) $(-1, 0)$

4. (a) $(-\sqrt{3}, -1)$ (b) $(3, 4)$

    (c) $(0, -2)$ (d) $(2, 0)$

In Exercises 5–10, graph the set of points whose polar coordinates satisfy the given equation.

5. $r = 3$ 6. $r = -3$

7. $r^2 = 4$ 8. $\theta = -\pi/4$

9. $|\theta| = \pi/6$ 10. $r^2 + 8 = 6r$

In Exercises 11–20, find an appropriate window and use a graphing calculator to produce the polar curve. Then sketch the complete curve and identify the type of curve by name.

11. $r = 1 + \cos\theta$ 12. $r = 2 - 2\cos\theta$

13. $r = 2\cos 3\theta$ 14. $r = -3\sin 2\theta$

15. $r = 1 - 2\sin\theta$ 16. $r = 3/2 + \cos\theta$

17. $r^2 = 4\cos 2\theta$ 18. $r^2 = \sin 2\theta$

19. $r = 4\sin\theta$ 20. $r = 3\cos\theta$

In Exercises 21–30, replace the polar equation by an equivalent Cartesian (rectangular) equation. Then identify or describe the graph.

21. $r = 4\csc\theta$ 22. $r = -3\sec\theta$

23. $r\cos\theta + r\sin\theta = 1$ 24. $r^2 = 1$

25. $r = \dfrac{5}{\sin\theta - 2\cos\theta}$ 26. $r^2\sin 2\theta = 2$

27. $\cos^2\theta = \sin^2\theta$ 28. $r^2 = -4r\cos\theta$

29. $r = 8\sin\theta$

30. $r = 2\cos\theta + 2\sin\theta$

In Exercises 31–38, find an appropriate window and use a graphing calculator to produce the polar curve. Then sketch the complete curve and identify the type of curve by name. (*Note:* You won't find these in the Polar Gallery.)

31. $r = \sec \theta \tan \theta$

32. $r = -\csc \theta \cot \theta$

33. $r = \dfrac{1}{1 + \cos \theta}$

34. $r = \dfrac{2}{1 - \sin \theta}$

35. $r = \dfrac{14}{5 + 9 \cos \theta}$

36. $r = \dfrac{12}{8 + 6 \cos \theta}$

37. $r = \dfrac{1}{1 - 0.8 \cos \theta}$

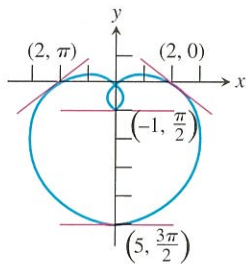**38.** $r = \dfrac{1}{1 - 1.3 \cos \theta}$

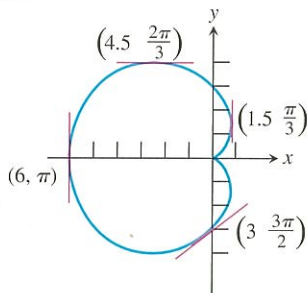In Exercises 39–42, find the slope of the curve at each indicated point.

39. $r = -1 + \sin \theta, \quad \theta = 0, \pi$

40. $r = \cos 2\theta, \quad \theta = 0, \pm \pi/2, \pi$

41. $r = 2 - 3 \sin \theta$

42. $r = 3(1 - \cos \theta)$

In Exercises 43–56, find the area of the region described.

43. inside the convex limaçon $r = 4 + 2 \cos \theta$

44. inside the cardioid $r = 2 + 2 \sin \theta$

45. inside one petal of the four-petaled rose $r = \cos 2\theta$

46. inside the eight-petaled rose $r = 2 \sin 4\theta$

47. inside one loop of the lemniscate $r^2 = 4 \cos 2\theta$

48. inside the six-petaled rose $r^2 = 2 \sin 3\theta$

49. inside the dimpled limaçon $r = 3 - 2 \cos \theta$

50. inside the inner loop of the limaçon $r = 2 \sin \theta - 1$

51. shared by the circles $r = 2 \cos \theta$ and $r = 2 \sin \theta$

52. shared by the circles $r = 1$ and $r = 2 \sin \theta$

53. shared by the circle $r = 2$ and the cardioid $r = 2(1 - \cos \theta)$

54. shared by the cardioids $r = 2(1 + \cos \theta)$ and $r = 2(1 - \cos \theta)$

55. inside the circle $r = 2$ and outside the cardioid $r = 2(1 - \sin \theta)$

56. inside the four-petaled rose $r = 4 \cos 2\theta$ and outside the circle $r = 2$

57. Sketch the polar curves $r = 3 \cos \theta$ and $r = 1 + \cos \theta$ and find the area that lies inside the circle and outside the cardioid.

58. Sketch the polar curves $r = 2$ and $r = 2(1 - \sin \theta)$ and find the area that lies inside the circle and outside the cardioid.

59. Sketch the polar curve $r = 2 \sin 3\theta$. Find the area enclosed by the curve and find the slope of the curve at the point where $\theta = \pi/4$.

60. The accompanying figure shows the parts of the graphs of the line $x = \frac{5}{3}y$ and the curve $x = \sqrt{1 + y^2}$ that lie in the first quadrant. Region R is enclosed by the line, the curve, and the x-axis.
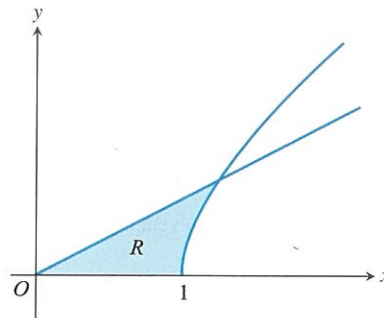
(a) Set up and evaluate an integral expression with respect to y that gives the area of R.

(b) Show that the curve $x = \sqrt{1 + y^2}$ can be described in polar coordinates by $r^2 = \dfrac{1}{\cos^2 \theta - \sin^2 \theta}$.

(c) Use the polar equation in part (b) to set up an integral expression with respect to θ that gives the area of R.

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

61. True or False There is exactly one point in the plane with polar coordinates (2, 2). Justify your answer.

62. True or False The total area enclosed by the 3-petaled rose $r = \sin 3\theta$ is $\int_0^{2\pi} \frac{1}{2} \sin^2 3\theta \, d\theta$. Justify your answer.

63. Multiple Choice The area of the region enclosed by the polar graph of $r = \sqrt{3 + \cos \theta}$ is given by which integral?

(A) $\int_0^{2\pi} \sqrt{3 + \cos \theta} \, d\theta$

(B) $\int_0^{\pi} \sqrt{3 + \cos \theta} \, d\theta$

(C) $2 \int_0^{\pi/2} (3 + \cos \theta) \, d\theta$

(D) $\int_0^{\pi} (3 + \cos \theta) \, d\theta$

(E) $\int_0^{\pi/2} \sqrt{3 + \cos \theta} \, d\theta$

64. Multiple Choice The area enclosed by one petal of the 3-petaled rose $r = 4 \cos(3\theta)$ is given by which integral?

(A) $16 \int_{-\pi/3}^{\pi/3} \cos(3\theta) \, d\theta$

(B) $8 \int_{-\pi/6}^{\pi/6} \cos(3\theta) \, d\theta$

(C) $8 \int_{-\pi/3}^{\pi/3} \cos^2(3\theta) \, d\theta$

(D) $16 \int_{-\pi/6}^{\pi/6} \cos^2(3\theta) \, d\theta$

(E) $8 \int_{-\pi/6}^{\pi/6} \cos^2(3\theta) \, d\theta$

65. Multiple Choice If $a \neq 0$ and $\theta \neq 0$, all of the following must necessarily represent the same point in polar coordinates *except* which ordered pair?

(A) (a, θ) (B) $(-a, -\theta)$ (C) $(-a, \theta - \pi)$

(D) $(-a, \theta + \pi)$ (E) $(a, \theta - 2\pi)$

66. Multiple Choice Which of the following gives the slope of the polar curve $r = f(\theta)$ graphed in the *xy*-plane?

(A) $\dfrac{dr}{d\theta}$ (B) $\dfrac{dy}{d\theta}$ (C) $\dfrac{dx}{d\theta}$ (D) $\dfrac{dy/d\theta}{dx/d\theta}$ (E) $\dfrac{dy}{dx}\dfrac{dr}{d\theta}$

Explorations

67. Rotating Curves Let $r_1(\theta) = 3(1 - \cos \theta)$ and $r_2(\theta) = r_1(\theta - \alpha)$.

(a) Graph r_2 for $\alpha = \pi/6, \pi/4, \pi/3,$ and $\pi/2$ and compare with the graph of r_1.

(b) Graph r_2 for $\alpha = -\pi/6, -\pi/4, -\pi/3,$ and $-\pi/2$ and compare with the graph of r_1.

(c) Based on your observations in parts (a) and (b), describe the relationship between the graphs of $r_1 = f(\theta)$ and $r_2 = f(\theta - \alpha)$.

68. Let $r = \dfrac{2}{1 + k \cos \theta}$.

(a) Graph r in a square viewing window for $k = 0.1, 0.3, 0.5,$ 0.7, and 0.9. Describe the graphs.

(b) Based on your observations in part (a), conjecture what happens to the graphs for $0 < k < 1$ and $k \to 0^+$.

69. Let $r = \dfrac{2}{1 + k \cos \theta}$.

(a) Graph r in a square viewing window for $k = 1.1, 1.3, 1.5, 1.7,$ and 1.9. Describe the graphs.

(b) Based on your observations in part (a), conjecture what happens to the graphs for $k > 1$ and $k \to 1^+$.

70. Let $r = \dfrac{k}{1 + \cos \theta}$.

(a) Graph r in a square viewing window for $k = 1, 3, 5, 7$, and 9. Describe the graphs.

(b) Based on your observations in part (a), conjecture what happens to the graphs for $k > 0$ and $k \to 0^+$.

Extending the Ideas

71. Distance Formula Show that the distance between two points (r_1, θ_1) and (r_2, θ_2) in polar coordinates is

$$d = \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos (\theta_1 - \theta_2)}.$$

72. Average Value If f is continuous, the average value of the polar coordinate r over the curve $r = f(\theta), \alpha \leq \theta \leq \beta$, with respect to θ is

$$r_{av} = \frac{1}{\beta - \alpha}\int_\alpha^\beta f(\theta)\, d\theta.$$

Use this formula to find the average value of r with respect to θ over the following curves $(a > 0)$.

(a) the cardioid $r = a(1 - \cos \theta)$

(b) the circle $r = a$

(c) the circle $r = a \cos \theta, \ -\pi/2 \leq \theta \leq \pi/2$

73. Length of a Polar Curve The parametric form of the arc length formula (Section 10.1) gives the length of a polar curve as

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta.$$

Assuming that the necessary derivatives are continuous, show that the substitutions $x = r \cos \theta$ and $y = r \sin\theta$ transform this expression into

$$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

74. Length of a Cardioid Use the formula in Exercise 73 to find the length of the cardioid $r = 1 + \cos \theta$.

Quick Quiz for AP* Preparation: Sections 10.1–10.3

You may use a graphing calculator to solve the following problems.

1. Multiple Choice Which of the following is equal to the area of the region inside the polar curve $r = 2 \cos \theta$ and outside the polar curve $r = \cos \theta$?

(A) $3\int_0^{\pi/2} \cos^2\theta\, d\theta$ (B) $3\int_0^{\pi} \cos^2\theta\, d\theta$

(C) $\dfrac{3}{2}\int_0^{\pi/2} \cos^2\theta\, d\theta$ (D) $3\int_0^{\pi/2} \cos\theta\, d\theta$

(E) $3\int_0^{\pi} \cos\theta\, d\theta$

2. Multiple Choice For what values of t does the curve given by the parametric equations $x = t^3 - t^2 - 1$ and $y = t^4 + 2t^2 - 8t$ have a vertical tangent?

(A) 0 only   (B) 1 only

(C) 0 and 2/3 only   (D) 0, 2/3, and 1

(E) No value

3. Multiple Choice The length of the path described by the parametric equations $x = t^2$ and $y = t$ from $t = 0$ to $t = 4$ is given by which integral?

(A) $\int_0^4 \sqrt{4t + 1}\, dt$ (B) $2\int_0^4 \sqrt{t^2 + 1}\, dt$ (C) $\int_0^4 \sqrt{2t^2 + 1}\, dt$

(D) $\int_0^4 \sqrt{4t^2 + 1}\, dt$ (E) $2\pi\int_0^4 \sqrt{4t^2 + 1}\, dt$

4. Free Response A polar curve is defined by the equation $r = \theta + \sin 2\theta$ for $0 \leq \theta \leq \pi$.

(a) Find the area bounded by the curve and the *x*-axis.

(b) Find the angle θ that corresponds to the point on the curve where $x = -2$.

(c) For $\dfrac{\pi}{3} < \theta < \dfrac{2\pi}{3}, \dfrac{dr}{d\theta}$ is negative. How can this be seen from the graph?

(d) At what angle θ in the interval $0 \leq \theta \leq \pi/2$ is the curve farthest away from the origin? Justify your answer.

Chapter 10 Key Terms

Absolute value of a vector (p. 538)
Acceleration vector (p. 542)
Archimedes spiral (p. 557)
Arc length of a parametrized curve (p. 533)
Arc length of a polar curve (p. 553)
Area between polar curves (p. 554)
Area differential (p. 553)
Area in polar coordinates (p. 553)
Arrow (p. 538)
Cardioid (p. 556)
Cartesian equation of a curve (p. 550)
Component form of a vector (p. 539)
Components of a vector (p. 538)
Convex limaçon (p. 556)
Cycloid (p. 533)
Dimpled limaçon (p. 556)
Directed distance (p. 548)
Directed line segment (p. 538)
Direction angle of a vector (p. 539)
Direction of motion (p. 542)
Direction vector (p. 538)
Displacement (p. 543)
Distance traveled (p. 544)

Dot product of vectors (p. 547)
Equivalent arrows (p. 539)
Head Minus Tail Rule (p. 539)
Huygens's clock (p. 534)
Initial point of an arrow (p. 539)
Initial ray of angle of direction (p. 548)
Lemniscate (p. 556)
Limaçon (p. 556)
Limaçon with inner loop (p. 556)
Magnitude of a vector (p. 538)
Opposite of a vector (p. 540)
Orthogonal vectors (p. 547)
Parallelogram representation of vector
  addition (p. 540)
Parametric equations of a polar
  curve (p. 551)
Parametric formula for dy/dx (p. 532)
Parametric formula for d^2y/dx^2 (p. 532)
Path of a particle (p. 542)
Polar coordinates (p. 548)
Polar equation of a curve (p. 549)
Polar graphing (p. 549)
Polar–rectangular conversion
  formulas (p. 551)

Pole (p. 548)
Position of a particle (p. 544)
Position vector (p. 538)
Properties of vectors (p. 541)
Rectangular coordinates (p. 548)
Resultant vector (p. 540)
Rose curve (p. 555)
Scalar (p. 539)
Scalar multiple of a vector (p. 540)
Speed (p. 542)
Standard representation of
  a vector (p. 538)
Sum of vectors (p. 540)
Tail-to-head representation of
  vector addition (p. 540)
Terminal point of an arrow (p. 539)
Unit vector (p. 540)
Vector (p. 538)
Vector addition (p. 539)
Velocity vector (p. 542)
Zero vector (p. 538)

Chapter 10 Review Exercises

In Exercises 1–4, let $\mathbf{u} = \langle -3, 4 \rangle$ and $\mathbf{v} = \langle 2, -5 \rangle$. Find
(a) the component form of the vector and (b) its magnitude.

1. $3\mathbf{u} - 4\mathbf{v}$ **2.** $\mathbf{u} + \mathbf{v}$ **3.** $-2\mathbf{u}$ **4.** $5\mathbf{v}$

In Exercises 5–8, find the component form of the vector.

5. the vector obtained by rotating $\langle 0, 1 \rangle$ through an angle of $2\pi/3$ radians

6. the unit vector that makes an angle of $\pi/6$ radian with the positive x-axis

7. the vector 2 units long in the direction $4\mathbf{i} - \mathbf{j}$

8. the vector 5 units long in the direction opposite to the direction of $\langle 3/5, 4/5 \rangle$

In Exercises 9 and 10, (a) find an equation for the tangent to the curve at the point corresponding to the given value of t, and
(b) find the value of d^2y/dx^2 at this point.

9. $x = (1/2) \tan t, \quad y = (1/2) \sec t; \quad t = \pi/3$

10. $x = 1 + 1/t^2, \quad y = 1 - 3/t; \quad t = 2$

In Exercises 11–14, find the points at which the tangent to the curve is (a) horizontal; (b) vertical.

11. $x = (1/2) \tan t, \quad y = (1/2) \sec t$

12. $x = -2 \cos t, \quad y = 2 \sin t$

13. $x = -\cos t, \quad y = \cos^2 t$ **14.** $x = 4 \cos t, \quad y = 9 \sin t$

In Exercises 15–20, find an appropriate window and graph the polar curve on a graphing calculator. Then sketch the curve on paper and identify the type of curve.

15. $r = 1 - \sin \theta$ **16.** $r = 2 + \cos \theta$

17. $r = \cos 2\theta$ **18.** $r \cos \theta = 1$

19. $r^2 = \sin 2\theta$ **20.** $r = -\sin \theta$

In Exercises 21 and 22, find the slope of the tangent lines at the point where $\theta = \pi/3$.

21. $r = \cos 2\theta$ **22.** $r = 2 + \cos 2\theta$

In Exercises 23 and 24, find equations for the horizontal and vertical tangent lines to the curves.

23. $r = 1 - \cos(\theta/2), \quad 0 \le \theta \le 4\pi$

24. $r = 2(1 - \sin \theta), \quad 0 \le \theta \le 2\pi$

25. Find equations for the lines that are tangent to the tips of the petals of the four-petaled rose $r = \sin 2\theta$.

26. Find equations for the lines that are tangent to the cardioid $r = 1 + \sin \theta$ at the points where it crosses the x-axis.

In Exercises 27–30, replace the polar equation by an equivalent Cartesian equation. Then identify or describe the graph.

27. $r \cos \theta = r \sin \theta$ **28.** $r = 3 \cos \theta$

29. $r = 4 \tan \theta \sec \theta$ **30.** $r \cos (\theta + \pi/3) = 2\sqrt{3}$

In Exercises 31–34, replace the Cartesian equation by an equivalent polar equation.

31. $x^2 + y^2 + 5y = 0$ **32.** $x^2 + y^2 - 2y = 0$

33. $x^2 + 4y^2 = 16$ **34.** $(x + 2)^2 + (y - 5)^2 = 16$

In Exercises 35–38, find the area of the region described.

35. enclosed by the limaçon $r = 2 - \cos \theta$

36. enclosed by one petal of the three-petaled rose $r = \sin 3\theta$

37. inside the "figure eight" $r = 1 + \cos 2\theta$ and outside the circle $r = 1$

38. inside the cardioid $r = 2(1 + \sin \theta)$ and outside the circle $r = 2 \sin \theta$

In Exercises 39 and 40, $\mathbf{r}(t)$ is the position vector of a particle moving in the plane at time t. Find **(a)** the velocity and acceleration vectors, and **(b)** the speed at the given value of t.

39. $\mathbf{r}(t) = \langle 4 \cos t, \ \sqrt{2} \sin t \rangle, t = \pi/4$

40. $\mathbf{r}(t) = \langle \sqrt{3} \sec t, \ \sqrt{3} \tan t \rangle, t = 0$

41. The position of a particle in the plane at time t is

$$\mathbf{r} = \left\langle \frac{1}{\sqrt{1 + t^2}}, \ \frac{t}{\sqrt{1 + t^2}} \right\rangle.$$ Find the particle's maximum speed.

42. Writing to Learn Suppose that $\mathbf{r}(t) = \langle e^t \cos t, e^t \sin t \rangle$. Show that the angle between $\mathbf{r}$ and the acceleration vector $\mathbf{a}$ never changes. What is the angle?

In Exercises 43–46, find the position vector.

43. $\mathbf{v}(t) = \langle -\sin t, \ \cos t \rangle$ and $\mathbf{r}(0) = \langle 0, 1 \rangle$

44. $\mathbf{v}(t) = \left\langle \dfrac{1}{t^2 + 1}, \ \dfrac{t}{\sqrt{t^2 + 1}} \right\rangle$ and $\mathbf{r}(0) = \langle 1, 1 \rangle$

45. $\mathbf{a}(t) = \langle 0, 2 \rangle$ and $\mathbf{v}(0) = \langle 0, 0 \rangle$ and $\mathbf{r}(0) = \langle 1, 0 \rangle$

46. $\mathbf{a}(t) = \langle -2, -2 \rangle$ and $\mathbf{v}(1) = \langle 4, 0 \rangle$ and $\mathbf{r}(1) = \langle 3, 3 \rangle$

47. Particle Motion A particle moves in the plane in such a manner that its coordinates at time t are

$$x = 3 \cos \frac{\pi}{4} t, \quad y = 5 \sin \frac{\pi}{4} t.$$

(a) Find the length of the velocity vector at $t = 3$.

(b) Find the x- and y-components of the acceleration of the particle at $t = 3$.

(c) Find a single equation in x and y for the path of the particle.

48. Particle Motion At time $t, \ 0 \le t \le 4$, the position of a particle moving along a path in the plane is given by the parametric equations

$$x = e^t \cos t, \quad y = e^t \sin t.$$

(a) Find the slope of the path of the particle at time $t = \pi$.

(b) Find the speed of the particle when $t = 3$.

(c) Find the distance traveled by the particle along the path from $t = 0$ to $t = 3$.

49. Particle Motion The position of a particle at any time $t \ge 0$ is given by

$$x(t) = t^2 - 2, \quad y(t) = \frac{2}{5} t^3.$$

(a) Find the magnitude of the velocity vector at $t = 4$.

(b) Find the total distance traveled by the particle from $t = 0$ to $t = 4$.

(c) Find dy/dx as a function of x.

50. Navigation An airplane, flying in the direction 80° east of north at 540 mph in still air, encounters a 55-mph tail wind acting in the direction 100° east of north. The airplane holds its compass heading but, because of the wind, acquires a different ground speed and direction. What are they?

AP* Examination Preparation

You may use a graphing calculator to solve the following problems.

51. A particle moves along the graph of $y = \cos x$ so that its x-component of acceleration is always 2. At time $t = 0$, the particle is at the point $(\pi, -1)$ and the velocity of the particle is $\langle 0, 0 \rangle$.

(a) Find the position vector of the particle.

(b) Find the speed of the particle when it is at the point $(4, \cos 4)$.

52. Two particles move in the xy-plane. For time $t \ge 0$, the position of particle A is given by $x = t - 2$ and $y = (t - 2)^2$, and the position of particle B is given by $x = \frac{3}{2} t - 4$ and $y = \frac{3}{2} t - 2$.

(a) Find the velocity vector for each particle at time $t = 3$.

(b) Find the distance traveled by particle A from $t = 0$ to $t = 3$.

(c) Determine the exact time when the particles collide.

53. A region R in the xy-plane is bounded below by the x-axis and above by the polar curve defined by $r = \dfrac{4}{1 + \sin \theta}$ for $0 \le \theta \le \pi$.

(a) Find the area of R by evaluating an integral in polar coordinates.

(b) The curve resembles an arch of the parabola $8y = 16 - x^2$. Convert the polar equation to rectangular coordinates and prove that the curves are the same.

(c) Set up an integral in rectangular coordinates that gives the area of R.