Chapter 1

Prerequisites for Calculus

Exponential functions are used to model situations in which growth or decay change dramatically. Such situations are found in nuclear power plants, which contain rods of plutonium-239; an extremely toxic radioactive isotope.

Operating at full capacity for one year, a 1,000 megawatt power plant discharges about 435 lb of plutonium-239. With a half-life of 24,400 years, how much of the isotope will remain after 1,000 years? This question can be answered with the mathematics covered in Section 1.3.

Chapter 1 Overview

This chapter reviews the most important things you need to know to start learning calculus. It also introduces the use of a graphing utility as a tool to investigate mathematical ideas, to support analytic work, and to solve problems with numerical and graphical methods. The emphasis is on functions and graphs, the main building blocks of calculus.

Functions and parametric equations are the major tools for describing the real world in mathematical terms, from temperature variations to planetary motions, from brain waves to business cycles, and from heartbeat patterns to population growth. Many functions have particular importance because of the behavior they describe. Trigonometric functions describe cyclic, repetitive activity; exponential, logarithmic, and logistic functions describe growth and decay; and polynomial functions can approximate these and most other functions.

1.1 Lines

What you'll learn about

- Increments
- Slope of a Line
- Parallel and Perpendicular Lines
- Equations of Lines
- Applications

. . . and why

Linear equations are used extensively in business and economic applications.

Increments

One reason calculus has proved to be so useful is that it is the right mathematics for relating the rate of change of a quantity to the graph of the quantity. Explaining that relationship is one goal of this book. It all begins with the slopes of lines.

When a particle in the plane moves from one point to another, the net changes or *increments* in its coordinates are found by subtracting the coordinates of its starting point from the coordinates of its stopping point.

DEFINITION Increments

If a particle moves from the point (x_1, y_1) to the point (x_2, y_2), the **increments** in its coordinates are

$$\Delta x = x_2 - x_1 \quad \text{and} \quad \Delta y = y_2 - y_1.$$

The symbols Δx and Δy are read "delta x" and "delta y." The letter Δ is a Greek capital d for "difference." Neither Δx nor Δy denotes multiplication; Δx is not "delta times x" nor is Δy "delta times y."

Increments can be positive, negative, or zero, as shown in Example 1.

EXAMPLE 1 Finding Increments

The coordinate increments from $(4, -3)$ to $(2, 5)$ are

$$\Delta x = 2 - 4 = -2, \qquad \Delta y = 5 - (-3) = 8.$$

From $(5, 6)$ to $(5, 1)$, the increments are

$$\Delta x = 5 - 5 = 0, \qquad \Delta y = 1 - 6 = -5.$$

Now try Exercise 1.

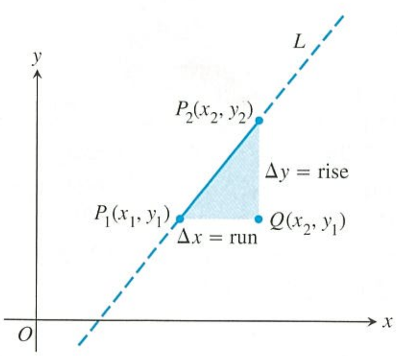


Figure 1.1 The slope of line L is $m = \dfrac{\text{rise}}{\text{run}} = \dfrac{\Delta y}{\Delta x}$.

Slope of a Line

Each nonvertical line has a *slope,* which we can calculate from increments in coordinates.

Let L be a nonvertical line in the plane and $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ two points on L (Figure 1.1). We call $\Delta y = y_2 - y_1$ the **rise** from P_1 to P_2 and $\Delta x = x_2 - x_1$ the **run** from

P_1 to P_2. Since L is not vertical, $\Delta x \neq 0$ and we define the slope of L to be the amount of rise per unit of run. It is conventional to denote the slope by the letter m.

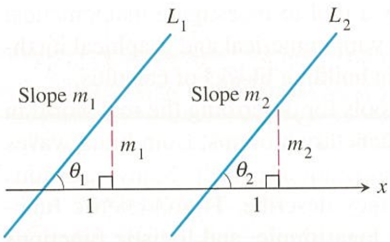


Figure 1.2 If $L_1 \parallel L_2$, then $\theta_1 = \theta_2$ and $m_1 = m_2$. Conversely, if $m_1 = m_2$, then $\theta_1 = \theta_2$ and $L_1 \parallel L_2$.

> **DEFINITION Slope**
>
> Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be points on a nonvertical line, L. The **slope** of L is
>
> $$m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}.$$

A line that goes uphill as x increases has a positive slope. A line that goes downhill as x increases has a negative slope. A horizontal line has slope zero since all of its points have the same y-coordinate, making $\Delta y = 0$. For vertical lines, $\Delta x = 0$ and the ratio $\Delta y/\Delta x$ is undefined. We express this by saying that vertical lines *have no slope.*

Parallel and Perpendicular Lines

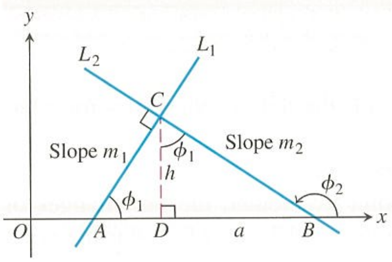


Figure 1.3 $\triangle ADC$ is similar to $\triangle CDB$. Hence ϕ_1 is also the upper angle in $\triangle CDB$, where $\tan \phi_1 = a/h$.

Parallel lines form equal angles with the x-axis (Figure 1.2). Hence, nonvertical parallel lines have the same slope. Conversely, lines with equal slopes form equal angles with the x-axis and are therefore parallel.

If two nonvertical lines L_1 and L_2 are perpendicular, their slopes m_1 and m_2 satisfy $m_1 m_2 = -1$, so each slope is the *negative reciprocal* of the other:

$$m_1 = -\frac{1}{m_2}, \qquad m_2 = -\frac{1}{m_1}.$$

The argument goes like this: In the notation of Figure 1.3, $m_1 = \tan\phi_1 = a/h$, while $m_2 = \tan\phi_2 = -h/a$. Hence, $m_1 m_2 = (a/h)(-h/a) = -1$.

Equations of Lines

The vertical line through the point (a, b) has equation $x = a$ since every x-coordinate on the line has the value a. Similarly, the horizontal line through (a, b) has equation $y = b$.

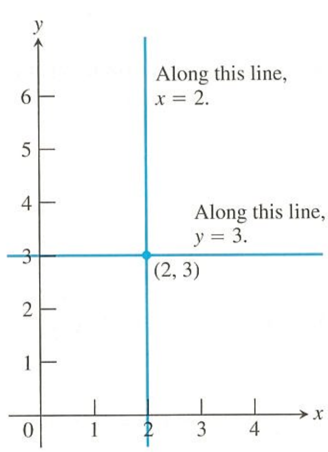


Figure 1.4 The standard equations for the vertical and horizontal lines through the point (2, 3) are $x = 2$ and $y = 3$. (Example 2)

EXAMPLE 2 Finding Equations of Vertical and Horizontal Lines

The vertical and horizontal lines through the point (2, 3) have equations $x = 2$ and $y = 3$, respectively (Figure 1.4).

Now try Exercise 9.

We can write an equation for any nonvertical line L if we know its slope m and the coordinates of one point $P_1(x_1, y_1)$ on it. If $P(x, y)$ is *any* other point on L, then

$$\frac{y - y_1}{x - x_1} = m,$$

so that

$$y - y_1 = m(x - x_1) \quad \text{or} \quad y = m(x - x_1) + y_1.$$

> **DEFINITION Point-Slope Equation**
>
> The equation
>
> $$y = m(x - x_1) + y_1$$
>
> is the **point-slope equation** of the line through the point (x_1, y_1) with slope m.

EXAMPLE 3 Using the Point-Slope Equation

Write the point-slope equation for the line through the point (2, 3) with slope −3/2.

SOLUTION

We substitute $x_1 = 2$, $y_1 = 3$, and $m = -3/2$ into the point-slope equation and obtain

$$y = -\frac{3}{2}(x - 2) + 3 \quad \text{or} \quad y = -\frac{3}{2}x + 6.$$

Now try Exercise 13.

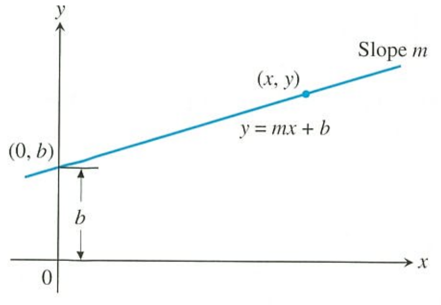


Figure 1.5 A line with slope m and y-intercept b.

The y-coordinate of the point where a nonvertical line intersects the y-axis is the **y-intercept** of the line. Similarly, the x-coordinate of the point where a nonhorizontal line intersects the x-axis is the **x-intercept** of the line. A line with slope m and y-intercept b passes through $(0, b)$ (Figure 1.5), so

$$y = m(x - 0) + b, \quad \text{or, more simply,} \quad y = mx + b.$$

DEFINITION Slope-Intercept Equation

The equation

$$y = mx + b$$

is the **slope-intercept equation** of the line with slope m and y-intercept b.

EXAMPLE 4 Writing the Slope-Intercept Equation

Write the slope-intercept equation for the line through (−2, −1) and (3, 4).

SOLUTION

The line's slope is

$$m = \frac{4 - (-1)}{3 - (-2)} = \frac{5}{5} = 1.$$

We can use this slope with either of the two given points in the point-slope equation. For $(x_1, y_1) = (-2, -1)$, we obtain

$$y = 1 \cdot (x - (-2)) + (-1)$$
$$y = x + 2 + (-1)$$
$$y = x + 1.$$

Now try Exercise 17.

If A and B are not both zero, the graph of the equation $Ax + By = C$ is a line. Every line has an equation in this form, even lines with undefined slopes.

DEFINITION General Linear Equation

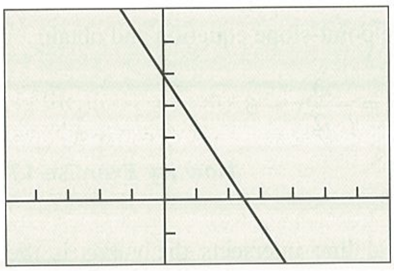
The equation

$$Ax + By = C \quad (A \text{ and } B \text{ not both } 0)$$

is a **general linear equation** in x and y.

Although the general linear form helps in the quick identification of lines, the slope-intercept form is the one to enter into a calculator for graphing.

$y = -\frac{8}{5}x + 4$



[–5, 7] by [–2, 6]

Figure 1.6 The line $8x + 5y = 20$. (Example 5)

EXAMPLE 5 Analyzing and Graphing a General Linear Equation

Find the slope and y-intercept of the line $8x + 5y = 20$. Graph the line.

SOLUTION

Solve the equation for y to put the equation in slope-intercept form:

$$8x + 5y = 20$$

$$5y = -8x + 20$$

$$y = -\frac{8}{5}x + 4$$

This form reveals the slope ($m = -8/5$) and y-intercept ($b = 4$), and puts the equation in a form suitable for graphing (Figure 1.6).

Now try Exercise 27.

EXAMPLE 6 Writing Equations for Lines

Write an equation for the line through the point $(-1, 2)$ that is **(a)** parallel, and **(b)** perpendicular to the line L: $y = 3x - 4$.

SOLUTION

The line L, $y = 3x - 4$, has slope 3.

(a) The line $y = 3(x + 1) + 2$, or $y = 3x + 5$, passes through the point $(-1, 2)$, and is parallel to L because it has slope 3.

(b) The line $y = (-1/3)(x + 1) + 2$, or $y = (-1/3)x + 5/3$, passes through the point $(-1, 2)$, and is perpendicular to L because it has slope $-1/3$.

Now try Exercise 31.

EXAMPLE 7 Determining a Function

The following table gives values for the linear function $f(x) = mx + b$. Determine m and b.

x	$f(x)$
-1	$14/3$
1	$-4/3$
2	$-13/3$

SOLUTION

The graph of f is a line. From the table we know that the following points are on the line: $(-1, 14/3)$, $(1, -4/3)$, $(2, -13/3)$.

Using the first two points, the slope m is

$$m = \frac{-4/3 - (14/3)}{1 - (-1)} = \frac{-6}{2} = -3.$$

So $f(x) = -3x + b$. Because $f(-1) = 14/3$, we have

$$f(-1) = -3(-1) + b$$

$$14/3 = 3 + b$$

$$b = 5/3.$$

continued

Thus, $m = -3$, $b = 5/3$, and $f(x) = -3x + 5/3$.

We can use either of the other two points determined by the table to check our work.

Now try Exercise 35.

Applications

Many important variables are related by linear equations. For example, the relationship between Fahrenheit temperature and Celsius temperature is linear, a fact we use to advantage in the next example.

EXAMPLE 8 Temperature Conversion

Find the relationship between Fahrenheit and Celsius temperature. Then find the Celsius equivalent of 90ºF and the Fahrenheit equivalent of -5ºC.

SOLUTION

Because the relationship between the two temperature scales is linear, it has the form $F = mC + b$. The freezing point of water is $F = 32°$ or $C = 0°$, while the boiling point is $F = 212°$ or $C = 100°$. Thus,

$$32 = m \cdot 0 + b \quad \text{and} \quad 212 = m \cdot 100 + b,$$

so $b = 32$ and $m = (212 - 32)/100 = 9/5$. Therefore,

$$F = \frac{9}{5}C + 32, \quad \text{or} \quad C = \frac{5}{9}(F - 32).$$

These relationships let us find equivalent temperatures. The Celsius equivalent of 90ºF is

$$C = \frac{5}{9}(90 - 32) \approx 32.2°.$$

The Fahrenheit equivalent of -5ºC is

$$F = \frac{9}{5}(-5) + 32 = 23°.$$

Now try Exercise 43.

Some graphing utilities have a feature that enables them to approximate the relationship between variables with a linear equation. We use this feature in Example 9.

It can be difficult to see patterns or trends in lists of paired numbers. For this reason, we sometimes begin by plotting the pairs (such a plot is called a **scatter plot**) to see whether the corresponding points lie close to a curve of some kind. If they do, and if we can find an equation $y = f(x)$ for the curve, then we have a formula that

1. summarizes the data with a simple expression, and
2. lets us predict values of y for other values of x.

The process of finding a curve to fit data is called **regression analysis** and the curve is called a **regression curve.**

There are many useful types of regression curves—power, exponential, logarithmic, sinusoidal, and so on. In the next example, we use the calculator's linear regression feature to fit the data in Table 1.1 with a line.

Table 1.1 World Population

Year	Population (millions)
1980	4454
1985	4853
1990	5285
1995	5696
2003	6305
2004	6378
2005	6450

Source: U.S. Bureau of the Census, *Statistical Abstract of the United States, 2004–2005.*

EXAMPLE 9 Regression Analysis—Predicting World Population

Starting with the data in Table 1.1, build a linear model for the growth of the world population. Use the model to predict the world population in the year 2010, and compare this prediction with the Statistical Abstract prediction of 6812 million.

continued

Why Not Round the Decimals in Equation 1 Even More?

If we do, our final calculation will be way off. Using $y = 80x - 153,849$, for instance, gives $y = 6951$ when $x = 2010$, as compared to $y = 6865$, an increase of 86 million. The rule is: *Retain all decimal places while working a problem. Round only at the end.* We rounded the coefficients in Equation 1 enough to make it readable, but not enough to hurt the outcome. However, we knew how much we could safely round *only from first having done the entire calculation with numbers unrounded.*

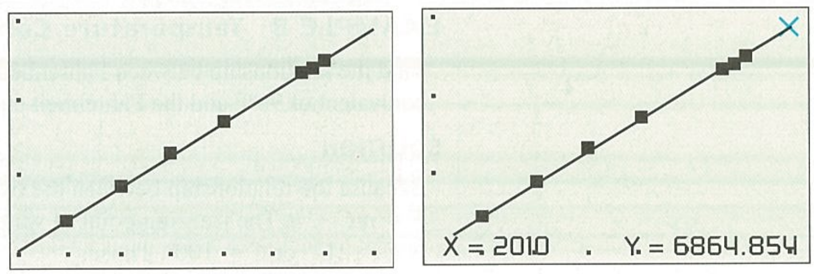
SOLUTION

Model Upon entering the data into the grapher, we find the regression equation to be approximately

$$y = 79.957x - 153848.716, \qquad (1)$$

where x represents the year and y the population *in millions.*

Figure 1.7a shows the scatter plot for Table 1.1 together with a graph of the regression line just found. You can see how well the line fits the data.



[1975, 2010] by [4000, 7000]
(a)

[1975, 2010] by [4000, 7000]
(b)

Figure 1.7 (Example 9)

Solve Graphically Our goal is to predict the population in the year 2010. Reading from the graph in Figure 1.7b, we conclude that when x is 2010, y is approximately 6865.

Rounding Rule

Round your answer as appropriate, but do not round the numbers in the calculations that lead to it.

Confirm Algebraically Evaluating Equation 1 for $x = 2010$ gives

$$y = 79.957(2010) - 153848.716$$
$$\approx 6865.$$

Interpret The linear regression equation suggests that the world population in the year 2010 will be about 6865 million, or approximately 53 million more than the Statistical Abstract prediction of 6812 million. *Now try Exercise 45.*

Regression Analysis

Regression analysis has four steps:

1. Plot the data (scatter plot).
2. Find the regression equation. For a line, it has the form $y = mx + b$.
3. Superimpose the graph of the regression equation on the scatter plot to see the fit.
4. Use the regression equation to predict y-values for particular values of x.

Quick Review 1.1 *(For help, go to Section 1.1.)*

1. Find the value of y that corresponds to $x = 3$ in $y = -2 + 4(x - 3)$.
2. Find the value of x that corresponds to $y = 3$ in $y = 3 - 2(x + 1)$.

In Exercises 3 and 4, find the value of m that corresponds to the values of x and y.

3. $x = 5,\quad y = 2,\quad m = \dfrac{y - 3}{x - 4}$
4. $x = -1,\quad y = -3,\quad m = \dfrac{2 - y}{3 - x}$

In Exercises 5 and 6, determine whether the ordered pair is a solution to the equation.

5. $3x - 4y = 5$
 (a) $(2, 1/4)$ **(b)** $(3, -1)$
6. $y = -2x + 5$
 (a) $(-1, 7)$ **(b)** $(-2, 1)$

In Exercises 7 and 8, find the distance between the points.

7. $(1, 0),\quad (0, 1)$
8. $(2, 1),\quad (1, -1/3)$

In Exercises 9 and 10, solve for y in terms of x.

9. $4x - 3y = 7$
10. $-2x + 5y = -3$

Section 1.1 Exercises

In Exercises 1–4, find the coordinate increments from A to B.

1. $A(1, 2),\quad B(-1, -1)$
2. $A(-3, 2),\quad B(-1, -2)$
3. $A(-3, 1),\quad B(-8, 1)$
4. $A(0, 4),\quad B(0, -2)$

In Exercises 5–8, let L be the line determined by points A and B.
 (a) Plot A and B. **(b)** Find the slope of L.
 (c) Draw the graph of L.

5. $A(1, -2),\quad B(2, 1)$
6. $A(-2, -1),\quad B(1, -2)$
7. $A(2, 3),\quad B(-1, 3)$
8. $A(1, 2),\quad B(1, -3)$

In Exercise 9–12, write an equation for **(a)** the vertical line and **(b)** the horizontal line through the point P.

9. $P(3, 2)$
10. $P(-1, 4/3)$
11. $P(0, -\sqrt{2})$
12. $P(-\pi, 0)$

In Exercises 13–16, write the point-slope equation for the line through the point P with slope m.

13. $P(1, 1),\quad m = 1$
14. $P(-1, 1),\quad m = -1$
15. $P(0, 3),\quad m = 2$
16. $P(-4, 0),\quad m = -2$

In Exercises 17–20, write the slope-intercept equation for the line with slope m and y-intercept b.

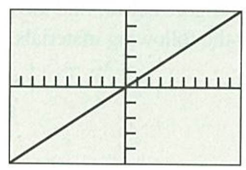
17. $m = 3,\quad b = -2$
18. $m = -1,\quad b = 2$
19. $m = -1/2,\quad b = -3$
20. $m = 1/3,\quad b = -1$

In Exercises 21–24, write a general linear equation for the line through the two points.

21. $(0, 0),\quad (2, 3)$
22. $(1, 1),\quad (2, 1)$
23. $(-2, 0),\quad (-2, -2)$
24. $(-2, 1),\quad (2, -2)$

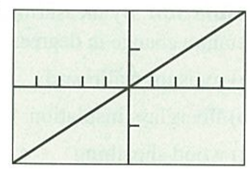
In Exercises 25 and 26, the line contains the origin and the point in the upper right corner of the grapher screen. Write an equation for the line.

25.



$[-10, 10]$ by $[-25, 25]$

26.



$[-5, 5]$ by $[-2, 2]$

In Exercises 27–30, find the **(a)** slope and **(b)** y-intercept, and **(c)** graph the line.

27. $3x + 4y = 12$
28. $x + y = 2$
29. $\dfrac{x}{3} + \dfrac{y}{4} = 1$
30. $y = 2x + 4$

In Exercises 31–34, write an equation for the line through P that is **(a)** parallel to L, and **(b)** perpendicular to L.

31. $P(0, 0),\quad L: y = -x + 2$
32. $P(-2, 2),\quad L: 2x + y = 4$
33. $P(-2, 4),\quad L: x = 5$
34. $P(-1, 1/2),\quad L: y = 3$

In Exercises 35 and 36, a table of values is given for the linear function $f(x) = mx + b$. Determine m and b.

35.

x	$f(x)$
1	2
3	9
5	16

36.

x	$f(x)$
2	-1
4	-4
6	-7

In Exercises 37 and 38, find the value of x or y for which the line through A and B has the given slope m.

37. $A(-2, 3), \quad B(4, y), \quad m = -2/3$

38. $A(-8, -2), \quad B(x, 2), \quad m = 2$

39. ***Revisiting Example 4*** Show that you get the same equation in Example 4 if you use the point (3, 4) to write the equation.

40. **Writing to Learn** ***x- and y-intercepts***

(a) Explain why c and d are the x-intercept and y-intercept, respectively, of the line

$$\frac{x}{c} + \frac{y}{d} = 1.$$

(b) How are the x-intercept and y-intercept related to c and d in the line

$$\frac{x}{c} + \frac{y}{d} = 2?$$

41. ***Parallel and Perpendicular Lines*** For what value of k are the two lines $2x + ky = 3$ and $x + y = 1$ **(a)** parallel? **(b)** perpendicular?

Group Activity In Exercises 42–44, work in groups of two or three to solve the problem.

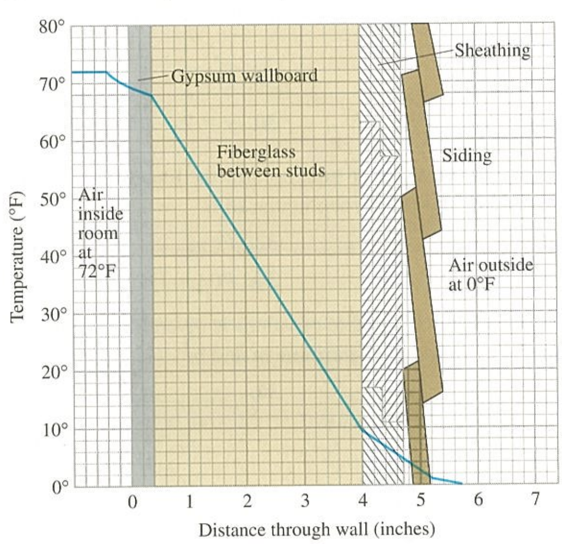
42. ***Insulation*** By measuring slopes in the figure below, find the temperature change in degrees per inch for the following materials.

(a) gypsum wallboard

(b) fiberglass insulation

(c) wood sheathing

(d) **Writing to Learn** Which of the materials in (a)–(c) is the best insulator? the poorest? Explain.



43. ***Pressure under Water*** The pressure p experienced by a diver under water is related to the diver's depth d by an equation of the form $p = kd + 1$ (k a constant). When $d = 0$ meters, the pressure is 1 atmosphere. The pressure at 100 meters is 10.94 atmospheres. Find the pressure at 50 meters.

44. ***Modeling Distance Traveled*** A car starts from point P at time $t = 0$ and travels at 45 mph.

(a) Write an expression $d(t)$ for the distance the car travels from P.

(b) Graph $y = d(t)$.

(c) What is the slope of the graph in (b)? What does it have to do with the car?

(d) **Writing to Learn** Create a scenario in which t could have negative values.

(e) **Writing to Learn** Create a scenario in which the y-intercept of $y = d(t)$ could be 30.

In Exercises 45 and 46, use linear regression analysis.

45. Table 1.2 shows the mean annual compensation of construction workers.

Table 1.2 Construction Workers' Average Annual Compensation

Year	Annual Total Compensation (dollars)
1999	42,598
2000	44,764
2001	47,822
2002	48,966

Source: U.S. Bureau of the Census, *Statistical Abstract of the United States, 2004–2005.*

(a) Find the linear regression equation for the data.

(b) Find the slope of the regression line. What does the slope represent?

(c) Superimpose the graph of the linear regression equation on a scatter plot of the data.

(d) Use the regression equation to predict the construction workers' average annual compensation in the year 2008.

46. Table 1.3 lists the ages and weights of nine girls.

Table 1.3 Girls' Ages and Weights

Age (months)	Weight (pounds)
19	22
21	23
24	25
27	28
29	31
31	28
34	32
38	34
43	39

(a) Find the linear regression equation for the data.

(b) Find the slope of the regression line. What does the slope represent?

(c) Superimpose the graph of the linear regression equation on a scatter plot of the data.

(d) Use the regression equation to predict the approximate weight of a 30-month-old girl.

Standardized Test Questions

You should solve the following problems without using a graphing calculator.

47. True or False The slope of a vertical line is zero. Justify your answer.

48. True or False The slope of a line perpendicular to the line $y = mx + b$ is $1/m$. Justify your answer.

49. Multiple Choice Which of the following is an equation of the line through $(-3, 4)$ with slope $1/2$?

(A) $y - 4 = \frac{1}{2}(x + 3)$ **(B)** $y + 3 = \frac{1}{2}(x - 4)$

(C) $y - 4 = -2(x + 3)$ **(D)** $y - 4 = 2(x + 3)$

(E) $y + 3 = 2(x - 4)$

50. Multiple Choice Which of the following is an equation of the vertical line through $(-2, 4)$?

(A) $y = 4$ **(B)** $x = 2$ **(C)** $y = -4$

(D) $x = 0$ **(E)** $x = -2$

51. Multiple Choice Which of the following is the x-intercept of the line $y = 2x - 5$?

(A) $x = -5$ **(B)** $x = 5$ **(C)** $x = 0$

(D) $x = 5/2$ **(E)** $x = -5/2$

52. Multiple Choice Which of the following is an equation of the line through $(-2, -1)$ parallel to the line $y = -3x + 1$?

(A) $y = -3x + 5$ **(B)** $y = -3x - 7$ **(C)** $y = \frac{1}{3}x - \frac{1}{3}$

(D) $y = -3x + 1$ **(E)** $y = -3x - 4$

Extending the Ideas

53. The median price of existing single-family homes has increased consistently during the past few years. However, the data in Table 1.4 show that there have been differences in various parts of the country.

Table 1.4 Median Price of Single-Family Homes

Year	South (dollars)	West (dollars)
1999	145,900	173,700
2000	148,000	196,400
2001	155,400	213,600
2002	163,400	238,500
2003	168,100	260,900

Source: U.S. Bureau of the Census, *Statistical Abstract of the United States, 2004–2005.*

(a) Find the linear regression equation for home cost in the South.

(b) What does the slope of the regression line represent?

(c) Find the linear regression equation for home cost in the West.

(d) Where is the median price increasing more rapidly, in the South or the West?

54. ***Fahrenheit versus Celsius*** We found a relationship between Fahrenheit temperature and Celsius temperature in Example 8.

(a) Is there a temperature at which a Fahrenheit thermometer and a Celsius thermometer give the same reading? If so, what is it?

(b) Writing to Learn Graph $y_1 = (9/5)x + 32$, $y_2 = (5/9)(x - 32)$, and $y_3 = x$ in the same viewing window. Explain how this figure is related to the question in part (a).

55. ***Parallelogram*** Three different parallelograms have vertices at $(-1, 1)$, $(2, 0)$, and $(2, 3)$. Draw the three and give the coordinates of the missing vertices.

56. ***Parallelogram*** Show that if the midpoints of consecutive sides of any quadrilateral are connected, the result is a parallelogram.

57. ***Tangent Line*** Consider the circle of radius 5 centered at $(0, 0)$. Find an equation of the line tangent to the circle at the point $(3, 4)$.

58. Group Activity ***Distance From a Point to a Line*** This activity investigates how to find the distance from a point $P(a, b)$ to a line $L: Ax + By = C$.

(a) Write an equation for the line M through P perpendicular to L.

(b) Find the coordinates of the point Q in which M and L intersect.

(c) Find the distance from P to Q.

1.2 Functions and Graphs

What you'll learn about

- Functions
- Domains and Ranges
- Viewing and Interpreting Graphs
- Even Functions and Odd Functions—Symmetry
- Functions Defined in Pieces
- Absolute Value Function
- Composite Functions

. . . and why

Functions and graphs form the basis for understanding mathematics and applications.

Functions

The values of one variable often depend on the values for another:

- The temperature at which water boils depends on elevation (the boiling point drops as you go up).
- The amount by which your savings will grow in a year depends on the interest rate offered by the bank.
- The area of a circle depends on the circle's radius.

In each of these examples, the value of one variable quantity depends on the value of another. For example, the boiling temperature of water, b, depends on the elevation, e; the amount of interest, I, depends on the interest rate, r. We call b and I **dependent variables** because they are determined by the values of the variables e and r on which they depend. The variables e and r are **independent variables.**

A rule that assigns to each element in one set a unique element in another set is called a *function.* The sets may be sets of any kind and do not have to be the same. A function is like a machine that assigns a unique output to every allowable input. The inputs make up the **domain** of the function; the outputs make up the **range** (Figure 1.8).

Figure 1.8 A "machine" diagram for a function.

DEFINITION Function

A **function** from a set D to a set R is a rule that assigns a unique element in R to each element in D.

In this definition, D is the domain of the function and R is a set *containing* the range (Figure 1.9).

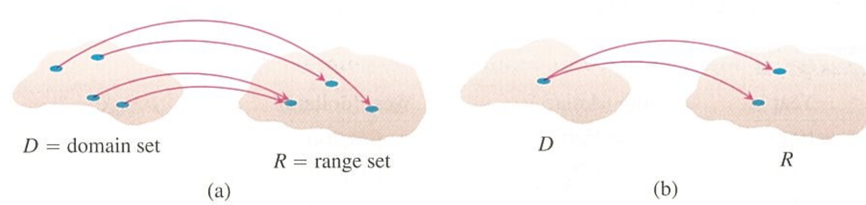


Figure 1.9 (a) A function from a set D to a set R. (b) *Not* a function. The assignment is not unique.

Leonhard Euler

(1707–1783)

Leonhard Euler, the dominant mathematical figure of his century and the most prolific mathematician ever, was also an astronomer, physicist, botanist, and chemist, and an expert in oriental languages. His work was the first to give the function concept the prominence that it has in mathematics today. Euler's collected books and papers fill 72 volumes. This does not count his enormous correspondence to approximately 300 addresses. His introductory algebra text, written originally in German (Euler was Swiss), is still available in English translation.

Euler invented a symbolic way to say "y is a function of x":

$$y = f(x),$$

which we read as "y equals f of x." This notation enables us to give different functions different names by changing the letters we use. To say that the boiling point of water is a function of elevation, we can write $b = f(e)$. To say that the area of a circle is a function of the circle's radius, we can write $A = A(r)$, giving the function the same name as the dependent variable.

The notation $y = f(x)$ gives a way to denote specific values of a function. The value of f at a can be written as $f(a)$, read "f of a."

EXAMPLE 1 The Circle-Area Function

Write a formula that expresses the area of a circle as a function of its radius. Use the formula to find the area of a circle of radius 2 in.

SOLUTION

If the radius of the circle is r, then the area $A(r)$ of the circle can be expressed as $A(r) = \pi r^2$. The area of a circle of radius 2 can be found by evaluating the function $A(r)$ at $r = 2$.

$$A(2) = \pi(2)^2 = 4\pi$$

The area of a circle of radius 2 is 4π in^2. *Now try Exercise 3.*

Domains and Ranges

In Example 1, the domain of the function is restricted by context: the independent variable is a radius and must be positive. When we define a function $y = f(x)$ with a formula and the domain is not stated explicitly or restricted by context, the domain is assumed to be the largest set of x-values for which the formula gives real y-values—the so-called **natural domain.** If we want to restrict the domain, we must say so. The domain of $y = x^2$ is understood to be the entire set of real numbers. We must write "$y = x^2, x > 0$" if we want to restrict the function to positive values of x.

The domains and ranges of many real-valued functions of a real variable are intervals or combinations of intervals. The intervals may be open, closed, or half-open (Figures 1.10 and 1.11) and finite or infinite (Figure 1.12).

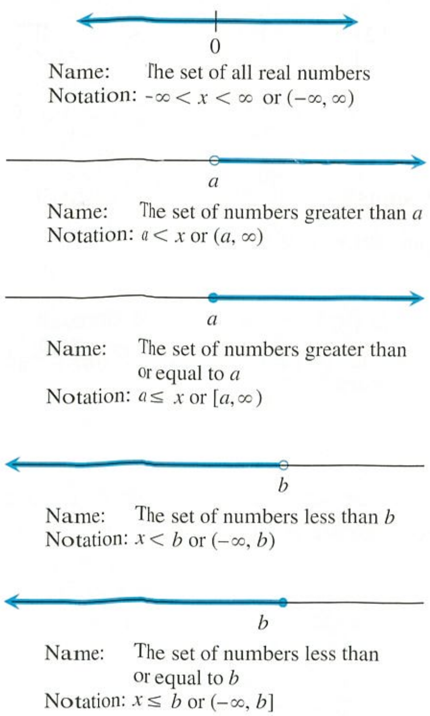


Figure 1.12 Infinite intervals—rays on the number line and the number line itself. The symbol ∞ (infinity) is used merely for convenience; it does not mean there is a number ∞.

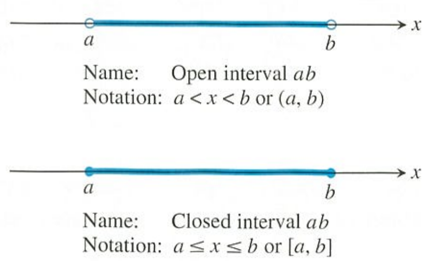


Figure 1.10 Open and closed finite intervals.

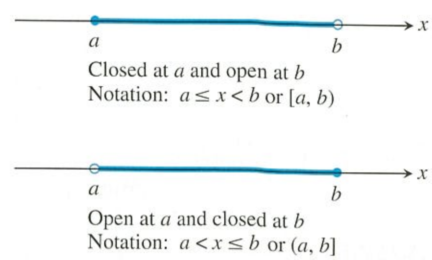


Figure 1.11 Half-open finite intervals.

The endpoints of an interval make up the interval's **boundary** and are called **boundary points.** The remaining points make up the interval's **interior** and are called **interior points. Closed intervals** contain their boundary points. **Open intervals** contain no boundary points. Every point of an open interval is an interior point of the interval.

Viewing and Interpreting Graphs

The points (x, y) in the plane whose coordinates are the input-output pairs of a function $y = f(x)$ make up the function's **graph.** The graph of the function $y = x + 2$, for example, is the set of points with coordinates (x, y) for which y equals $x + 2$.

EXAMPLE 2 Identifying Domain and Range of a Function

Identify the domain and range, and then sketch a graph of the function.

(a) $y = \dfrac{1}{x}$ **(b)** $y = \sqrt{x}$

SOLUTION

(a) The formula gives a real y-value for every real x-value except $x = 0$. (*We cannot divide any number by* 0.) The domain is $(-\infty, 0) \cup (0, \infty)$. The value y takes on every real number except $y = 0$. ($y = c \neq 0$ if $x = 1/c$) The range is also $(-\infty, 0) \cup (0, \infty)$. A sketch is shown in Figure 1.13a.

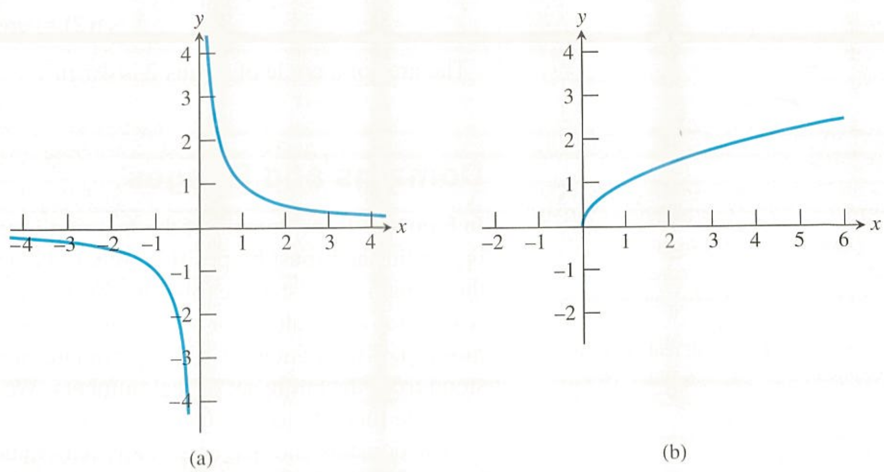


Figure 1.13 A sketch of the graph of (a) $y = 1/x$ and (b) $y = \sqrt{x}$. (Example 2)

(b) The formula gives a real number only when x is positive or zero. The domain is $[0, \infty)$. Because $\sqrt{x}$ denotes the principal square root of x, y is greater than or equal to zero. The range is also $[0, \infty)$. A sketch is shown in Figure 1.13b.

Now try Exercise 9.

Graphing with pencil and paper requires that you develop graph *drawing* skills. Graphing with a grapher (graphing calculator) requires that you develop graph *viewing* skills.

Power Function

Any function that can be written in the form $f(x) = kx^a$, where k and a are nonzero constants, is a **power function**.

Graph Viewing Skills

1. Recognize that the graph is reasonable.
2. See all the important characteristics of the graph.
3. Interpret those characteristics.
4. Recognize grapher failure.

Being able to recognize that a graph is reasonable comes with experience. You need to know the basic functions, their graphs, and how changes in their equations affect the graphs.

Grapher failure occurs when the graph produced by a grapher is less than precise—or even incorrect—usually due to the limitations of the screen resolution of the grapher.

EXAMPLE 3 Identifying Domain and Range of a Function

Use a grapher to identify the domain and range, and then draw a graph of the function.

(a) $y = \sqrt{4 - x^2}$ **(b)** $y = x^{2/3}$

SOLUTION

(a) Figure 1.14a shows a graph of the function for $-4.7 \le x \le 4.7$ and $-3.1 \le y \le 3.1$, that is, the viewing window $[-4.7, 4.7]$ by $[-3.1, 3.1]$, with x-scale $= y$-scale $= 1$. The graph appears to be the upper half of a circle. The domain appears to be $[-2, 2]$. This observation is correct because we must have $4 - x^2 \ge 0$, or equivalently, $-2 \le x \le 2$. The range appears to be $[0, 2]$, which can also be verified algebraically.

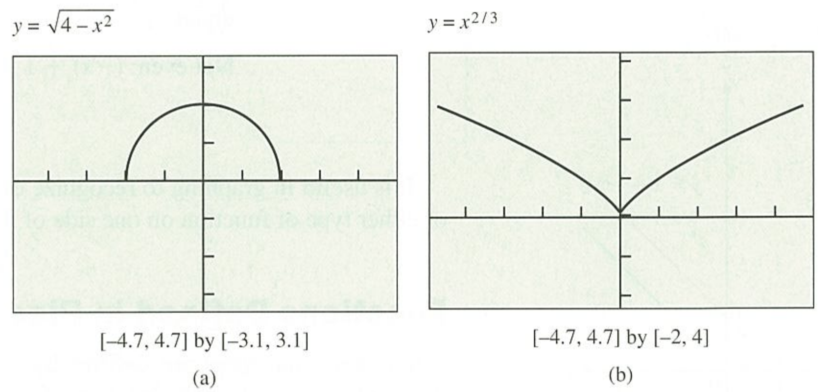


Figure 1.14 The graph of (a) $y = \sqrt{4 - x^2}$ and (b) $y = x^{2/3}$. (Example 3)

(b) Figure 1.14b shows a graph of the function in the viewing window $[-4.7, 4.7]$ by $[-2, 4]$, with x-scale $= y$-scale $= 1$. The domain appears to be $(-\infty, \infty)$, which we can verify by observing that $x^{2/3} = (\sqrt[3]{x})^2$. Also the range is $[0, \infty)$ by the same observation. ***Now try Exercise 15.***

Graphing $y = x^{2/3}$—Possible Grapher Failure

On some graphing calculators you need to enter this function as $y = (x^2)^{1/3}$ or $y = (x^{1/3})^2$ to obtain a correct graph. Try graphing this function on your grapher.

Even Functions and Odd Functions—Symmetry

The graphs of *even* and *odd* functions have important symmetry properties.

DEFINITIONS Even Function, Odd Function

A function $y = f(x)$ is an

even function of x if $f(-x) = f(x)$,

odd function of x if $f(-x) = -f(x)$,

for every x in the function's domain.

The names even and odd come from powers of x. If y is an even power of x, as in $y = x^2$ or $y = x^4$, it is an even function of x (because $(-x)^2 = x^2$ and $(-x)^4 = x^4$). If y is an odd power of x, as in $y = x$ or $y = x^3$, it is an odd function of x (because $(-x)^1 = -x$ and $(-x)^3 = -x^3$).

The graph of an even function is **symmetric about the y-axis.** Since $f(-x) = f(x)$, a point (x, y) lies on the graph if and only if the point $(-x, y)$ lies on the graph (Figure 1.15a).

The graph of an odd function is **symmetric about the origin.** Since $f(-x) = -f(x)$, a point (x, y) lies on the graph if and only if the point $(-x, -y)$ lies on the graph (Figure 1.15b).

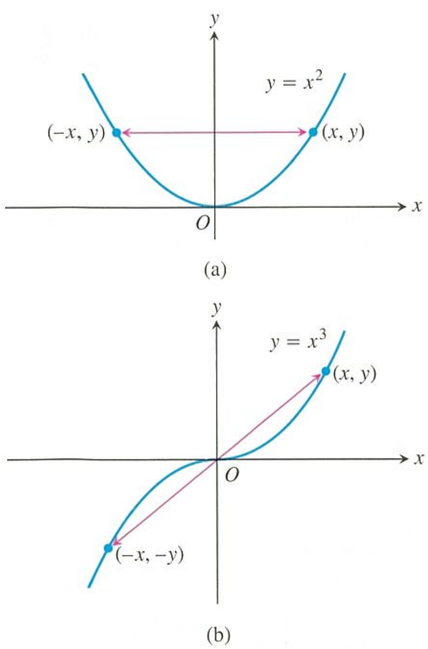


Figure 1.15 (a) The graph of $y = x^2$ (an even function) is symmetric about the y-axis. (b) The graph of $y = x^3$ (an odd function) is symmetric about the origin.

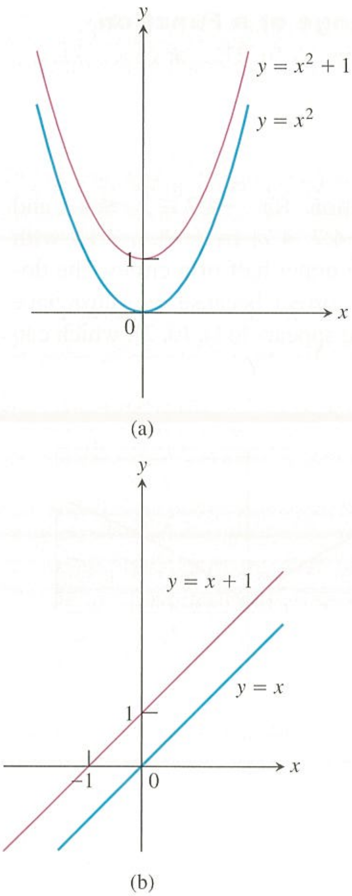
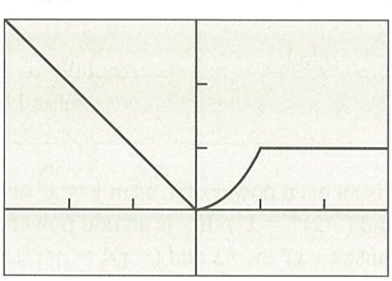


Figure 1.16 (a) When we add the constant term 1 to the function $y = x^2$, the resulting function $y = x^2 + 1$ is still even and its graph is still symmetric about the y-axis. (b) When we add the constant term 1 to the function $y = x$, the resulting function $y = x + 1$ is no longer odd. The symmetry about the origin is lost. (Example 4)

$$y = \begin{cases} -x, x < 0 \\ x^2, 0 \le x \le 1 \\ 1, x > 1 \end{cases}$$



[–3, 3] by [–1, 3]

Figure 1.17 The graph of a piecewise defined function. (Example 5).

Equivalently, a graph is symmetric about the origin if a rotation of 180° about the origin leaves the graph unchanged.

EXAMPLE 4 Recognizing Even and Odd Functions

$f(x) = x^2$	Even function: $(-x)^2 = x^2$ for all x; symmetry about y-axis.
$f(x) = x^2 + 1$	Even function: $(-x)^2 + 1 = x^2 + 1$ for all x; symmetry about y-axis (Figure 1.16a).
$f(x) = x$	Odd function: $(-x) = -x$ for all x; symmetry about the origin.
$f(x) = x + 1$	Not odd: $f(-x) = -x + 1$, but $-f(x) = -x - 1$. The two are not equal.
	Not even: $(-x) + 1 \neq x + 1$ for all $x \neq 0$ (Figure 1.16b).

Now try Exercises 21 and 23.

It is useful in graphing to recognize even and odd functions. Once we know the graph of either type of function on one side of the y-axis, we know its graph on both sides.

Functions Defined in Pieces

While some functions are defined by single formulas, others are defined by applying different formulas to different parts of their domains.

EXAMPLE 5 Graphing Piecewise-Defined Functions

$$\text{Graph } y = f(x) = \begin{cases} -x, & x < 0 \\ x^2, & 0 \le x \le 1 \\ 1, & x > 1. \end{cases}$$

SOLUTION

The values of f are given by three separate formulas: $y = -x$ when $x < 0$, $y = x^2$ when $0 \le x \le 1$, and $y = 1$ when $x > 1$. However, the function is *just one function,* whose domain is the entire set of real numbers (Figure 1.17). ***Now try Exercise 33.***

EXAMPLE 6 Writing Formulas for Piecewise Functions

Write a formula for the function $y = f(x)$ whose graph consists of the two line segments in Figure 1.18.

SOLUTION

We find formulas for the segments from (0, 0) to (1, 1) and from (1, 0) to (2, 1) and piece them together in the manner of Example 5.

Segment from (0, 0) to (1, 1) The line through (0, 0) and (1, 1) has slope $m = (1 - 0)/(1 - 0) = 1$ and y-intercept $b = 0$. Its slope-intercept equation is $y = x$. The segment from (0, 0) to (1, 1) that includes the point (0, 0) but not the point (1, 1) is the graph of the function $y = x$ restricted to the half-open interval $0 \le x < 1$, namely,

$$y = x, \quad 0 \le x < 1.$$

continued

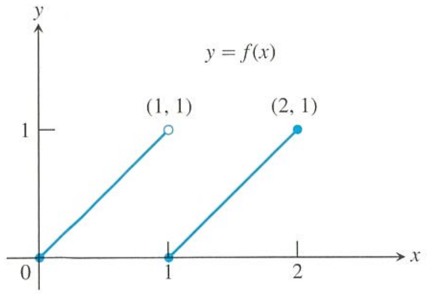


Figure 1.18 The segment on the left contains (0, 0) but not (1, 1). The segment on the right contains both of its endpoints. (Example 6)

Segment from (1, 0) to (2, 1) The line through (1, 0) and (2, 1) has slope $m = (1 - 0)/(2 - 1) = 1$ and passes through the point (1, 0). The corresponding point-slope equation for the line is

$$y = 1(x - 1) + 0, \quad \text{or} \quad y = x - 1.$$

The segment from (1, 0) to (2, 1) that includes both endpoints is the graph of $y = x - 1$ restricted to the closed interval $1 \le x \le 2$, namely,

$$y = x - 1, \quad 1 \le x \le 2.$$

Piecewise Formula Combining the formulas for the two pieces of the graph, we obtain

$$f(x) = \begin{cases} x, & 0 \le x < 1 \\ x - 1, & 1 \le x \le 2. \end{cases}$$

Now try Exercise 43.

Absolute Value Function

The **absolute value function** $y = |x|$ is defined piecewise by the formula

$$|x| = \begin{cases} -x, & x < 0 \\ x, & x \ge 0. \end{cases}$$

The function is even, and its graph (Figure 1.19) is symmetric about the y-axis.

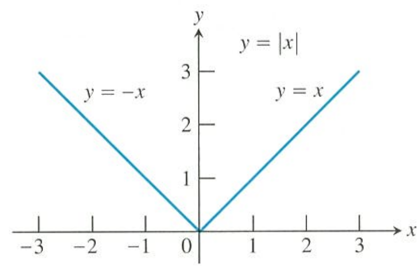
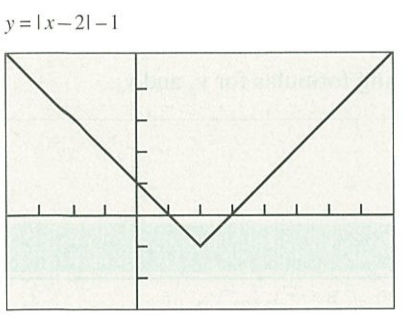


Figure 1.19 The absolute value function has domain $(-\infty, \infty)$ and range $[0, \infty)$.



[–4, 8] by [–3, 5]

Figure 1.20 The lowest point of the graph of $f(x) = |x - 2| - 1$ is $(2, -1)$. (Example 7)

EXAMPLE 7 Using Transformations

Draw the graph of $f(x) = |x - 2| - 1$. Then find the domain and range.

SOLUTION

The graph of f is the graph of the absolute value function shifted 2 units horizontally to the right and 1 unit vertically downward (Figure 1.20). The domain of f is $(-\infty, \infty)$ and the range is $[-1, \infty)$. *Now try Exercise 49.*

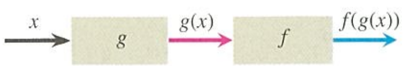


Figure 1.21 Two functions can be composed when a portion of the range of the first lies in the domain of the second.

Composite Functions

Suppose that some of the outputs of a function g can be used as inputs of a function f. We can then link g and f to form a new function whose inputs x are inputs of g and whose outputs are the numbers $f(g(x))$, as in Figure 1.21. We say that the function $f(g(x))$ (read

"f of g of x") is the **composite of g and f.** It is made by *composing* g and f in the order of first g, then f. The usual "stand-alone" notation for this composite is $f \circ g$, which is read as "f of g." Thus, the value of $f \circ g$ at x is $(f \circ g)(x) = f(g(x))$.

EXAMPLE 8 Composing Functions

Find a formula for $f(g(x))$ if $g(x) = x^2$ and $f(x) = x - 7$. Then find $f(g(2))$.

SOLUTION

To find $f(g(x))$, we replace x in the formula $f(x) = x - 7$ by the expression given for $g(x)$.

$$f(x) = x - 7$$

$$f(g(x)) = g(x) - 7 = x^2 - 7$$

We then find the value of $f(g(2))$ by substituting 2 for x.

$$f(g(2)) = (2)^2 - 7 = -3$$

Now try Exercise 51.

EXPLORATION 1 Composing Functions

Some graphers allow a function such as y_1 to be used as the independent variable of another function. With such a grapher, we can compose functions.

1. Enter the functions $y_1 = f(x) = 4 - x^2$, $y_2 = g(x) = \sqrt{x}$, $y_3 = y_2(y_1(x))$, and $y_4 = y_1(y_2(x))$. Which of y_3 and y_4 corresponds to $f \circ g$? to $g \circ f$?
2. Graph y_1, y_2, and y_3 and make conjectures about the domain and range of y_3.
3. Graph y_1, y_2, and y_4 and make conjectures about the domain and range of y_4.
4. Confirm your conjectures algebraically by finding formulas for y_3 and y_4.

Quick Review 1.2 *(For help, go to Appendix A1 and Section 1.2.)*

In Exercises 1–6, solve for x.

1. $3x - 1 \le 5x + 3$ **2.** $x(x - 2) > 0$

3. $|x - 3| \le 4$ **4.** $|x - 2| \ge 5$

5. $x^2 < 16$ **6.** $9 - x^2 \ge 0$

In Exercises 7 and 8, describe how the graph of f can be transformed to the graph of g.

7. $f(x) = x^2$, $g(x) = (x + 2)^2 - 3$

8. $f(x) = |x|$, $g(x) = |x - 5| + 2$

In Exercises 9–12, find all real solutions to the equations.

9. $f(x) = x^2 - 5$

(a) $f(x) = 4$ **(b)** $f(x) = -6$

10. $f(x) = 1/x$

(a) $f(x) = -5$ **(b)** $f(x) = 0$

11. $f(x) = \sqrt{x + 7}$

(a) $f(x) = 4$ **(b)** $f(x) = 1$

12. $f(x) = \sqrt[3]{x - 1}$

(a) $f(x) = -2$ **(b)** $f(x) = 3$

Section 1.2 Exercises

In Exercises 1–4, **(a)** write a formula for the function and **(b)** use the formula to find the indicated value of the function.

1. the area A of a circle as a function of its diameter d; the area of a circle of diameter 4 in.
2. the height h of an equilateral triangle as a function of its side length s; the height of an equilateral triangle of side length 3 m
3. the surface area S of a cube as a function of the length of the cube's edge e; the surface area of a cube of edge length 5 ft
4. the volume V of a sphere as a function of the sphere's radius r; the volume of a sphere of radius 3 cm

In Exercises 5–12, **(a)** identify the domain and range and **(b)** sketch the graph of the function.

5. $y = 4 - x^2$ **6.** $y = x^2 - 9$

7. $y = 2 + \sqrt{x-1}$ **8.** $y = -\sqrt{-x}$

9. $y = \dfrac{1}{x-2}$ **10.** $y = \sqrt[4]{-x}$

11. $y = 1 + \dfrac{1}{x}$ **12.** $y = 1 + \dfrac{1}{x^2}$

In Exercises 13–20, use a grapher to **(a)** identify the domain and range and **(b)** draw the graph of the function.

13. $y = \sqrt[3]{x}$ **14.** $y = 2\sqrt{3-x}$

15. $y = \sqrt[3]{1-x^2}$ **16.** $y = \sqrt{9-x^2}$

17. $y = x^{2/5}$ **18.** $y = x^{3/2}$

19. $y = \sqrt[3]{x-3}$ **20.** $y = \dfrac{1}{\sqrt{4-x^2}}$

In Exercises 21–30, determine whether the function is even, odd, or neither. Try to answer without writing anything (except the answer).

21. $y = x^4$ **22.** $y = x + x^2$

23. $y = x + 2$ **24.** $y = x^2 - 3$

25. $y = \sqrt{x^2+2}$ **26.** $y = x + x^3$

27. $y = \dfrac{x^3}{x^2-1}$ **28.** $y = \sqrt[3]{2-x}$

29. $y = \dfrac{1}{x-1}$ **30.** $y = \dfrac{1}{x^2-1}$

In Exercises 31–34, graph the piecewise-defined functions.

31. $f(x) = \begin{cases} 3-x, & x \le 1 \\ 2x, & 1 < x \end{cases}$ **32.** $f(x) = \begin{cases} 1, & x < 0 \\ \sqrt{x}, & x \ge 0 \end{cases}$

33. $f(x) = \begin{cases} 4-x^2, & x < 1 \\ (3/2)x + 3/2, & 1 \le x \le 3 \\ x+3, & x > 3 \end{cases}$

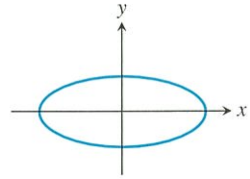
34. $f(x) = \begin{cases} x^2, & x < 0 \\ x^3, & 0 \le x \le 1 \\ 2x-1, & x > 1 \end{cases}$

35. Writing to Learn The *vertical line test* to determine whether a curve is the graph of a function states: If every vertical line in the xy-plane intersects a given curve in at most one point, then the curve is the graph of a function. Explain why this is true.

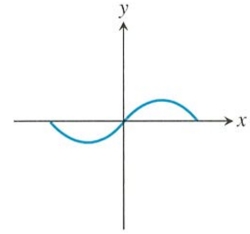
36. Writing to Learn For a curve to be *symmetric about the x-axis*, the point (x, y) must lie on the curve if and only if the point $(x, -y)$ lies on the curve. Explain why a curve that is symmetric about the x-axis is not the graph of a function, unless the function is $y = 0$.

In Exercises 37–40, use the vertical line test (see Exercise 35) to determine whether the curve is the graph of a function.

37.



38.



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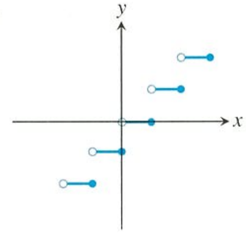

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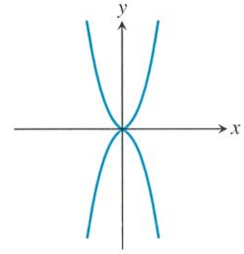
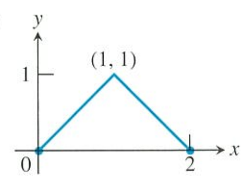


In Exercises 41–48, write a piecewise formula for the function.

41.



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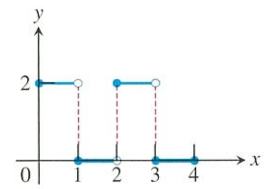

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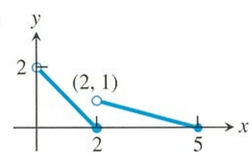

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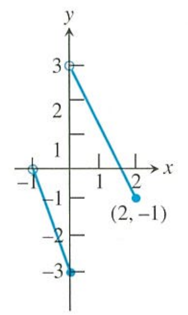

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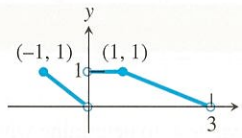

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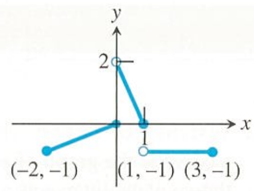

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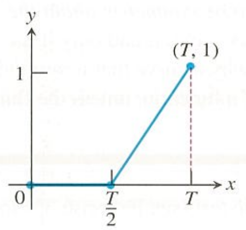

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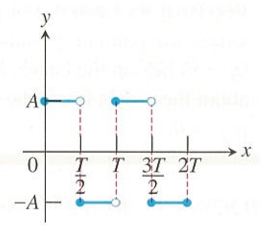


In Exercises 49 and 50, **(a)** draw the graph of the function. Then find its **(b)** domain and **(c)** range.

49. $f(x) = -|3 - x| + 2$ **50.** $f(x) = 2|x + 4| - 3$

In Exercises 51 and 52, find

(a) $f(g(x))$ **(b)** $g(f(x))$ **(c)** $f(g(0))$

(d) $g(f(0))$ **(e)** $g(g(-2))$ **(f)** $f(f(x))$

51. $f(x) = x + 5, \quad g(x) = x^2 - 3$

52. $f(x) = x + 1, \quad g(x) = x - 1$

53. Copy and complete the following table.

	$g(x)$	$f(x)$	$(f \circ g)(x)$
(a)	?	$\sqrt{x - 5}$	$\sqrt{x^2 - 5}$
(b)	?	$1+1/x$	x
(c)	$1/x$	?	x
(d)	$\sqrt{x}$	?	$\lvert x \rvert, x \geq 0$

54. ***Broadway Season Statistics*** Table 1.5 shows the gross revenue for the Broadway season in millions of dollars for several years.

Table 1.5 Broadway Season Revenue

Year	Amount ($ millions)
1997	558
1998	588
1999	603
2000	666
2001	643
2002	721
2003	771

Source: The League of American Theatres and Producers, Inc., New York, NY, as reported in *The World Almanac and Book of Facts, 2005.*

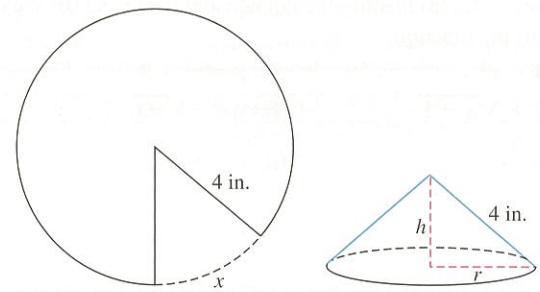
(a) Find the quadratic regression for the data in Table 1.5. Let $x = 0$ represent 1990, $x = 1$ represent 1991, and so forth.

(b) Superimpose the graph of the quadratic regression equation on a scatter plot of the data.

(c) Use the quadratic regression to predict the amount of revenue in 2008.

(d) Now find the linear regression for the data and use it to predict the amount of revenue in 2008.

55. ***The Cone Problem*** Begin with a circular piece of paper with a 4-in. radius as shown in (a). Cut out a sector with an arc length of x. Join the two edges of the remaining portion to form a cone with radius r and height h, as shown in (b).



(a) (b)

(a) Explain why the circumference of the base of the cone is $8\pi - x$.

(b) Express the radius r as a function of x.

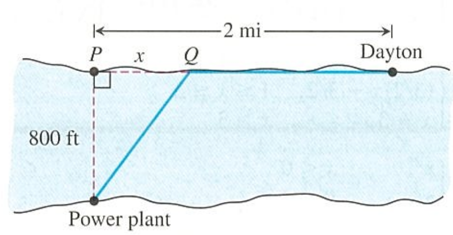
(c) Express the height h as a function of x.

(d) Express the volume V of the cone as a function of x.

56. ***Industrial Costs*** Dayton Power and Light, Inc., has a power plant on the Miami River where the river is 800 ft wide. To lay a new cable from the plant to a location in the city 2 mi downstream on the opposite side costs $180 per foot across the river and $100 per foot along the land.

(a) Suppose that the cable goes from the plant to a point Q on the opposite side that is x ft from the point P directly opposite the plant. Write a function $C(x)$ that gives the cost of laying the cable in terms of the distance x.

(b) Generate a table of values to determine if the least expensive location for point Q is less than 2000 ft or greater than 2000 ft from point P.



(Not to scale)

Standardized Test Questions

You should solve the following problems without using a graphing calculator.

57. True or False The function $f(x) = x^4 + x^2 + x$ is an even function. Justify your answer.

58. True or False The function $f(x) = x^{-3}$ is an odd function. Justify your answer.

59. Multiple Choice Which of the following gives the domain of $f(x) = \dfrac{x}{\sqrt{9 - x^2}}$?

(A) $x \neq \pm 3$ **(B)** $(-3, 3)$ **(C)** $[-3, 3]$
(D) $(-\infty, -3) \cup (3, \infty)$ **(E)** $(3, \infty)$

60. Multiple Choice Which of the following gives the range of $f(x) = 1 + \dfrac{1}{x - 1}$?

(A) $(-\infty, 1) \cup (1, \infty)$ **(B)** $x \neq 1$ **(C)** all real numbers
(D) $(-\infty, 0) \cup (0, \infty)$ **(E)** $x \neq 0$

61. Multiple Choice If $f(x) = 2x - 1$ and $g(x) = x + 3$, which of the following gives $(f \circ g)(2)$?

(A) 2 **(B)** 6 **(C)** 7 **(D)** 9 **(E)** 10

62. Multiple Choice The length L of a rectangle is twice as long as its width W. Which of the following gives the area A of the rectangle as a function of its width?

(A) $A(W) = 3W$ **(B)** $A(W) = \dfrac{1}{2}W^2$ **(C)** $A(W) = 2W^2$
(D) $A(W) = W^2 + 2W$ **(E)** $A(W) = W^2 - 2W$

Explorations

In Exercises 63–66, **(a)** graph $f \circ g$ and $g \circ f$ and make a conjecture about the domain and range of each function. **(b)** Then confirm your conjectures by finding formulas for $f \circ g$ and $g \circ f$.

63. $f(x) = x - 7$, $g(x) = \sqrt{x}$

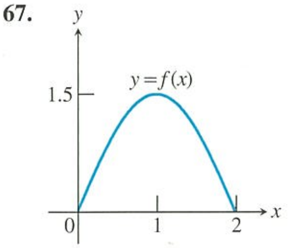
64. $f(x) = 1 - x^2$, $g(x) = \sqrt{x}$

65. $f(x) = x^2 - 3$, $g(x) = \sqrt{x + 2}$

66. $f(x) = \dfrac{2x - 1}{x + 3}$, $g(x) = \dfrac{3x + 1}{2 - x}$

Group Activity In Exercises 67–70, a portion of the graph of a function defined on $[-2, 2]$ is shown. Complete each graph assuming that the graph is **(a)** even, **(b)** odd.

67.



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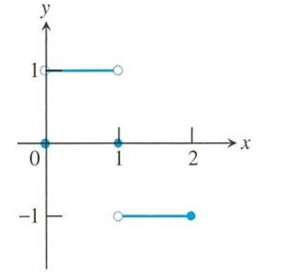

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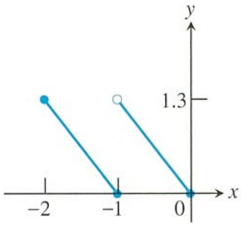

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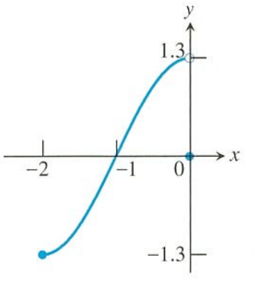


Extending the Ideas

71. Enter $y_1 = \sqrt{x}$, $y_2 = \sqrt{1 - x}$ and $y_3 = y_1 + y_2$ on your grapher.

(a) Graph y_3 in $[-3, 3]$ by $[-1, 3]$.

(b) Compare the domain of the graph of y_3 with the domains of the graphs of y_1 and y_2.

(c) Replace y_3 by

$$y_1 - y_2,\quad y_2 - y_1,\quad y_1 \cdot y_2,\quad y_1/y_2,\quad \text{and}\quad y_2/y_1,$$

in turn, and repeat the comparison of part (b).

(d) Based on your observations in (b) and (c), what would you conjecture about the domains of sums, differences, products, and quotients of functions?

72. ***Even and Odd Functions***

(a) Must the product of two even functions always be even? Give reasons for your answer.

(b) Can anything be said about the product of two odd functions? Give reasons for your answer.

1.3 Exponential Functions

What you'll learn about

- Exponential Growth
- Exponential Decay
- Applications
- The Number e

. . . and why

Exponential functions model many growth patterns.

Exponential Growth

Table 1.6 shows the growth of \$100 invested in 1996 at an interest rate of 5.5%, compounded annually.

Table 1.6 Savings Account Growth

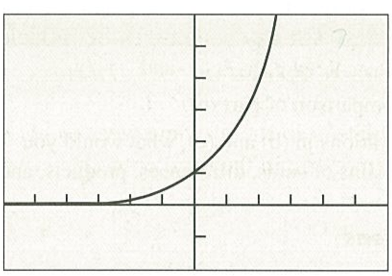
Year	Amount (dollars)	Increase (dollars)
1996	100	
1997	$100(1.055) = 105.50$	5.50
1998	$100(1.055)^2 = 111.30$	5.80
1999	$100(1.055)^3 = 117.42$	6.12
2000	$100(1.055)^4 = 123.88$	6.46

After the first year, the value of the account is always 1.055 times its value in the previous year. After n years, the value is $y = 100 \cdot (1.055)^n$.

Compound interest provides an example of *exponential growth* and is modeled by a function of the form $y = P \cdot a^x$, where P is the initial investment and a is equal to 1 plus the interest rate expressed as a decimal.

The equation $y = P \cdot a^x$, $a > 0$, $a \neq 1$, identifies a family of functions called *exponential functions.* Notice that the ratio of consecutive amounts in Table 1.6 is always the same: $111.30/105.30 = 117.42/111.30 = 123.88/117.42 \approx 1.055$. This fact is an important feature of exponential curves that has widespread application, as we will see.

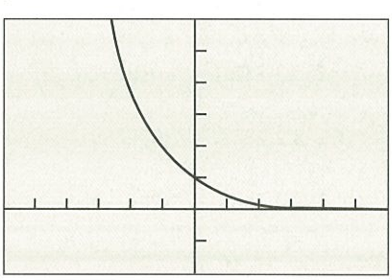
$y = 2^x$



[–6, 6] by [–2, 6]

(a)

$y = 2^{-x}$



[–6, 6] by [–2, 6]

(b)

Figure 1.22 A graph of (a) $y = 2^x$ and (b) $y = 2^{-x}$.

EXPLORATION 1 Exponential Functions

1. Graph the function $y = a^x$ for $a = 2, 3, 5$, in a $[-5, 5]$ by $[-2, 5]$ viewing window.
2. For what values of x is it true that $2^x < 3^x < 5^x$?
3. For what values of x is it true that $2^x > 3^x > 5^x$?
4. For what values of x is it true that $2^x = 3^x = 5^x$?
5. Graph the function $y = (1/a)^x = a^{-x}$ for $a = 2, 3, 5$.
6. Repeat parts 2–4 for the functions in part 5.

DEFINITION Exponential Function

Let a be a positive real number other than 1. The function

$$f(x) = a^x$$

is the **exponential function with base *a*.**

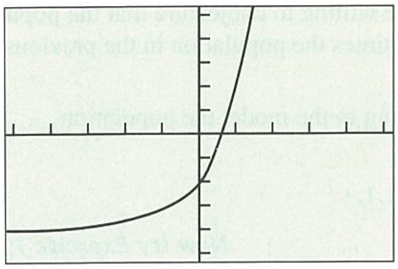
The domain of $f(x) = a^x$ is $(-\infty, \infty)$ and the range is $(0, \infty)$. If $a > 1$, the graph of f looks like the graph of $y = 2^x$ in Figure 1.22a. If $0 < a < 1$, the graph of f looks like the graph of $y = 2^{-x}$ in Figure 1.22b.

EXAMPLE 1 Graphing an Exponential Function

Graph the function $y = 2(3^x) - 4$. State its domain and range.

continued

$y = 2(3^x) - 4$



[–5, 5] by [–5, 5]

Figure 1.23 The graph of $y = 2(3^x) - 4$. (Example 1)

SOLUTION

Figure 1.23 shows the graph of the function y. It appears that the domain is $(-\infty, \infty)$. The range is $(-4, \infty)$ because $2(3^x) > 0$ for all x.

Now try Exercise 1.

EXAMPLE 2 Finding Zeros

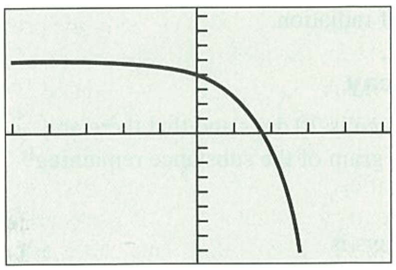
Find the zeros of $f(x) = 5 - 2.5^x$ graphically.

SOLUTION

Figure 1.24a suggests that f has a zero between $x = 1$ and $x = 2$, closer to 2. We can use our grapher to find that the zero is approximately 1.756 (Figure 1.24b).

Now try Exercise 9.

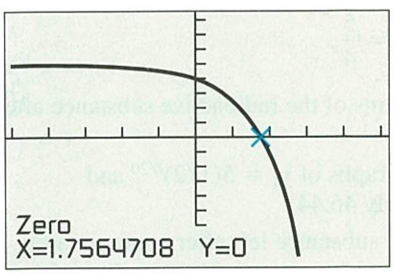
$y = 5 - 2.5^x$



[–5, 5] by [–8, 8]

(a)

$y = 5 - 2.5^x$



[–5, 5] by [–8, 8]

(b)

Figure 1.24 (a) A graph of $f(x) = 5 - 2.5^x$. (b) Showing the use of the ZERO feature to approximate the zero of f. (Example 2)

Exponential functions obey the rules for exponents.

Rules for Exponents

If $a > 0$ and $b > 0$, the following hold for all real numbers x and y.

1. $a^x \cdot a^y = a^{x+y}$ **2.** $\dfrac{a^x}{a^y} = a^{x-y}$ **3.** $(a^x)^y = (a^y)^x = a^{xy}$

4. $a^x \cdot b^x = (ab)^x$ **5.** $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$

In Table 1.6 we observed that the ratios of the amounts in consecutive years were always the same, namely the interest rate. Population growth can sometimes be modeled with an exponential function, as we see in Table 1.7 and Example 3.

Table 1.7 gives the United States population for several recent years. In this table we have divided the population in one year by the population in the previous year to get an idea of how the population is growing. These ratios are given in the third column.

Table 1.7 United States Population

Year	Population (millions)	Ratio
1998	276.1	
		$279.3/276.1 \approx 1.0116$
1999	279.3	
		$282.4/279.3 \approx 1.0111$
2000	282.4	
		$285.3/282.4 \approx 1.0102$
2001	285.3	
		$288.2/285.3 \approx 1.0102$
2002	288.2	
		$291.0/288.2 \approx 1.0097$
2003	291.0	

Source: Statistical Abstract of the United States, 2004–2005.

EXAMPLE 3 Predicting United States Population

Use the data in Table 1.7 and an exponential model to predict the population of the United States in the year 2010.

continued

SOLUTION

Based on the third column of Table 1.7, we might be willing to conjecture that the population of the United States in any year is about 1.01 times the population in the previous year.

If we start with the population in 1998, then according to the model the population (in millions) in 2010 would be about

$$276.1(1.01)^{12} \approx 311.1,$$

or about 311.1 million people. *Now try Exercise 19.*

Exponential Decay

Exponential functions can also model phenomena that produce a decrease over time, such as happens with radioactive decay. The **half-life** of a radioactive substance is the amount of time it takes for half of the substance to change from its original radioactive state to a nonradioactive state by emitting energy in the form of radiation.

EXAMPLE 4 Modeling Radioactive Decay

Suppose the half-life of a certain radioactive substance is 20 days and that there are 5 grams present initially. When will there be only 1 gram of the substance remaining?

SOLUTION

Model The number of grams remaining after 20 days is

$$5\left(\frac{1}{2}\right) = \frac{5}{2}.$$

The number of grams remaining after 40 days is

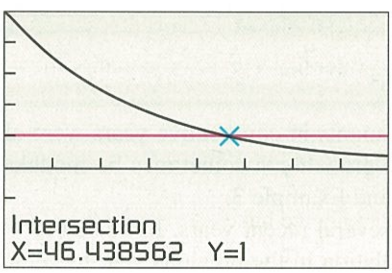
$$5\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = 5\left(\frac{1}{2}\right)^2 = \frac{5}{4}.$$

The function $y = 5(1/2)^{t/20}$ models the mass in grams of the radioactive substance after t days.

Solve Graphically Figure 1.25 shows that the graphs of $y_1 = 5(1/2)^{t/20}$ and $y_2 = 1$ (for 1 gram) intersect when t is approximately 46.44.

Interpret There will be 1 gram of the radioactive substance left after approximately 46.44 days, or about 46 days 10.5 hours. *Now try Exercise 23.*

$y = 5\left(\frac{1}{2}\right)^{t/20}, y = 1$



[0, 80] by [–3, 5]

Figure 1.25 (Example 4)

Compound interest investments, population growth, and radioactive decay are all examples of *exponential growth and decay.*

DEFINITIONS Exponential Growth, Exponential Decay

The function $y = k \cdot a^x$, $k > 0$ is a model for **exponential growth** if $a > 1$, and a model for **exponential decay** if $0 < a < 1$.

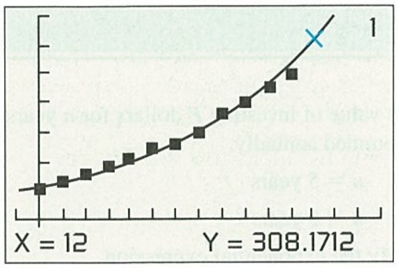
Applications

Most graphers have the exponential growth and decay model $y = k \cdot a^x$ built in as an exponential regression equation. We use this feature in Example 5 to analyze the U.S. population from the data in Table 1.8.

Table 1.8 U.S. Population

Year	Population (millions)
1880	50.2
1890	63.0
1900	76.2
1910	92.2
1920	106.0
1930	123.2
1940	132.1
1950	151.3
1960	179.3
1970	203.3
1980	226.5
1990	248.7

Source: The Statistical Abstract of the United States, 2004–2005.



[–1, 15] by [–50, 350]

Figure 1.26 (Example 5)

EXAMPLE 5 Predicting the U.S. Population

Use the population data in Table 1.8 to estimate the population for the year 2000. Compare the result with the actual 2000 population of approximately 281.4 million.

SOLUTION

Model Let $x = 0$ represent 1880, $x = 1$ represent 1890, and so on. We enter the data into the grapher and find the exponential regression equation to be

$$f(x) = (56.4696)(1.1519)^x.$$

Figure 1.26 shows the graph of f superimposed on the scatter plot of the data.

Solve Graphically The year 2000 is represented by $x = 12$. Reading from the curve, we find

$$f(12) \approx 308.2.$$

The exponential model estimates the 2000 population to be 308.2 million, an overestimate of approximately 26.8 million, or about 9.5%.

Now try Exercise 39(a, b).

EXAMPLE 6 Interpreting Exponential Regression

What *annual* rate of growth can we infer from the exponential regression equation in Example 5?

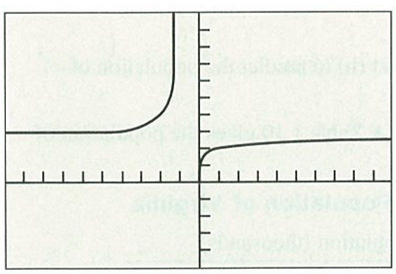
SOLUTION

Let r be the annual rate of growth of the U.S. population, expressed as a decimal. Because the time increments we used were 10-year intervals, we have

$$(1 + r)^{10} \approx 1.1519$$
$$r \approx \sqrt[10]{1.1519} - 1$$
$$r \approx 0.014$$

The annual rate of growth is about 1.4%. *Now try Exercise 39(c).*

$y = (1 + 1/x)^x$



[–10, 10] by [–5, 10]

X	Y1
1000	2.7169
2000	2.7176
3000	2.7178
4000	2.7179
5000	2.718
6000	2.7181
7000	2.7181

Y1 = (1+1/X)^X

Figure 1.27 A graph and table of values for $f(x) = (1 + 1/x)^x$ both suggest that as $x \to \infty$, $f(x) \to e \approx 2.718$.

The Number *e*

Many natural, physical, and economic phenomena are best modeled by an exponential function whose base is the famous number e, which is 2.718281828 to nine decimal places. We can define e to be the number that the function $f(x) = (1 + 1/x)^x$ approaches as x approaches infinity. The graph and table in Figure 1.27 strongly suggest that such a number exists.

The exponential functions $y = e^x$ and $y = e^{-x}$ are frequently used as models of exponential growth or decay. For example, interest **compounded continuously** uses the model $y = P \cdot e^{rt}$, where P is the initial investment, r is the interest rate as a decimal, and t is time in years.

Quick Review 1.3 *(For help, go to Section 1.3.)*

In Exercises 1–3, evaluate the expression. Round your answers to 3 decimal places.

1. $5^{2/3}$ **2.** $3^{\sqrt{2}}$

3. $3^{-1.5}$

In Exercises 4–6, solve the equation. Round your answers to 4 decimal places.

4. $x^3 = 17$ **5.** $x^5 = 24$

6. $x^{10} = 1.4567$

In Exercises 7 and 8, find the value of investing P dollars for n years with the interest rate r compounded annually.

7. $P = \$500, \quad r = 4.75\%, \quad n = 5$ years

8. $P = \$1000, \quad r = 6.3\%, \quad n = 3$ years

In Exercises 9 and 10, simplify the exponential expression.

9. $\dfrac{(x^{-3}y^2)^2}{(x^4y^3)^3}$ **10.** $\left(\dfrac{a^3b^{-2}}{c^4}\right)^2\left(\dfrac{a^4c^{-2}}{b^3}\right)^{-1}$

Section 1.3 Exercises

In Exercises 1–4, graph the function. State its domain and range.

1. $y = -2^x + 3$ **2.** $y = e^x + 3$

3. $y = 3 \cdot e^{-x} - 2$ **4.** $y = -2^{-x} - 1$

In Exercises 5–8, rewrite the exponential expression to have the indicated base.

5. 9^{2x}, base 3 **6.** 16^{3x}, base 2

7. $(1/8)^{2x}$, base 2 **8.** $(1/27)^x$, base 3

In Exercises 9–12, use a graph to find the zeros of the function.

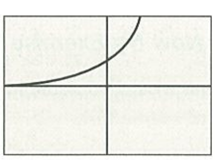
9. $f(x) = 2^x - 5$ **10.** $f(x) = e^x - 4$

11. $f(x) = 3^x - 0.5$ **12.** $f(x) = 3 - 2^x$

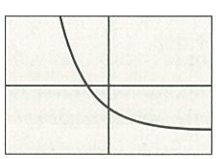
In Exercises 13–18, match the function with its graph. Try to do it without using your grapher.

13. $y = 2^x$ **14.** $y = 3^{-x}$ **15.** $y = -3^{-x}$

16. $y = -0.5^{-x}$ **17.** $y = 2^{-x} - 2$ **18.** $y = 1.5^x - 2$



(a)



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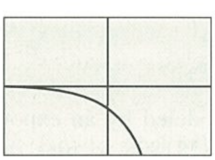
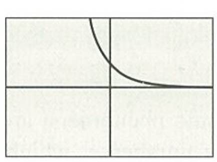
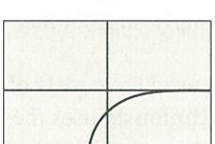


(c)



(d)



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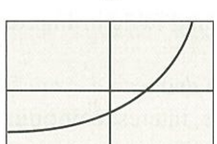


(f)

19. ***Population of Nevada*** Table 1.9 gives the population of Nevada for several years.

Table 1.9 Population of Nevada

Year	Population (thousands)
1998	1,853
1999	1,935
2000	1,998
2001	2,095
2002	2,167
2003	2,241

Source: Statistical Abstract of the United States, 2004–2005.

(a) Compute the ratios of the population in one year by the population in the previous year.

(b) Based on part (a), create an exponential model for the population of Nevada.

(c) Use your model in part (b) to predict the population of Nevada in 2010.

20. ***Population of Virginia*** Table 1.10 gives the population of Virginia for several years.

Table 1.10 Population of Virginia

Year	Population (thousands)
1998	6,901
1999	7,000
2000	7,078
2001	7,193
2002	7,288
2003	7,386

Source: Statistical Abstract of the United States, 2004–2005.

(a) Compute the ratios of the population in one year by the population in the previous year.

(b) Based on part (a), create an exponential model for the population of Virginia.

(c) Use your model in part (b) to predict the population of Virginia in 2008.

In Exercises 21–32, use an exponential model to solve the problem.

21. *Population Growth* The population of Knoxville is 500,000 and is increasing at the rate of 3.75% each year. Approximately when will the population reach 1 million?

22. *Population Growth* The population of Silver Run in the year 1890 was 6250. Assume the population increased at a rate of 2.75% per year.

(a) Estimate the population in 1915 and 1940.

(b) Approximately when did the population reach 50,000?

23. *Radioactive Decay* The half-life of phosphorus-32 is about 14 days. There are 6.6 grams present initially.

(a) Express the amount of phosphorus-32 remaining as a function of time t.

(b) When will there be 1 gram remaining?

24. *Finding Time* If John invests $2300 in a savings account with a 6% interest rate compounded annually, how long will it take until John's account has a balance of $4150?

25. *Doubling Your Money* Determine how much time is required for an investment to double in value if interest is earned at the rate of 6.25% compounded annually.

26. *Doubling Your Money* Determine how much time is required for an investment to double in value if interest is earned at the rate of 6.25% compounded monthly.

27. *Doubling Your Money* Determine how much time is required for an investment to double in value if interest is earned at the rate of 6.25% compounded continuously.

28. *Tripling Your Money* Determine how much time is required for an investment to triple in value if interest is earned at the rate of 5.75% compounded annually.

29. *Tripling Your Money* Determine how much time is required for an investment to triple in value if interest is earned at the rate of 5.75% compounded daily.

30. *Tripling Your Money* Determine how much time is required for an investment to triple in value if interest is earned at the rate of 5.75% compounded continuously.

31. *Cholera Bacteria* Suppose that a colony of bacteria starts with 1 bacterium and doubles in number every half hour. How many bacteria will the colony contain at the end of 24 h?

32. *Eliminating a Disease* Suppose that in any given year, the number of cases of a disease is reduced by 20%. If there are 10,000 cases today, how many years will it take

(a) to reduce the number of cases to 1000?

(b) to eliminate the disease; that is, to reduce the number of cases to less than 1?

Group Activity In Exercises 33–36, copy and complete the table for the function.

33. $y = 2x - 3$

x	y	Change (Δy)
1	?	
		?
2	?	
		?
3	?	
		?
4	?	

34. $y = -3x + 4$

x	y	Change (Δy)
1	?	
		?
2	?	
		?
3	?	
		?
4	?	

35. $y = x^2$

x	y	Change (Δy)
1	?	
		?
2	?	
		?
3	?	
		?
4	?	

36. $y = 3e^x$

x	y	Ratio (y_i/y_{i-1})
1	?	
		?
2	?	
		?
3	?	
		?
4	?	

37. Writing to Learn Explain how the change Δy is related to the slopes of the lines in Exercises 33 and 34. If the changes in x are constant for a linear function, what would you conclude about the corresponding changes in y?

38. *Bacteria Growth* The number of bacteria in a petri dish culture after t hours is

$$B = 100e^{0.693t}.$$

(a) What was the initial number of bacteria present?

(b) How many bacteria are present after 6 hours?

(c) Approximately when will the number of bacteria be 200? Estimate the doubling time of the bacteria.

39. ***Population of Texas*** Table 1.11 gives the population of Texas for several years.

Table 1.11 Population of Texas

Year	Population (thousands)
1980	14,229
1990	16,986
1995	18,959
1998	20,158
1999	20,558
2000	20,852

Source: Statistical Abstract of the United States, 2004–2005.

(a) Let $x = 0$ represent 1980, $x = 1$ represent 1981, and so forth. Find an exponential regression for the data, and superimpose its graph on a scatter plot of the data.

(b) Use the exponential regression equation to estimate the population of Texas in 2003. How close is the estimate to the actual population of 22,119,000 in 2003?

(c) Use the exponential regression equation to estimate the annual rate of growth of the population of Texas.

40. ***Population of California*** Table 1.12 gives the population of California for several years.

Table 1.12 Population of California

Year	Population (thousands)
1980	23,668
1990	29,811
1995	31,697
1998	32,988
1999	33,499
2000	33,872

Source: Statistical Abstract of the United States, 2004–2005.

(a) Let $x = 0$ represent 1980, $x = 1$ represent 1981, and so forth. Find an exponential regression for the data, and superimpose its graph on a scatter plot of the data.

(b) Use the exponential regression equation to estimate the population of California in 2003. How close is the estimate to the actual population of 35,484,000 in 2003?

(c) Use the exponential regression equation to estimate the annual rate of growth of the population of California.

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

41. True or False The number 3^{-2} is negative. Justify your answer.

42. True or False If $4^3 = 2^a$, then $a = 6$. Justify your answer.

43. Multiple Choice John invests \$200 at 4.5% compounded annually. About how long will it take for John's investment to double in value?

(A) 6 yrs **(B)** 9 yrs **(C)** 12 yrs **(D)** 16 yrs **(E)** 20 yrs

44. Multiple Choice Which of the following gives the domain of $y = 2e^{-x} - 3$?

(A) $(-\infty, \infty)$ **(B)** $[-3, \infty)$ **(C)** $[-1, \infty)$ **(D)** $(-\infty, 3]$
(E) $x \neq 0$

45. Multiple Choice Which of the following gives the range of $y = 4 - 2^{-x}$?

(A) $(-\infty, \infty)$ **(B)** $(-\infty, 4)$ **(C)** $[-4, \infty)$
(D) $(-\infty, 4]$ **(E)** all reals

46. Multiple Choice Which of the following gives the best approximation for the zero of $f(x) = 4 - e^x$?

(A) $x = -1.386$ **(B)** $x = 0.386$ **(C)** $x = 1.386$
(D) $x = 3$ **(E)** there are no zeros

Exploration

47. Let $y_1 = x^2$ and $y_2 = 2^x$.

(a) Graph y_1 and y_2 in $[-5, 5]$ by $[-2, 10]$. How many times do you think the two graphs cross?

(b) Compare the corresponding changes in y_1 and y_2 as x changes from 1 to 2, 2 to 3, and so on. How large must x be for the changes in y_2 to overtake the changes in y_1?

(c) Solve for x: $x^2 = 2^x$. **(d)** Solve for x: $x^2 < 2^x$.

Extending the Ideas

In Exercises 48 and 49, assume that the graph of the exponential function $f(x) = k \cdot a^x$ passes through the two points. Find the values of a and k.

48. $(1, 4.5), (-1, 0.5)$ **49.** $(1, 1.5), (-1, 6)$

Quick Quiz for AP* Preparation: Sections 1.1–1.3

You may use graphing calculator to solve the following problems.

1. Multiple Choice Which of the following gives an equation for the line through $(3, -1)$ and parallel to the line $y = -2x + 1$?

(A) $= \frac{1}{2}x + \frac{7}{2}$ **(B)** $y = \frac{1}{2}x - \frac{5}{2}$ **(C)** $y = -2x + 5$

(D) $y = -2x - 7$ **(E)** $y = -2x + 1$

2. Multiple Choice If $f(x) = x^2 + 1$ and $g(x) = 2x - 1$, which of the following gives $f \circ g(2)$?

(A) 2 **(B)** 5 **(C)** 9 **(D)** 10 **(E)** 15

3. Multiple Choice The half-life of a certain radioactive substance is 8 hrs. There are 5 grams present initially. Which of the following gives the best approximation when there will be 1 gram remaining?

(A) 2 **(B)**10 **(C)** 15 **(D)** 16 **(E)** 19

4. Free Response Let $f(x) = e^{-x} - 2$.

(a) Find the domain of f. **(b)** Find the range of f.

(c) Find the zeros of f.

1.4 Parametric Equations

What you'll learn about

- Relations
- Circles
- Ellipses
- Lines and Other Curves

. . . and why

Parametric equations can be used to obtain graphs of relations and functions.

Relations

A **relation** is a set of ordered pairs (x, y) of real numbers. The **graph of a relation** is the set of points in the plane that correspond to the ordered pairs of the relation. If x and y are *functions* of a third variable t, called a *parameter*, then we can use the *parametric mode* of a grapher to obtain a graph of the relation.

EXAMPLE 1 Graphing Half a Parabola

Describe the graph of the relation determined by

$$x = \sqrt{t}, \quad y = t, \quad t \geq 0.$$

Indicate the direction in which the curve is traced. Find a Cartesian equation for a curve that contains the parametrized curve.

SOLUTION

Set $x_1 = \sqrt{t}$, $y_1 = t$, and use the parametric mode of the grapher to draw the graph in Figure 1.28. The graph appears to be the right half of the parabola $y = x^2$. Notice that there is no information about t on the graph itself. The curve appears to be traced to the upper right with starting point $(0, 0)$.

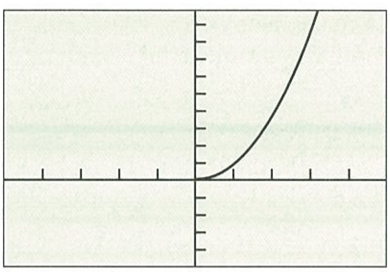
Confirm Algebraically Both x and y will be greater than or equal to zero because $t \geq 0$. Eliminating t we find that for every value of t,

$$y = t = (\sqrt{t})^2 = x^2.$$

Thus, the relation is the function $y = x^2$, $x \geq 0$.

Now try Exercise 5.

$x = \sqrt{t}$, $y = t$



$[-5, 5]$ by $[-5, 10]$

Figure 1.28 You must choose a *smallest* and *largest* value for t in parametric mode. Here we used 0 and 10, respectively. (Example 1)

DEFINITIONS Parametric Curve, Parametric Equations

If x and y are given as functions

$$x = f(t), \quad y = g(t)$$

over an interval of t-values, then the set of points $(x, y) = (f(t), g(t))$ defined by these equations is a **parametric curve.** The equations are **parametric equations** for the curve.

The variable t is a **parameter** for the curve and its domain I is the **parameter interval.** If I is a closed interval, $a \leq t \leq b$, the point $(f(a), g(a))$ is the **initial point of the curve** and the point $(f(b), g(b))$ is the **terminal point of the curve.** When we give parametric equations and a parameter interval for a curve, we say that we have **parametrized** the curve. The equations and interval constitute a **parametrization of the curve.**

In Example 1, the parameter interval is $[0, \infty)$, so $(0, 0)$ is the initial point and there is no terminal point.

A grapher can draw a parametrized curve only over a closed interval, so the portion it draws has endpoints even when the curve being graphed does not. Keep this in mind when you graph.

Circles

In applications, t often denotes time, an angle, or the distance a particle has traveled along its path from its starting point. In fact, parametric graphing can be used to simulate the motion of the particle.

EXPLORATION 1 Parametrizing Circles

Let $x = a \cos t$ and $y = a \sin t$.

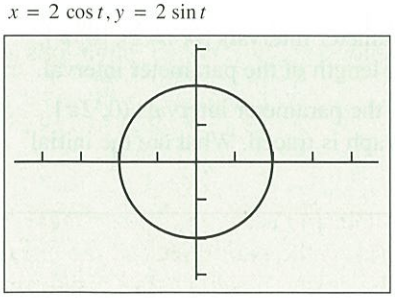
1. Let $a = 1, 2,$ or 3 and graph the parametric equations in a *square viewing window* using the parameter interval $[0, 2\pi]$. How does changing a affect this graph?
2. Let $a = 2$ and graph the parametric equations using the following parameter intervals: $[0, \pi/2]$, $[0, \pi]$, $[0, 3\pi/2]$, $[2\pi, 4\pi]$, and $[0, 4\pi]$. Describe the role of the length of the parameter interval.
3. Let $a = 3$ and graph the parametric equations using the intervals $[\pi/2, 3\pi/2]$, $[\pi, 2\pi]$, $[3\pi/2, 3\pi]$, and $[\pi, 5\pi]$. What are the initial point and terminal point in each case?
4. Graph $x = 2\cos(-t)$ and $y = 2\sin(-t)$ using the parameter intervals $[0, 2\pi]$, $[\pi, 3\pi]$, and $[\pi/2, 3\pi/2]$. In each case, describe how the graph is traced.

For $x = a \cos t$ and $y = a \sin t$, we have

$$x^2 + y^2 = a^2 \cos^2 t + a^2 \sin^2 t = a^2(\cos^2 t + \sin^2 t) = a^2(1) = a^2,$$

using the identity $\cos^2 t + \sin^2 t = 1$. Thus, the curves in Exploration 1 were either circles or portions of circles, each with center at the origin.

$x = 2\cos t,\ y = 2\sin t$



$[-4.7, 4.7]$ by $[-3.1, 3.1]$

Figure 1.29 A graph of the parametric curve $x = 2\cos t$, $y = 2\sin t$, with Tmin $= 0$, Tmax $= 2\pi$, and Tstep $= \pi/24 \approx 0.131$. (Example 2)

EXAMPLE 2 Graphing a Circle

Describe the graph of the relation determined by

$$x = 2\cos t, \quad y = 2\sin t, \quad 0 \le t \le 2\pi.$$

Find the initial and terminal points, if any, and indicate the direction in which the curve is traced. Find a Cartesian equation for a curve that contains the parametrized curve.

SOLUTION

Figure 1.29 shows that the graph appears to be a circle with radius 2. By watching the graph develop we can see that the curve is traced exactly once counterclockwise. The initial point at $t = 0$ is $(2, 0)$, and the terminal point at $t = 2\pi$ is also $(2, 0)$.

Next we eliminate the variable t.

$$\begin{aligned} x^2 + y^2 &= 4\cos^2 t + 4\sin^2 t \\ &= 4(\cos^2 t + \sin^2 t) \\ &= 4 \qquad \text{Because } \cos^2 t + \sin^2 t = 1 \end{aligned}$$

The parametrized curve is a circle centered at the origin of radius 2.

Now try Exercise 9.

Ellipses

Parametrizations of ellipses are similar to parametrizations of circles. Recall that the standard form of an ellipse centered at (0, 0) is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

EXAMPLE 3 Graphing an Ellipse

Graph the parametric curve $x = 3 \cos t$, $y = 4 \sin t$, $0 \leq t \leq 2\pi$.

Find a Cartesian equation for a curve that contains the parametric curve. What portion of the graph of the Cartesian equation is traced by the parametric curve? Indicate the direction in which the curve is traced and the initial and terminal points, if any.

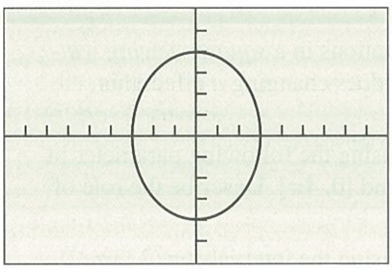
SOLUTION

Figure 1.30 suggests that the curve is an ellipse. The Cartesian equation is

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 = \cos^2 t + \sin^2 t = 1,$$

so the parametrized curve lies along an ellipse with major axis endpoints $(0, \pm 4)$ and minor axis endpoints $(\pm 3, 0)$. As t increases from 0 to 2π, the point $(x, y) = (3 \cos t, 4 \sin t)$ starts at (3, 0) and traces the entire ellipse once counterclockwise. Thus, (3, 0) is both the initial point and the terminal point. *Now try Exercise 13.*

$x = 3 \cos t$, $y = 4 \sin t$



[–9, 9] by [–6, 6]

Figure 1.30 A graph of the parametric equations $x = 3 \cos t$, $y = 4 \sin t$ for $0 \leq t \leq 2\pi$. (Example 3)

EXPLORATION 2 Parametrizing Ellipses

Let $x = a \cos t$ and $y = b \sin t$.

1. Let $a = 2$ and $b = 3$. Then graph using the parameter interval $[0, 2\pi]$. Repeat, changing b to 4, 5, and 6.
2. Let $a = 3$ and $b = 4$. Then graph using the parameter interval $[0, 2\pi]$. Repeat, changing a to 5, 6, and 7.
3. Based on parts 1 and 2, how do you identify the axis that contains the major axis of the ellipse? the minor axis?
4. Let $a = 4$ and $b = 3$. Then graph using the parameter intervals $[0, \pi/2]$, $[0, \pi]$, $[0, 3\pi/2]$, and $[0, 4\pi]$. Describe the role of the length of the parameter interval.
5. Graph $x = 5 \cos(-t)$ and $y = 2 \sin(-t)$ using the parameter intervals $(0, 2\pi]$, $[\pi, 3\pi]$, and $[\pi/2, 3\pi/2]$. Describe how the graph is traced. What are the initial point and terminal point in each case?

For $x = a \cos t$ and $y = b \sin t$, we have $(x/a)^2 + (y/b)^2 = \cos^2 t + \sin^2 t = 1$. Thus, the curves in Exploration 2 were either ellipses or portions of ellipses, each with center at the origin.

In the exercises you will see how to graph hyperbolas parametrically.

Lines and Other Curves

Lines, line segments, and many other curves can be defined parametrically.

EXAMPLE 4 Graphing a Line Segment

Draw and identify the graph of the parametric curve determined by

$$x = 3t, \quad y = 2 - 2t, \quad 0 \le t \le 1.$$

SOLUTION

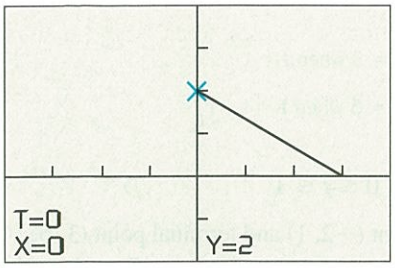
The graph (Figure 1.31) appears to be a line segment with endpoints (0, 2) and (3, 0).

Confirm Algebraically When $t = 0$, the equations give $x = 0$ and $y = 2$. When $t = 1$, they give $x = 3$ and $y = 0$. When we substitute $t = x/3$ into the y equation, we obtain

$$y = 2 - 2\left(\frac{x}{3}\right) = -\frac{2}{3}x + 2.$$

Thus, the parametric curve traces the segment of the line $y = -(2/3)x + 2$ from the point (0, 2) to (3, 0). *Now try Exercise 17.*

$x = 3t,\ y = 2 - 2t$



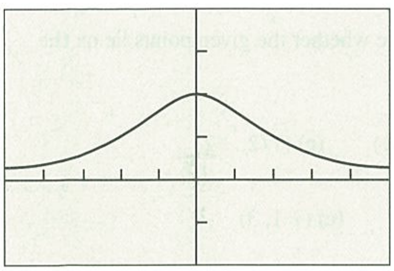
[–4, 4] by [–2, 4]

Figure 1.31 The graph of the line segment $x = 3t$, $y = 2 - 2t$, $0 \le t \le 1$, with trace on the initial point (0, 2). (Example 4)

If we change the parameter interval [0, 1] in Example 4 to $(-\infty, \infty)$, the parametrization will trace the entire line $y = -(2/3)x + 2$.

The bell-shaped curve in Exploration 3 is the famous witch of Agnesi. You will find more information about this curve in Exercise 47.

$x = 2 \cot t,\ y = 2 \sin^2 t$



[–5, 5] by [–2, 4]

Figure 1.32 The witch of Agnesi (Exploration 3)

EXPLORATION 3 Graphing the Witch of Agnesi

The witch of Agnesi is the curve

$$x = 2 \cot t, \quad y = 2 \sin^2 t, \quad 0 < t < \pi.$$

1. Draw the curve using the window in Figure 1.32. What did you choose as a closed parameter interval for your grapher? In what direction is the curve traced? How far to the left and right of the origin do you think the curve extends?
2. Graph the same parametric equations using the parameter intervals $(-\pi/2, \pi/2)$, $(0, \pi/2)$, and $(\pi/2, \pi)$. In each case, describe the curve you see and the direction in which it is traced by your grapher.
3. What happens if you replace $x = 2 \cot t$ by $x = -2 \cot t$ in the original parametrization? What happens if you use $x = 2 \cot (\pi - t)$?

Maria Agnesi (1718–1799)

The first text to include differential and integral calculus along with analytic geometry, infinite series, and differential equations was written in the 1740s by the Italian mathematician Maria Gaetana Agnesi. Agnesi, a gifted scholar and linguist whose Latin essay defending higher education for women was published when she was only nine years old, was a well-published scientist by age 20, and an honorary faculty member of the University of Bologna by age 30.

Today, Agnesi is remembered chiefly for a bell-shaped curve called *the witch of Agnesi.* This name, found only in English texts, is the result of a mistranslation. Agnesi's own name for the curve was *versiera* or "turning curve." John Colson, a noted Cambridge mathematician, probably confused versiera with *avversiera*, which means "wife of the devil" and translated it into "witch."

EXAMPLE 5 Parametrizing a Line Segment

Find a parametrization for the line segment with endpoints $(-2, 1)$ and $(3, 5)$.

SOLUTION

Using $(-2, 1)$ we create the parametric equations

$$x = -2 + at, \qquad y = 1 + bt.$$

These represent a line, as we can see by solving each equation for t and equating to obtain

$$\frac{x + 2}{a} = \frac{y - 1}{b}.$$

continued

This line goes through the point $(-2, 1)$ when $t = 0$. We determine a and b so that the line goes through $(3, 5)$ when $t = 1$.

$$3 = -2 + a \quad \Rightarrow \quad a = 5 \qquad x = 3 \text{ when } t = 1.$$

$$5 = 1 + b \quad \Rightarrow \quad b = 4 \qquad y = 5 \text{ when } t = 1.$$

Therefore,

$$x = -2 + 5t, \quad y = 1 + 4t, \quad 0 \le t \le 1$$

is a parametrization of the line segment with initial point $(-2, 1)$ and terminal point $(3, 5)$.

Now try Exercise 23.

Quick Review 1.4 *(For help, go to Section 1.1 and Appendix A1.)*

In Exercises 1–3, write an equation for the line.

1. the line through the points $(1, 8)$ and $(4, 3)$

2. the horizontal line through the point $(3, -4)$

3. the vertical line through the point $(2, -3)$

In Exercises 4–6, find the x- and y-intercepts of the graph of the relation.

4. $\frac{x^2}{9} + \frac{y^2}{16} = 1$ **5.** $\frac{x^2}{16} - \frac{y^2}{9} = 1$

6. $2y^2 = x + 1$

In Exercises 7 and 8, determine whether the given points lie on the graph of the relation.

7. $2x^2y + y^2 = 3$

(a) $(1, 1)$ **(b)** $(-1, -1)$ **(c)** $(1/2, -2)$

8. $9x^2 - 18x + 4y^2 = 27$

(a) $(1, 3)$ **(b)** $(1, -3)$ **(c)** $(-1, 3)$

9. Solve for t.

(a) $2x + 3t = -5$ **(b)** $3y - 2t = -1$

10. For what values of a is each equation true?

(a) $\sqrt{a^2} = a$ **(b)** $\sqrt{a^2} = \pm a$ **(c)** $\sqrt{4a^2} = 2|a|$

Section 1.4 Exercises

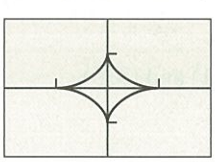
In Exercises 1–4, match the parametric equations with their graph. State the approximate dimensions of the viewing window. Give a parameter interval that traces the curve exactly once.

1. $x = 3 \sin(2t), \quad y = 1.5 \cos t$

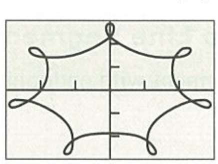
2. $x = \sin^3 t, \quad y = \cos^3 t$

3. $x = 7 \sin t - \sin(7t), \quad y = 7 \cos t - \cos(7t)$

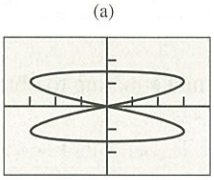
4. $x = 12 \sin t - 3 \sin(6t), \quad y = 12 \cos t + 3 \cos(6t)$



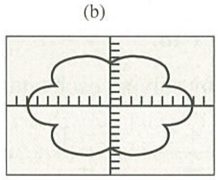
(a)



(b)



(c)



(d)

In Exercises 5–22, a parametrization is given for a curve.

(a) Graph the curve. What are the initial and terminal points, if any? Indicate the direction in which the curve is traced.

(b) Find a Cartesian equation for a curve that contains the parametrized curve. What portion of the graph of the Cartesian equation is traced by the parametrized curve?

5. $x = 3t, \quad y = 9t^2, \quad -\infty < t < \infty$

6. $x = -\sqrt{t}, \quad y = t, \quad t \ge 0$

7. $x = t, \quad y = \sqrt{t}, \quad t \ge 0$

8. $x = (\sec^2 t) - 1, \quad y = \tan t, \quad -\pi/2 < t < \pi/2$

9. $x = \cos t, \quad y = \sin t, \quad 0 \le t \le \pi$

10. $x = \sin(2\pi t), \quad y = \cos(2\pi t), \quad 0 \le t \le 1$

11. $x = \cos(\pi - t), \quad y = \sin(\pi - t), \quad 0 \le t \le \pi$

12. $x = 4 \cos t, \quad y = 2 \sin t, \quad 0 \le t \le 2\pi$

13. $x = 4 \sin t, \quad y = 2 \cos t, \quad 0 \le t \le \pi$

14. $x = 4 \sin t, \quad y = 5 \cos t, \quad 0 \le t \le 2\pi$

15. $x = 2t - 5, \quad y = 4t - 7, \quad -\infty < t < \infty$

16. $x = 1 - t, \quad y = 1 + t, \quad -\infty < t < \infty$

17. $x = t, \quad y = 1 - t, \quad 0 \le t \le 1$

18. $x = 3 - 3t, \quad y = 2t, \quad 0 \le t \le 1$

19. $x = 4 - \sqrt{t}, \quad y = \sqrt{t}, \quad 0 \le t$

20. $x = t^2, \quad y = \sqrt{4 - t^2}, \quad 0 \le t \le 2$

21. $x = \sin t, \quad y = \cos 2t, \quad -\infty < t < \infty$

22. $x = t^2 - 3, \quad y = t, \quad t \le 0$

In Exercises 23–28, find a parametrization for the curve.

23. the line segment with endpoints $(-1, -3)$ and $(4, 1)$

24. the line segment with endpoints $(-1, 3)$ and $(3, -2)$

25. the lower half of the parabola $x - 1 = y^2$

26. the left half of the parabola $y = x^2 + 2x$

27. the ray (half line) with initial point $(2, 3)$ that passes through the point $(-1, -1)$

28. the ray (half line) with initial point $(-1, 2)$ that passes through the point $(0, 0)$

Group Activity In Exercises 29–32, refer to the graph of

$$x = 3 - |t|, \quad y = t - 1, \quad -5 \le t \le 5,$$

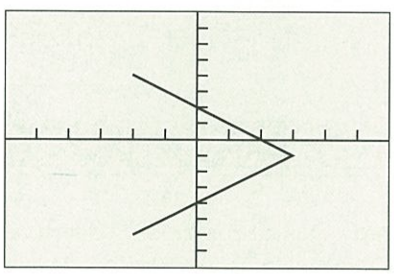
shown in the figure. Find the values of t that produce the graph in the given quadrant.

29. Quadrant I

30. Quadrant II

31. Quadrant III

32. Quadrant IV



[–6, 6] by [–8, 8]

In Exercises 33 and 34, find a parametrization for the part of the graph that lies in Quadrant I.

33. $y = x^2 + 2x + 2$

34. $y = \sqrt{x + 3}$

35. ***Circles*** Find parametrizations to model the motion of a particle that starts at $(a, 0)$ and traces the circle $x^2 + y^2 = a^2, a > 0$, as indicated.

(a) once clockwise **(b)** once counterclockwise

(c) twice clockwise **(d)** twice counterclockwise

36. ***Ellipses*** Find parametrizations to model the motion of a particle that starts at $(-a, 0)$ and traces the ellipse

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1, \quad a > 0, b > 0,$$

as indicated.

(a) once clockwise **(b)** once counterclockwise

(c) twice clockwise **(d)** twice counterclockwise

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

37. True or False The graph of the parametric curve $x = 3 \cos t$, $y = 4 \sin t$ is a circle. Justify your answer.

38. True or False The parametric curve $x = 2 \cos(-t)$, $y = 2 \sin(-t)$, $0 \le t \le 2\pi$ is traced clockwise. Justify your answer.

In Exercises 39 and 40, use the parametric curve $x = 5t$, $y = 3 - 3t$, $0 \le t \le 1$.

39. Multiple Choice Which of the following describes its graph?

(A) circle **(B)** parabola **(C)** ellipse

(D) line segment **(E)** line

40. Multiple Choice Which of the following is the initial point of the curve?

(A) $(-5, 6)$ **(B)** $(0, -3)$ **(C)** $(0, 3)$ **(D)** $(5, 0)$

(E) $(10, -3)$

41. Multiple Choice Which of the following describes the graph of the parametric curve $x = -3 \sin t$, $y = -3 \cos t$?

(A) circle **(B)** parabola **(C)** ellipse

(D) hyperbola **(E)** line

42. Multiple Choice Which of the following describes the graph of the parametric curve $x = 3t$, $y = 2t$, $t \ge 1$?

(A) circle **(B)** parabola **(C)** line segment

(D) line **(E)** ray

Explorations

43. ***Hyperbolas*** Let $x = a \sec t$ and $y = b \tan t$.

(a) Writing to Learn Let $a = 1, 2$, or 3, $b = 1, 2$, or 3, and graph using the parameter interval $(-\pi/2, \pi/2)$. Explain what you see, and describe the role of a and b in these parametric equations. (Caution: If you get what appear to be asymptotes, try using the approximation $[-1.57, 1.57]$ for the parameter interval.)

(b) Let $a = 2$, $b = 3$, and graph in the parameter interval $(\pi/2, 3\pi/2)$. Explain what you see.

(c) Writing to Learn Let $a = 2$, $b = 3$, and graph using the parameter interval $(-\pi/2, 3\pi/2)$. Explain why you must be careful about graphing in this interval or any interval that contains $\pm\pi/2$.

(d) Use algebra to explain why

$$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1.$$

(e) Let $x = a \tan t$ and $y = b \sec t$. Repeat (a), (b), and (d) using an appropriate version of (d).

44. ***Transformations*** Let $x = (2 \cos t) + h$ and $y = (2 \sin t) + k$.

(a) Writing to Learn Let $k = 0$ and $h = -2, -1, 1$, and 2, in turn. Graph using the parameter interval $[0, 2\pi]$. Describe the role of h.

(b) Writing to Learn Let $h = 0$ and $k = -2, -1, 1,$ and 2, in turn. Graph using the parameter interval $[0, 2\pi]$. Describe the role of k.

(c) Find a parametrization for the circle with radius 5 and center at $(2, -3)$.

(d) Find a parametrization for the ellipse centered at $(-3, 4)$ with semimajor axis of length 5 parallel to the x-axis and semiminor axis of length 2 parallel to the y-axis.

In Exercises 45 and 46, a parametrization is given for a curve.

(a) Graph the curve. What are the initial and terminal points, if any? Indicate the direction in which the curve is traced.

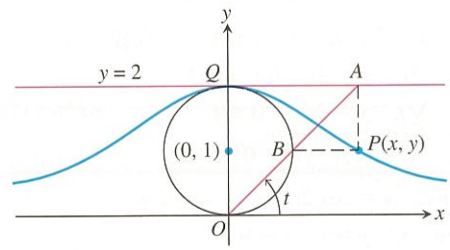
(b) Find a Cartesian equation for a curve that contains the parametrized curve. What portion of the graph of the Cartesian equation is traced by the parametrized curve?

45. $x = -\sec t, \quad y = \tan t, \quad -\pi/2 < t < \pi/2$

46. $x = \tan t, \quad y = -2 \sec t, \quad -\pi/2 < t < \pi/2$

Extending the Ideas

47. *The Witch of Agnesi* The bell-shaped witch of Agnesi can be constructed as follows. Start with the circle of radius 1, centered at the point (0, 1) as shown in the figure.



Choose a point A on the line $y = 2$, and connect it to the origin with a line segment. Call the point where the segment crosses the circle B. Let P be the point where the vertical line through A crosses the horizontal line through B. The witch is the curve traced by P as A moves along the line $y = 2$.

Find a parametrization for the witch by expressing the coordinates of P in terms of t, the radian measure of the angle that segment OA makes with the positive x-axis. The following equalities (which you may assume) will help:

(i) $x = AQ$ **(ii)** $y = 2 - AB \sin t$ **(iii)** $AB \cdot AO = (AQ)^2$

48. *Parametrizing Lines and Segments*

(a) Show that $x = x_1 + (x_2 - x_1)t, \quad y = y_1 + (y_2 - y_1)t,$ $-\infty < t < \infty$ is a parametrization for the line through the points (x_1, y_1) and (x_2, y_2).

(b) Find a parametrization for the line segment with endpoints (x_1, y_1) and (x_2, y_2).

1.5 Functions and Logarithms

What you'll learn about

- One-to-One Functions
- Inverses
- Finding Inverses
- Logarithmic Functions
- Properties of Logarithms
- Applications

. . . and why

Logarithmic functions are used in many applications, including finding time in investment problems.

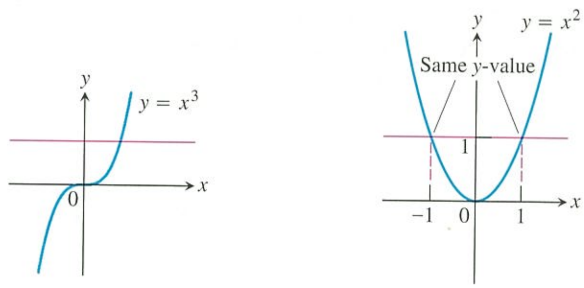
One-to-One Functions

As you know, a function is a rule that assigns a single value in its range to each point in its domain. Some functions assign the same output to more than one input. For example, $f(x) = x^2$ assigns the output 4 to both 2 and -2. Other functions never output a given value more than once. For example, the cubes of different numbers are always different.

If each output value of a function is associated with exactly one input value, the function is *one-to-one.*

> **DEFINITION One-to-One Function**
>
> A function $f(x)$ is **one-to-one** on a domain D if $f(a) \neq f(b)$ whenever $a \neq b$.

The graph of a one-to-one function $y = f(x)$ can intersect any horizontal line at most once (the *horizontal line test*). If it intersects such a line more than once it assumes the same y-value more than once, and is therefore not one-to-one (Figure 1.33).

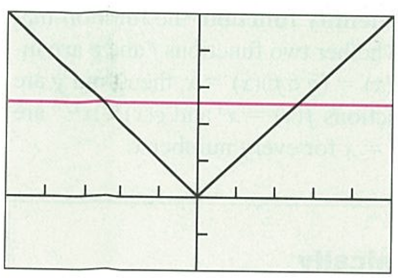


One-to-one: Graph meets each horizontal line once.

Not one-to-one: Graph meets some horizontal lines more than once.

Figure 1.33 Using the horizontal line test, we see that $y = x^3$ is one-to-one and $y = x^2$ is not.

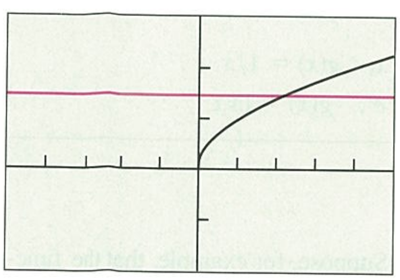
$y = |x|$



[–5, 5] by [–2, 5]

(a)

$y = \sqrt{x}$



[–5, 5] by [–2, 3]

(b)

Figure 1.34 (a) The graph of $f(x) = |x|$ and a horizontal line. (b) The graph of $g(x) = \sqrt{x}$ and a horizontal line. (Example 1)

EXAMPLE 1 Using the Horizontal Line Test

Determine whether the functions are one-to-one.

(a) $f(x) = |x|$ **(b)** $g(x) = \sqrt{x}$

SOLUTION

(a) As Figure 1.34a suggests, each horizontal line $y = c$, $c > 0$, intersects the graph of $f(x) = |x|$ twice. So f is not one-to-one.

(b) As Figure 1.34b suggests, each horizontal line intersects the graph of $g(x) = \sqrt{x}$ either once or not at all. The function g is one-to-one.

Now try Exercise 1.

Inverses

Since each output of a one-to-one function comes from just one input, a one-to-one function can be reversed to send outputs back to the inputs from which they came. The function defined by reversing a one-to-one function f is the **inverse of f.** The functions in Tables 1.13 and 1.14 are inverses of one another. The symbol for the inverse of f is f^{-1}, read "f inverse." The -1 in f^{-1} is not an exponent; $f^{-1}(x)$ does not mean $1/f(x)$.

Table 1.13 Rental Charge versus Time

Time x (hours)	Charge y (dollars)
1	5.00
2	7.50
3	10.00
4	12.50
5	15.00
6	17.50

Table 1.14 Time versus Rental Charge

Charge x (dollars)	Time y (hours)
5.00	1
7.50	2
10.00	3
12.50	4
15.00	5
17.50	6

As Tables 1.13 and 1.14 suggest, composing a function with its inverse in either order sends each output back to the input from which it came. In other words, the result of composing a function and its inverse in either order is the **identity function,** the function that assigns each number to itself. This gives a way to test whether two functions f and g are inverses of one another. Compute $f \circ g$ and $g \circ f$. If $(f \circ g)(x) = (g \circ f)(x) = x$, then f and g are inverses of one another; otherwise they are not. The functions $f(x) = x^3$ and $g(x) = x^{1/3}$ are inverses of one another because $(x^3)^{1/3} = x$ and $(x^{1/3})^3 = x$ for every number x.

EXPLORATION 1 Testing for Inverses Graphically

For each of the function pairs below,

(a) Graph f and g together in a square window.

(b) Graph $f \circ g$. **(c)** Graph $g \circ f$.

What can you conclude from the graphs?

1. $f(x) = x^3, \quad g(x) = x^{1/3}$

2. $f(x) = x, \quad g(x) = 1/x$

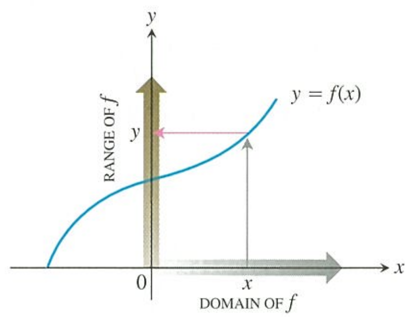
3. $f(x) = 3x, \quad g(x) = x/3$

4. $f(x) = e^x, \quad g(x) = \ln x$

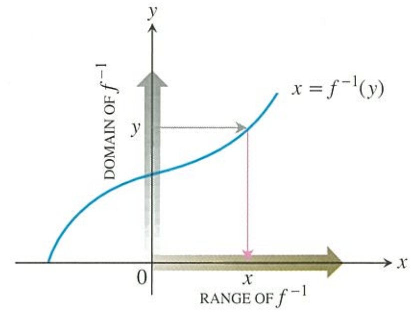
Finding Inverses

How do we find the graph of the inverse of a function? Suppose, for example, that the function is the one pictured in Figure 1.35a. To read the graph, we start at the point x on the x-axis, go up to the graph, and then move over to the y-axis to read the value of y. If we start with y and want to find the x from which it came, we reverse the process (Figure 1.35b).

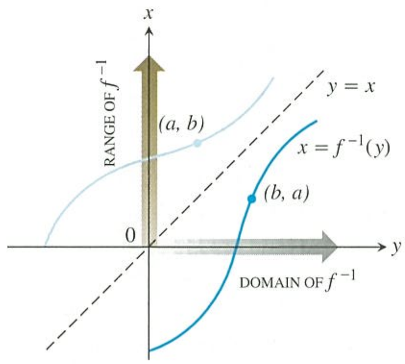
The graph of f is already the graph of f^{-1}, although the latter graph is not drawn in the usual way with the domain axis horizontal and the range axis vertical. For f^{-1}, the input-output pairs are reversed. To display the graph of f^{-1} in the usual way, we have to reverse the pairs by reflecting the graph across the 45° line $y = x$ (Figure 1.35c) and interchanging the letters x and y (Figure 1.35d). This puts the independent variable, now called x, on the horizontal axis and the dependent variable, now called y, on the vertical axis.



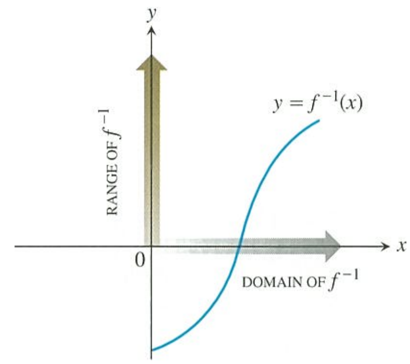
(a) To find the value of f at x, we start at x, go up to the curve, and then over to the y-axis.



(b) The graph of f is also the graph of f^{-1}. To find the x that gave y, we start at y and go over to the curve and down to the x-axis. The domain of f^{-1} is the range of f. The range of f^{-1} is the domain of f.



(c) To draw the graph of f^{-1} in the usual way, we reflect the system across the line $y = x$.



(d) Then we interchange the letters x and y. We now have a normal-looking graph of f^{-1} as a function of x.

Figure 1.35 The graph of $y = f^{-1}(x)$.

The fact that the graphs of f and f^{-1} are reflections of each other across the line $y = x$ is to be expected because the input-output pairs (a, b) of f have been reversed to produce the input-output pairs (b, a) of f^{-1}.

The pictures in Figure 1.35 tell us how to express f^{-1} as a function of x algebraically.

Writing f^{-1} as a Function of x

1. Solve the equation $y = f(x)$ for x in terms of y.
2. Interchange x and y. The resulting formula will be $y = f^{-1}(x)$.

EXAMPLE 2 Finding the Inverse Function

Show that the function $y = f(x) = -2x + 4$ is one-to-one and find its inverse function.

SOLUTION

Every horizontal line intersects the graph of f exactly once, so f is one-to-one and has an inverse.

Step 1:

Solve for x in terms of y: $y = -2x + 4$

$$x = -\frac{1}{2}y + 2$$

continued

Step 2:

Interchange x and y: $y = -\frac{1}{2}x + 2$

The inverse of the function $f(x) = -2x + 4$ is the function $f^{-1}(x) = -(1/2)x + 2$. We can verify that both composites are the identity function.

$$f^{-1}(f(x)) = -\frac{1}{2}(-2x + 4) + 2 = x - 2 + 2 = x$$

$$f(f^{-1}(x)) = -2\left(-\frac{1}{2}x + 2\right) + 4 = x - 4 + 4 = x$$

Now try Exercise 13.

Graphing $y = f(x)$ and $y = f^{-1}(x)$ Parametrically

We can graph any function $y = f(x)$ as

$$x_1 = t, \quad y_1 = f(t).$$

Interchanging t and $f(t)$ produces parametric equations for the inverse:

$$x_2 = f(t), \quad y_2 = t.$$

We can use parametric graphing to graph the inverse of a function without finding an explicit rule for the inverse, as illustrated in Example 3.

EXAMPLE 3 Graphing the Inverse Parametrically

(a) Graph the one-to-one function $f(x) = x^2$, $x \geq 0$, together with its inverse and the line $y = x$, $x \geq 0$.

(b) Express the inverse of f as a function of x.

SOLUTION

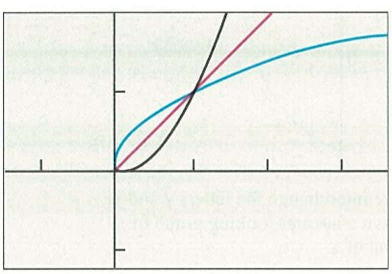
(a) We can graph the three functions parametrically as follows:

Graph of f: $x_1 = t$, $y_1 = t^2$, $t \geq 0$

Graph of f^{-1}: $x_2 = t^2$, $y_2 = t$

Graph of $y = x$: $x_3 = t$, $y_3 = t$

Figure 1.36 shows the three graphs.



[–1.5, 3.5] by [–1, 2]

Figure 1.36 The graphs of f and f^{-1} are reflections of each other across the line $y = x$. (Example 3)

(b) Next we find a formula for $f^{-1}(x)$.

Step 1:

Solve for x in terms of y.

$$y = x^2$$

$$\sqrt{y} = \sqrt{x^2}$$

$$\sqrt{y} = x \qquad \text{Because } x \geq 0.$$

Step 2:

Interchange x and y.

$$\sqrt{x} = y$$

Thus, $f^{-1}(x) = \sqrt{x}$.

Now try Exercise 27.

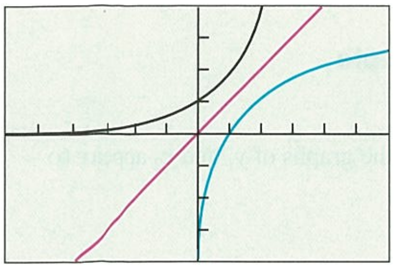
Logarithmic Functions

If a is any positive real number other than 1, the base a exponential function $f(x) = a^x$ is one-to-one. It therefore has an inverse. Its inverse is called the *base a logarithm function.*

DEFINITION Base *a* Logarithm Function

The **base *a* logarithm function** $y = \log_a x$ is the inverse of the base a exponential function $y = a^x$ $(a > 0, a \neq 1)$.

The domain of $\log_a x$ is $(0, \infty)$, the range of a^x. The range of $\log_a x$ is $(-\infty, \infty)$, the domain of a^x.



[–6, 6] by [–4, 4]

Figure 1.37 The graphs of $y = 2^x$ ($x_1 = t$, $y_1 = 2^t$), its inverse $y = \log_2 x$ ($x_2 = 2^t$, $y_2 = t$), and $y = x$ ($x_3 = t$, $y_3 = t$).

Because we have no technique for solving for x in terms of y in the equation $y = a^x$, we do not have an explicit formula for the logarithm function as a function of x. However, the graph of $y = \log_a x$ can be obtained by reflecting the graph of $y = a^x$ across the line $y = x$, or by using parametric graphing (Figure 1.37).

Logarithms with base e and base 10 are so important in applications that calculators have special keys for them. They also have their own special notation and names:

$$\log_e x = \ln x,$$

$$\log_{10} x = \log x.$$

The function $y = \ln x$ is called the **natural logarithm function** and $y = \log x$ is often called the **common logarithm function.**

Properties of Logarithms

Because a^x and $\log_a x$ are inverses of each other, composing them in either order gives the identity function. This gives two useful properties.

Inverse Properties for a^x and $\log_a x$

1. Base a: $a^{\log_a x} = x, \quad \log_a a^x = x, \quad a > 1, x > 0$
2. Base e: $e^{\ln x} = x, \quad \ln e^x = x, \quad x > 0$

These properties help us with the solution of equations that contain logarithms and exponential functions.

EXAMPLE 4 Using the Inverse Properties

Solve for x: **(a)** $\ln x = 3t + 5$ **(b)** $e^{2x} = 10$

SOLUTION

(a) $\ln x = 3t + 5$

$e^{\ln x} = e^{3t+5}$ Exponentiate both sides.

$x = e^{3t+5}$ Inverse Property

(b) $e^{2x} = 10$

$\ln e^{2x} = \ln 10$ Take logarithms of both sides.

$2x = \ln 10$ Inverse Property

$x = \frac{1}{2} \ln 10 \approx 1.15$

Now try Exercises 33 and 37.

The logarithm function has the following useful arithmetic properties.

Properties of Logarithms

For any real numbers $x > 0$ and $y > 0$,

1. *Product Rule*: $\log_a xy = \log_a x + \log_a y$
2. *Quotient Rule*: $\log_a \frac{x}{y} = \log_a x - \log_a y$
3. *Power Rule*: $\log_a x^y = y \log_a x$

EXPLORATION 2 **Supporting the Product Rule**

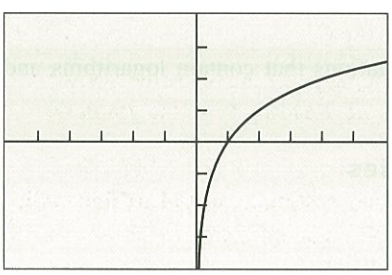
Let $y_1 = \ln(ax)$, $y_2 = \ln x$, and $y_3 = y_1 - y_2$.

1. Graph y_1 and y_2 for $a = 2, 3, 4$, and 5. How do the graphs of y_1 and y_2 appear to be related?
2. Support your finding by graphing y_3.
3. Confirm your finding algebraically.

The following formula allows us to evaluate $\log_a x$ for any base $a > 0$, $a \neq 1$, and to obtain its graph using the natural logarithm function on our grapher.

Change of Base Formula

$$\log_a x = \frac{\ln x}{\ln a}$$



[–6, 6] by [–4, 4]

Figure 1.38 The graph of $f(x) = \log_2 x$ using $f(x) = (\ln x)/(\ln 2)$. (Example 5)

EXAMPLE 5 Graphing a Base *a* Logarithm Function

Graph $f(x) = \log_2 x$.

SOLUTION

We use the change of base formula to rewrite $f(x)$.

$$f(x) = \log_2 x = \frac{\ln x}{\ln 2}.$$

Figure 1.38 gives the graph of f. *Now try Exercise 41.*

Applications

In Section 1.3 we used graphical methods to solve exponential growth and decay problems. Now we can use the properties of logarithms to solve the same problems algebraically.

EXAMPLE 6 Finding Time

Sarah invests \$1000 in an account that earns 5.25% interest compounded annually. How long will it take the account to reach \$2500?

SOLUTION

Model The amount in the account at any time t in years is $1000(1.0525)^t$, so we need to solve the equation

$$1000(1.0525)^t = 2500.$$

continued

Table 1.15 Saudi Arabia's Natural Gas Production

Year	Cubic Feet (trillions)
1997	1.60
1998	1.65
1999	1.63
2000	1.76
2001	1.90

Source: Statistical Abstract of the United States, 2004–2005.

Solve Algebraically

$$(1.0525)^t = 2.5 \quad \text{Divide by 1000.}$$
$$\ln (1.0525)^t = \ln 2.5 \quad \text{Take logarithms of both sides.}$$
$$t \ln 1.0525 = \ln 2.5 \quad \text{Power Rule}$$
$$t = \frac{\ln 2.5}{\ln 1.0525} \approx 17.9$$

Interpret The amount in Sarah's account will be \$2500 in about 17.9 years, or about 17 years and 11 months. *Now try Exercise 47.*

EXAMPLE 7 Estimating Natural Gas Production

Table 1.15 shows the annual number of cubic feet in trillions of natural gas produced by Saudi Arabia for several years.

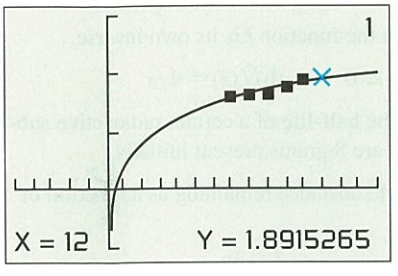
Find the natural logarithm regression equation for the data in Table 1.15 and use it to estimate the number of cubic feet of natural gas produced by Saudi Arabia in 2002. Compare with the actual amount of 2.00 trillion cubic feet in 2002.

SOLUTION

Model We let $x = 0$ represent 1990, $x = 1$ represent 1991, and so forth. We compute the natural logarithm regression equation to be

$$f(x) = 0.3730 + (0.611)\ln(x).$$

$f(x) = 0.3730 + (0.611) \ln x$



[−5, 15] by [−1, 3]

Figure 1.39 The value of f at $x = 12$ is about 1.89. (Example 7)

Solve Graphically Figure 1.39 shows the graph of f superimposed on the scatter plot of the data. The year 2002 is represented by $x = 12$. Reading from the graph we find $f(12) = 1.89$ trillion cubic feet.

Interpret The natural logarithmic model gives an underestimate of 0.11 trillion cubic feet of the 2002 natural gas production. *Now try Exercise 49.*

Quick Review 1.5 *(For help, go to Sections 1.2, 1.3, and 1.4.)*

In Exercises 1–4, let $f(x) = \sqrt[3]{x-1}$, $g(x) = x^2 + 1$, and evaluate the expression.

1. $(f \circ g)(1)$

2. $(g \circ f)(-7)$

3. $(f \circ g)(x)$

4. $(g \circ f)(x)$

In Exercises 5 and 6, choose parametric equations and a parameter interval to represent the function on the interval specified.

5. $y = \dfrac{1}{x-1}, \quad x \geq 2$

6. $y = x, \quad x < -3$

In Exercises 7–10, find the points of intersection of the two curves. Round your answers to 2 decimal places.

7. $y = 2x - 3, \quad y = 5$

8. $y = -3x + 5, \quad y = -3$

9. **(a)** $y = 2^x, \quad y = 3$

(b) $y = 2^x, \quad y = -1$

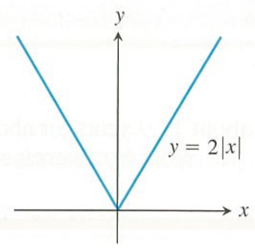
10. **(a)** $y = e^{-x}, \quad y = 4$

(b) $y = e^{-x}, \quad y = -1$

Section 1.5 Exercises

In Exercises 1–6, determine whether the function is one-to-one.

1.



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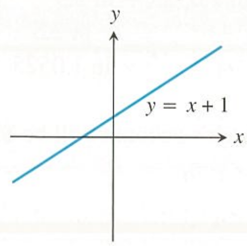

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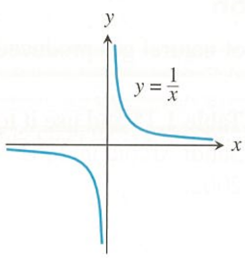

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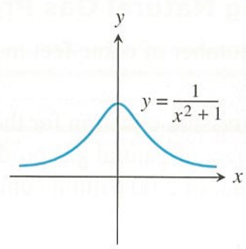
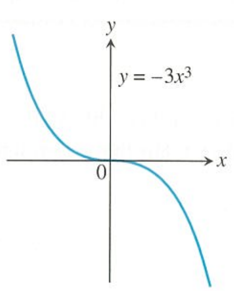
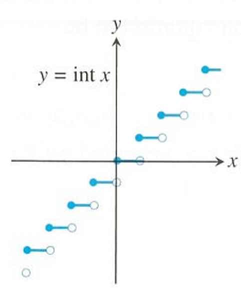


5.



6.



In Exercises 7–12, determine whether the function has an inverse function.

7. $y = \frac{3}{x-2} - 1$ **8.** $y = x^2 + 5x$ **9.** $y = x^3 - 4x + 6$

10. $y = x^3 + x$ **11.** $y = \ln x^2$ **12.** $y = 2^{3-x}$

In Exercises 13–24, find f^{-1} and verify that

$$(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x.$$

13. $f(x) = 2x + 3$ **14.** $f(x) = 5 - 4x$

15. $f(x) = x^3 - 1$ **16.** $f(x) = x^2 + 1, \quad x \geq 0$

17. $f(x) = x^2, \quad x \leq 0$ **18.** $f(x) = x^{2/3}, \quad x \geq 0$

19. $f(x) = -(x - 2)^2, \quad x \leq 2$

20. $f(x) = x^2 + 2x + 1, \quad x \geq -1$

21. $f(x) = \frac{1}{x^2}, \quad x > 0$ **22.** $f(x) = \frac{1}{x^3}$

23. $f(x) = \frac{2x + 1}{x + 3}$ **24.** $f(x) = \frac{x + 3}{x - 2}$

In Exercises 25–32, use parametric graphing to graph f, f^{-1}, and $y = x$.

25. $f(x) = e^x$ **26.** $f(x) = 3^x$ **27.** $f(x) = 2^{-x}$

28. $f(x) = 3^{-x}$ **29.** $f(x) = \ln x$ **30.** $f(x) = \log x$

31. $f(x) = \sin^{-1} x$ **32.** $f(x) = \tan^{-1} x$

In Exercises 33–36, solve the equation algebraically. Support your solution graphically.

33. $(1.045)^t = 2$ **34.** $e^{0.05t} = 3$

35. $e^x + e^{-x} = 3$ **36.** $2^x + 2^{-x} = 5$

In Exercises 37 and 38, solve for y.

37. $\ln y = 2t + 4$ **38.** $\ln (y - 1) - \ln 2 = x + \ln x$

In Exercises 39–42, draw the graph and determine the domain and range of the function.

39. $y = 2 \ln (3 - x) - 4$ **40.** $y = -3 \log (x + 2) + 1$

41. $y = \log_2 (x + 1)$ **42.** $y = \log_3 (x - 4)$

In Exercises 43 and 44, find a formula for f^{-1} and verify that $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$.

43. $f(x) = \frac{100}{1 + 2^{-x}}$ **44.** $f(x) = \frac{50}{1 + 1.1^{-x}}$

45. ***Self-inverse*** Prove that the function f is its own inverse.

(a) $f(x) = \sqrt{1 - x^2}, \quad x \geq 0$ **(b)** $f(x) = 1/x$

46. ***Radioactive Decay*** The half-life of a certain radioactive substance is 12 hours. There are 8 grams present initially.

(a) Express the amount of substance remaining as a function of time t.

(b) When will there be 1 gram remaining?

47. ***Doubling Your Money*** Determine how much time is required for a \$500 investment to double in value if interest is earned at the rate of 4.75% compounded annually.

48. ***Population Growth*** The population of Glenbrook is 375,000 and is increasing at the rate of 2.25% per year. Predict when the population will be 1 million.

In Exercises 49 and 50, let $x = 0$ represent 1990, $x = 1$ represent 1991, and so forth.

49. ***Natural Gas Production***

(a) Find a natural logarithm regression equation for the data in Table 1.16 and superimpose its graph on a scatter plot of the data.

Table 1.16 Canada's Natural Gas Production

Year	Cubic Feet (trillions)
1997	5.76
1998	5.98
1999	6.26
2000	6.47
2001	6.60

Source: Statistical Abstract of the United States, 2004–2005.

(b) Estimate the number of cubic feet of natural gas produced by Canada in 2002. Compare with the actual amount of 6.63 trillion cubic feet in 2002.

(c) Predict when Canadian natural gas production will reach 7 trillion cubic feet.

50. ***Natural Gas Production***

(a) Find a natural logarithm regression equation for the data in Table 1.17 and superimpose its graph on a scatter plot of the data.

Table 1.17 China's Natural Gas Production

Year	Cubic Feet (trillions)
1997	0.75
1998	0.78
1999	0.85
2000	0.96
2001	1.07

Source: Statistical Abstract of the United States, 2004–2005.

(b) Estimate the number of cubic feet of natural gas produced by China in 2002. Compare with the actual amount of 1.15 trillion cubic feet in 2002.

(c) Predict when China's natural gas production will reach 1.5 trillion cubic feet.

51. Group Activity ***Inverse Functions*** Let $y = f(x) = mx + b$, $m \neq 0$.

(a) Writing to Learn Give a convincing argument that f is a one-to-one function.

(b) Find a formula for the inverse of f. How are the slopes of f and f^{-1} related?

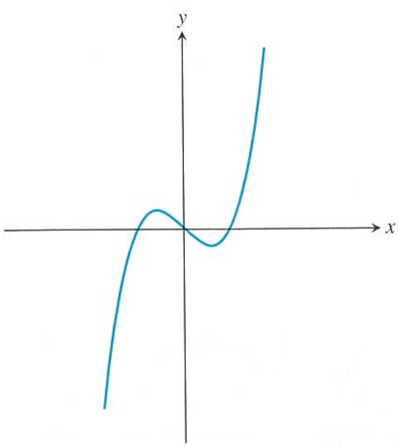
(c) If the graphs of two functions are parallel lines with a nonzero slope, what can you say about the graphs of the inverses of the functions?

(d) If the graphs of two functions are perpendicular lines with a nonzero slope, what can you say about the graphs of the inverses of the functions?

Standardized Test Questions

You should solve the following problems without using a graphing calculator.

52. True or False The function displayed in the graph below is one-to-one. Justify your answer.



53. True or False If $(f \circ g)(x) = x$, then g is the inverse function of f. Justify your answer.

In Exercises 54 and 55, use the function $f(x) = 3 - \ln(x + 2)$.

54. Multiple Choice Which of the following is the domain of f?

(A) $x \neq -2$ **(B)** $(-\infty, \infty)$ **(C)** $(-2, \infty)$
(D) $[-1.9, \infty)$ **(E)** $(0, \infty)$

55. Multiple Choice Which of the following is the range of f?

(A) $(-\infty, \infty)$ **(B)** $(-\infty, 0)$ **(C)** $(-2, \infty)$
(D) $(0, \infty)$ **(E)** $(0, 5.3)$

56. Multiple Choice Which of the following is the inverse of $f(x) = 3x - 2$?

(A) $g(x) = \dfrac{1}{3x - 2}$ **(B)** $g(x) = x$ **(C)** $g(x) = 3x - 2$
(D) $g(x) = \dfrac{x - 2}{3}$ **(E)** $g(x) = \dfrac{x + 2}{3}$

57. Multiple Choice Which of the following is a solution of the equation $2 - 3^{-x} = -1$?

(A) $x = -2$ **(B)** $x = -1$ **(C)** $x = 0$
(D) $x = 1$ **(E)** There are no solutions.

Exploration

58. ***Supporting the Quotient Rule*** Let $y_1 = \ln(x/a)$, $y_2 = \ln x$, $y_3 = y_2 - y_1$, and $y_4 = e^{y_3}$.

(a) Graph y_1 and y_2 for $a = 2, 3, 4$, and 5. How are the graphs of y_1 and y_2 related?

(b) Graph y_3 for $a = 2, 3, 4$, and 5. Describe the graphs.

(c) Graph y_4 for $a = 2, 3, 4$, and 5. Compare the graphs to the graph of $y = a$.

(d) Use $e^{y_3} = e^{y_2 - y_1} = a$ to solve for y_1.

Extending the Ideas

59. ***One-to-One Functions*** If f is a one-to-one function, prove that $g(x) = -f(x)$ is also one-to-one.

60. ***One-to-One Functions*** If f is a one-to-one function and $f(x)$ is never zero, prove that $g(x) = 1/f(x)$ is also one-to-one.

61. ***Domain and Range*** Suppose that $a \neq 0$, $b \neq 1$, and $b > 0$. Determine the domain and range of the function.

(a) $y = a(b^{c-x}) + d$ **(b)** $y = a \log_b (x - c) + d$

62. Group Activity ***Inverse Functions***

Let $f(x) = \dfrac{ax + b}{cx + d}$, $c \neq 0$, $ad - bc \neq 0$.

(a) Writing to Learn Give a convincing argument that f is one-to-one.

(b) Find a formula for the inverse of f.

(c) Find the horizontal and vertical asymptotes of f.

(d) Find the horizontal and vertical asymptotes of f^{-1}. How are they related to those of f?

1.6 Trigonometric Functions

What you'll learn about

- Radian Measure
- Graphs of Trigonometric Functions
- Periodicity
- Even and Odd Trigonometric Functions
- Transformations of Trigonometric Graphs
- Inverse Trigonometric Functions

. . . and why

Trigonometric functions can be used to model periodic behavior and applications such as musical notes.

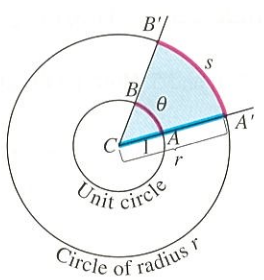


Figure 1.40 The radian measure of angle ACB is the length θ of arc AB on the unit circle centered at C. The value of θ can be found from any other circle, however, as the ratio s/r.

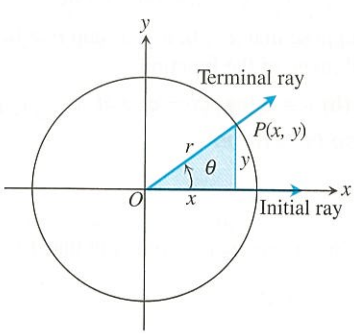


Figure 1.41 An angle θ in standard position.

Radian Measure

The **radian measure** of the angle ACB at the center of the unit circle (Figure 1.40) equals the length of the arc that ACB cuts from the unit circle.

EXAMPLE 1 Finding Arc Length

Find the length of an arc subtended on a circle of radius 3 by a central angle of measure $2\pi/3$.

SOLUTION

According to Figure 1.40, if s is the length of the arc, then

$$s = r\theta = 3(2\pi/3) = 2\pi.$$

Now try Exercise 1.

When an angle of measure θ is placed in *standard position* at the center of a circle of radius r (Figure 1.41), the six basic trigonometric functions of θ are defined as follows:

sine: $\sin\theta = \dfrac{y}{r}$	**cosecant:** $\csc\theta = \dfrac{r}{y}$
cosine: $\cos\theta = \dfrac{x}{r}$	**secant:** $\sec\theta = \dfrac{r}{x}$
tangent: $\tan\theta = \dfrac{y}{x}$	**cotangent:** $\cot\theta = \dfrac{x}{y}$

Graphs of Trigonometric Functions

When we graph trigonometric functions in the coordinate plane, we usually denote the independent variable (radians) by x instead of θ. Figure 1.42 on the next page shows sketches of the six trigonometric functions. It is a good exercise for you to compare these with what you see in a grapher viewing window. (Some graphers have a "trig viewing window.")

EXPLORATION 1 Unwrapping Trigonometric Functions

Set your grapher in *radian mode,* parametric mode, and *simultaneous mode* (all three). Enter the parametric equations

$$x_1 = \cos t, \quad y_1 = \sin t \quad \text{and} \quad x_2 = t, \quad y_2 = \sin t.$$

1. Graph for $0 \le t \le 2\pi$ in the window $[-1.5, 2\pi]$ by $[-2.5, 2.5]$. Describe the two curves. (You may wish to make the viewing window square.)
2. Use trace to compare the y-values of the two curves.
3. Repeat part 2 in the window $[-1.5, 4\pi]$ by $[-5, 5]$, using the parameter interval $0 \le t \le 4\pi$.
4. Let $y_2 = \cos t$. Use trace to compare the x-values of curve 1 (the unit circle) with the y-values of curve 2 using the parameter intervals $[0, 2\pi]$ and $[0, 4\pi]$.
5. Set $y_2 = \tan t$, $\csc t$, $\sec t$, and $\cot t$. Graph each in the window $[-1.5, 2\pi]$ by $[-2.5, 2.5]$ using the interval $0 \le t \le 2\pi$. How is a y-value of curve 2 related to the corresponding point on curve 1? (Use trace to explore the curves.)

Angle Convention: Use Radians

From now on in this book it is assumed that all angles are measured in radians unless degrees or some other unit is stated explicitly. When we talk about the angle $\pi/3$, we mean $\pi/3$ radians (which is 60°), not $\pi/3$ degrees. When you do calculus, keep your calculator in radian mode.

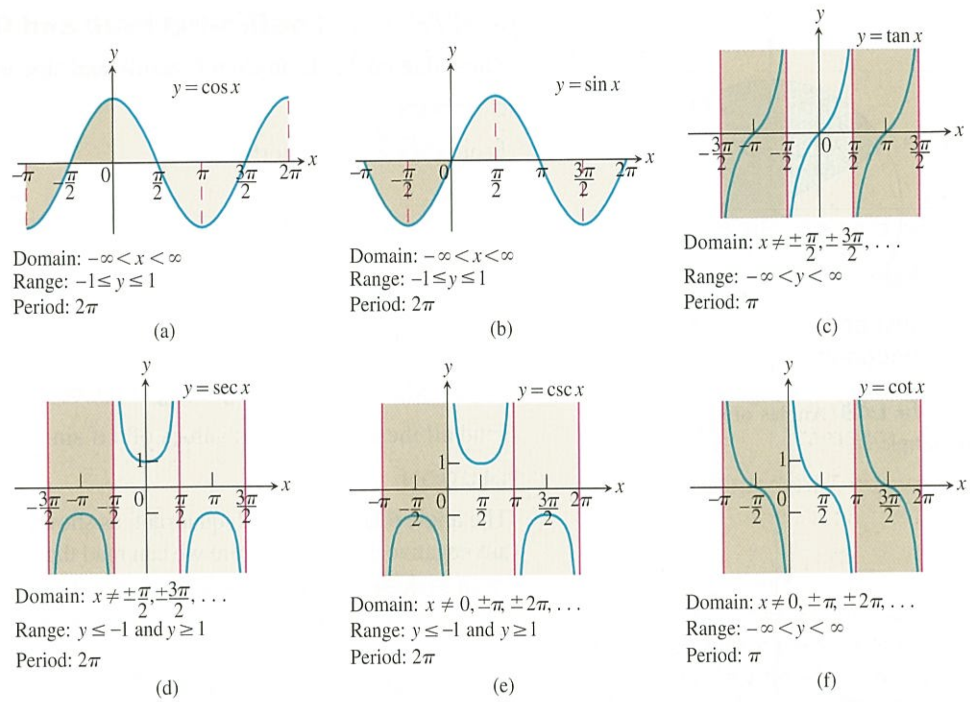


Figure 1.42 Graphs of the (a) cosine, (b) sine, (c) tangent, (d) secant, (e) cosecant, and (f) cotangent functions using radian measure.

Periods of Trigonometric Functions

Period π: $\tan(x + \pi) = \tan x$
$\cot(x + \pi) = \cot x$

Period 2π: $\sin(x + 2\pi) = \sin x$
$\cos(x + 2\pi) = \cos x$
$\sec(x + 2\pi) = \sec x$
$\csc(x + 2\pi) = \csc x$

Periodicity

When an angle of measure θ and an angle of measure $\theta + 2\pi$ are in standard position, their terminal rays coincide. The two angles therefore have the same trigonometric function values:

$$\cos(\theta + 2\pi) = \cos\theta \quad \sin(\theta + 2\pi) = \sin\theta \quad \tan(\theta + 2\pi) = \tan\theta$$
$$\sec(\theta + 2\pi) = \sec\theta \quad \csc(\theta + 2\pi) = \csc\theta \quad \cot(\theta + 2\pi) = \cot\theta \tag{1}$$

Similarly, $\cos(\theta - 2\pi) = \cos\theta$, $\sin(\theta - 2\pi) = \sin\theta$, and so on.

We see the values of the trigonometric functions repeat at regular intervals. We describe this behavior by saying that the six basic trigonometric functions are *periodic*.

> **DEFINITION Periodic Function, Period**
>
> A function $f(x)$ is **periodic** if there is a positive number p such that $f(x + p) = f(x)$ for every value of x. The smallest such value of p is the **period** of f.

As we can see in Figure 1.42, the functions $\cos x$, $\sin x$, $\sec x$, and $\csc x$ are periodic with period 2π. The functions $\tan x$ and $\cot x$ are periodic with period π.

Even and Odd Trigonometric Functions

The graphs in Figure 1.42 suggest that $\cos x$ and $\sec x$ are even functions because their graphs are symmetric about the y-axis. The other four basic trigonometric functions are odd.

EXAMPLE 2 Confirming Even and Odd

Show that cosine is an even function and sine is odd.

SOLUTION

From Figure 1.43 it follows that

$$\cos(-\theta) = \frac{x}{r} = \cos\theta, \quad \sin(-\theta) = \frac{-y}{r} = -\sin\theta,$$

so cosine is an even function and sine is odd. *Now try Exercise 5.*

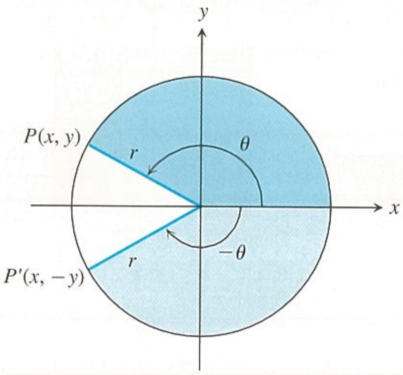


Figure 1.43 Angles of opposite sign. (Example 2)

EXAMPLE 3 Finding Trigonometric Values

Find all the trigonometric values of θ if $\sin\theta = -3/5$ and $\tan\theta < 0$.

SOLUTION

The angle θ is in the fourth quadrant, as shown in Figure 1.44, because its sine and tangent are negative. From this figure we can read that $\cos\theta = 4/5$, $\tan\theta = -3/4$, $\csc\theta = -5/3$, $\sec\theta = 5/4$, and $\cot\theta = -4/3$. *Now try Exercise 9.*

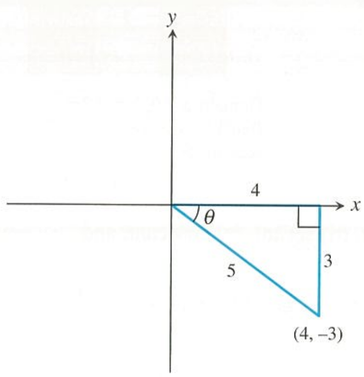


Figure 1.44 The angle θ in standard position. (Example 3)

Transformations of Trigonometric Graphs

The rules for shifting, stretching, shrinking, and reflecting the graph of a function apply to the trigonometric functions. The following diagram will remind you of the controlling parameters.

Vertical stretch or shrink; reflection about x-axis — Vertical shift

$$y = af(b(x + c)) + d$$

Horizontal stretch or shrink; reflection about y-axis — Horizontal shift

The general sine function or **sinusoid** can be written in the form

$$f(x) = A\sin\left[\frac{2\pi}{B}(x - C)\right] + D,$$

where $|A|$ is the *amplitude*, $|B|$ is the *period*, C is the *horizontal shift*, and D is the *vertical shift*.

EXAMPLE 4 Graphing a Trigonometric Function

Determine the **(a)** period, **(b)** domain, **(c)** range, and **(d)** draw the graph of the function $y = 3\cos(2x - \pi) + 1$.

SOLUTION

We can rewrite the function in the form

$$y = 3\cos\left[2\left(x - \frac{\pi}{2}\right)\right] + 1.$$

(a) The period is given by $2\pi/B$, where $2\pi/B = 2$. The period is π.

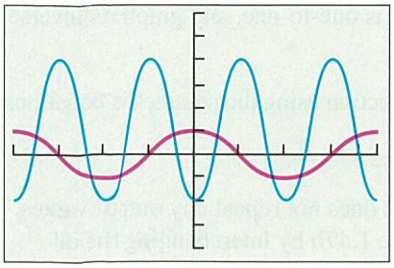
(b) The domain is $(-\infty, \infty)$.

(c) The graph is a basic cosine curve with amplitude 3 that has been shifted up 1 unit. Thus, the range is $[-2, 4]$.

continued

(d) The graph has been shifted to the right $\pi/2$ units. The graph is shown in Figure 1.45 together with the graph of $y = \cos x$. Notice that four periods of $y = 3\cos(2x - \pi) + 1$ are drawn in this window. ***Now try Exercise 13.***

$y = 3\cos(2x - \pi) + 1,\ y = \cos x$



$[-2\pi, 2\pi]$ by $[-4, 6]$

Figure 1.45 The graph of $y = 3\cos(2x - \pi) + 1$ (blue) and the graph of $y = \cos x$ (red). (Example 4)

Musical notes are pressure waves in the air. The wave behavior can be modeled with great accuracy by general sine curves. Devices called Calculator Based Laboratory™ (CBL) systems can record these waves with the aid of a microphone. The data in Table 1.18 give pressure displacement versus time in seconds of a musical note produced by a tuning fork and recorded with a CBL system.

Table 1.18 Tuning Fork Data

Time	Pressure	Time	Pressure	Time	Pressure
0.00091	−0.080	0.00271	−0.141	0.00453	0.749
0.00108	0.200	0.00289	−0.309	0.00471	0.581
0.00125	0.480	0.00307	−0.348	0.00489	0.346
0.00144	0.693	0.00325	−0.248	0.00507	0.077
0.00162	0.816	0.00344	−0.041	0.00525	−0.164
0.00180	0.844	0.00362	0.217	0.00543	−0.320
0.00198	0.771	0.00379	0.480	0.00562	−0.354
0.00216	0.603	0.00398	0.681	0.00579	−0.248
0.00234	0.368	0.00416	0.810	0.00598	−0.035
0.00253	0.099	0.00435	0.827		

EXAMPLE 5 Finding the Frequency of a Musical Note

Consider the tuning fork data in Table 1.18.

(a) Find a sinusoidal regression equation (general sine curve) for the data and superimpose its graph on a scatter plot of the data.

(b) The *frequency* of a musical note, or wave, is measured in cycles per second, or hertz (1 Hz = 1 cycle per second). The frequency is the reciprocal of the *period* of the wave, which is measured in seconds per cycle. Estimate the frequency of the note produced by the tuning fork.

SOLUTION

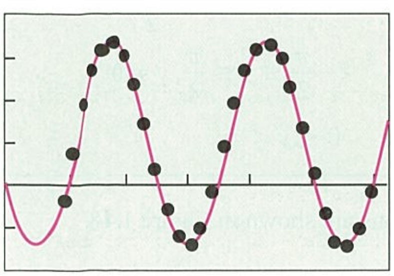
(a) The sinusoidal regression equation produced by our calculator is approximately

$$y = 0.6\sin(2488.6x - 2.832) + 0.266.$$

Figure 1.46 shows its graph together with a scatter plot of the tuning fork data.

(b) The period is $\dfrac{2\pi}{2488.6}$ sec, so the frequency is $\dfrac{2488.6}{2\pi} \approx 396$ Hz.

$y = 0.6\sin(2488.6x - 2.832) + 0.266$



$[0, 0.0062]$ by $[-0.5, 1]$

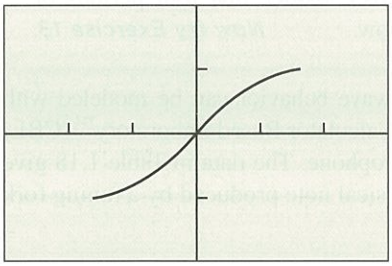
Figure 1.46 A sinusoidal regression model for the tuning fork data in Table 1.18. (Example 5)

Interpretation The tuning fork is vibrating at a frequency of about 396 Hz. On the pure tone scale, this is the note G above middle C. It is a few cycles per second different from the frequency of the G we hear on a piano's tempered scale, 392 Hz.
Now try Exercise 23.

Inverse Trigonometric Functions

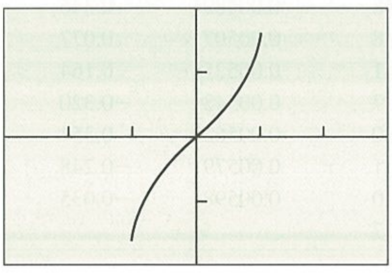
None of the six basic trigonometric functions graphed in Figure 1.42 is one-to-one. These functions do not have inverses. However, in each case the domain can be restricted to produce a new function that does have an inverse, as illustrated in Example 6.

$x = t,\ y = \sin t,\ -\frac{\pi}{2} \le t \le \frac{\pi}{2}$



[–3, 3] by [–2, 2]
(a)

$x = \sin t,\ y = t,\ -\frac{\pi}{2} \le t \le \frac{\pi}{2}$



[–3, 3] by [–2, 2]
(b)

Figure 1.47 (a) A restricted sine function and (b) its inverse. (Example 6)

EXAMPLE 6 Restricting the Domain of the Sine

Show that the function $y = \sin x,\ -\pi/2 \le x \le \pi/2$, is one-to-one, and graph its inverse.

SOLUTION

Figure 1.47a shows the graph of this restricted sine function using the parametric equations

$$x_1 = t, \quad y_1 = \sin t, \quad -\frac{\pi}{2} \le t \le \frac{\pi}{2}.$$

This restricted sine function is one-to-one because it does not repeat any output values. It therefore has an inverse, which we graph in Figure 1.47b by interchanging the ordered pairs using the parametric equations

$$x_2 = \sin t, \quad y_2 = t, \quad -\frac{\pi}{2} \le t \le \frac{\pi}{2}.$$

Now try Exercise 25.

The inverse of the restricted sine function of Example 6 is called the *inverse sine function*. The inverse sine of x is the angle whose sine is x. It is denoted by $\sin^{-1} x$ or $\arcsin x$. Either notation is read "arcsine of x" or "the inverse sine of x."

The domains of the other basic trigonometric functions can also be restricted to produce a function with an inverse. The domains and ranges of the resulting inverse functions become parts of their definitions.

DEFINITIONS Inverse Trigonometric Functions

Function	Domain	Range
$y = \cos^{-1} x$	$-1 \le x \le 1$	$0 \le y \le \pi$
$y = \sin^{-1} x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$y = \tan^{-1} x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
$y = \sec^{-1} x$	$\lvert x \rvert \ge 1$	$0 \le y \le \pi,\ y \ne \frac{\pi}{2}$
$y = \csc^{-1} x$	$\lvert x \rvert \ge 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2},\ y \ne 0$
$y = \cot^{-1} x$	$-\infty < x < \infty$	$0 < y < \pi$

The graphs of the six inverse trigonometric functions are shown in Figure 1.48.

EXAMPLE 7 Finding Angles in Degrees and Radians

Find the measure of $\cos^{-1}(-0.5)$ in degrees and radians.

SOLUTION

Put the calculator in degree mode and enter $\cos^{-1}(-0.5)$. The calculator returns 120, which means 120 degrees. Now put the calculator in radian mode and enter $\cos^{-1}(-0.5)$. The calculator returns 2.094395102, which is the measure of the angle in radians. You can check that $2\pi/3 \approx 2.094395102$.

Now try Exercise 27.

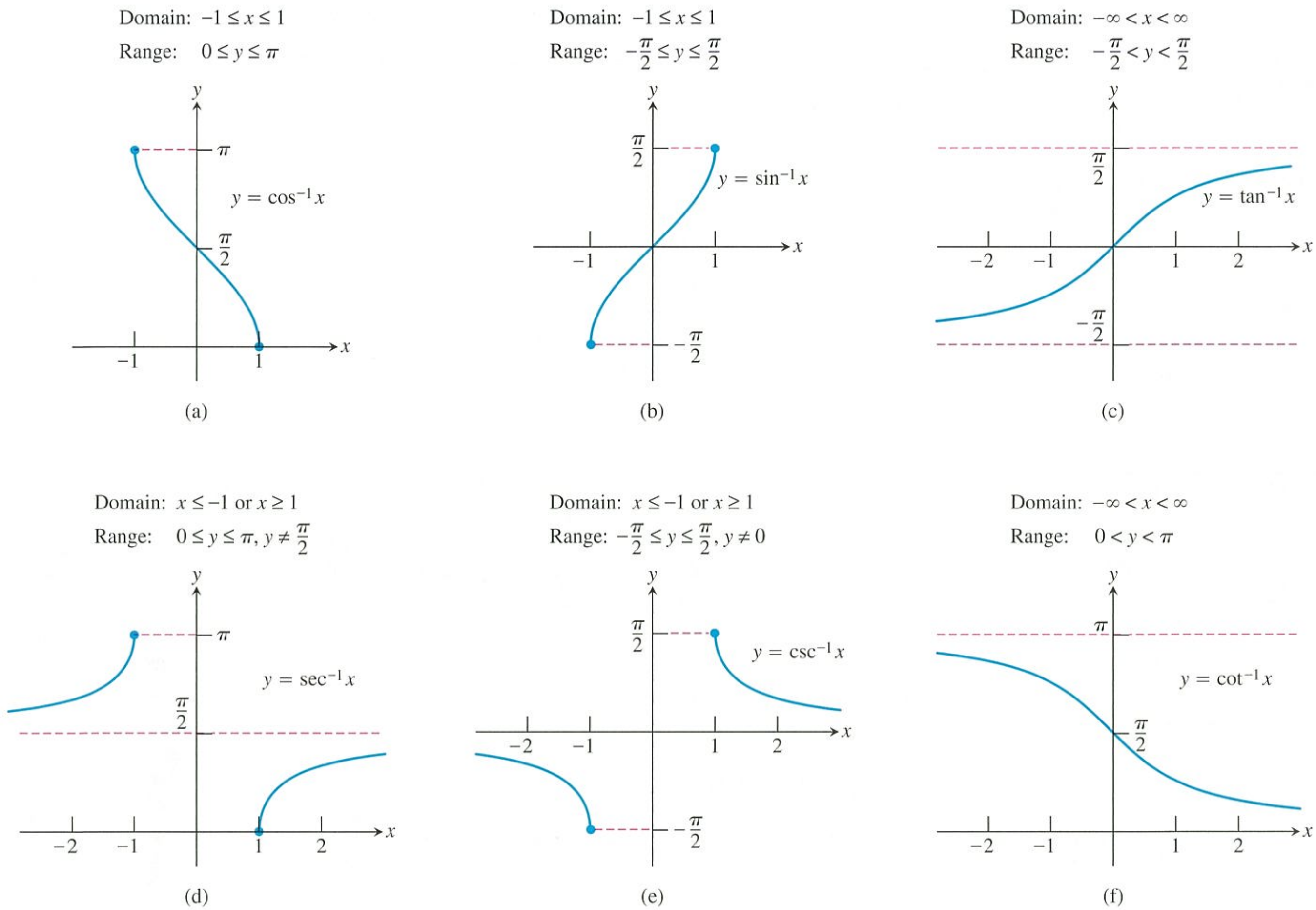


Figure 1.48 Graphs of (a) $y = \cos^{-1} x$, (b) $y = \sin^{-1} x$, (c) $y = \tan^{-1} x$, (d) $y = \sec^{-1} x$, (e) $y = \csc^{-1} x$, and (f) $y = \cot^{-1} x$.

EXAMPLE 8 Using the Inverse Trigonometric Functions

Solve for x.

(a) $\sin x = 0.7$ in $0 \le x < 2\pi$

(b) $\tan x = -2$ in $-\infty < x < \infty$

SOLUTION

(a) Notice that $x = \sin^{-1}(0.7) \approx 0.775$ is in the first quadrant, so 0.775 is one solution of this equation. The angle $\pi - x$ is in the second quadrant and has sine equal to 0.7. Thus two solutions in this interval are

$$\sin^{-1}(0.7) \approx 0.775 \quad \text{and} \quad \pi - \sin^{-1}(0.7) \approx 2.366.$$

(b) The angle $x = \tan^{-1}(-2) \approx -1.107$ is in the fourth quadrant and is the only solution to this equation in the interval $-\pi/2 < x < \pi/2$ where $\tan x$ is one-to-one. Since $\tan x$ is periodic with period π, the solutions to this equation are of the form

$$\tan^{-1}(-2) + k\pi \quad \approx -1.107 + k\pi$$

where k is any integer.

Now try Exercise 31.

Quick Review 1.6 *(For help, go to Sections 1.2 and 1.6.)*

In Exercises 1–4, convert from radians to degrees or degrees to radians.

1. $\pi/3$ **2.** -2.5 **3.** $-40°$ **4.** $45°$

In Exercises 5–7, solve the equation graphically in the given interval.

5. $\sin x = 0.6,\quad 0 \le x \le 2\pi$ **6.** $\cos x = -0.4,\quad 0 \le x \le 2\pi$

7. $\tan x = 1,\quad -\frac{\pi}{2} \le x < \frac{3\pi}{2}$

8. Show that $f(x) = 2x^2 - 3$ is an even function. Explain why its graph is symmetric about the y-axis.

9. Show that $f(x) = x^3 - 3x$ is an odd function. Explain why its graph is symmetric about the origin.

10. Give one way to restrict the domain of the function $f(x) = x^4 - 2$ to make the resulting function one-to-one.

Section 1.6 Exercises

In Exercises 1–4, the angle lies at the center of a circle and subtends an arc of the circle. Find the missing angle measure, circle radius, or arc length.

	Angle	Radius	Arc Length
1.	$5\pi/8$	2	?
2.	175°	?	10
3.	?	14	7
4.	?	6	$3\pi/2$

In Exercises 5–8, determine if the function is even or odd.

5. secant **6.** tangent

7. cosecant **8.** cotangent

In Exercises 9 and 10, find all the trigonometric values of θ with the given conditions.

9. $\cos\theta = -\frac{15}{17},\quad \sin\theta > 0$

10. $\tan\theta = -1,\quad \sin\theta < 0$

In Exercises 11–14, determine **(a)** the period, **(b)** the domain, **(c)** the range, and **(d)** draw the graph of the function.

11. $y = 3\csc(3x + \pi) - 2$ **12.** $y = 2\sin(4x + \pi) + 3$

13. $y = -3\tan(3x + \pi) + 2$

14. $y = 2\sin\left(2x + \frac{\pi}{3}\right)$

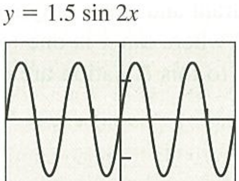
In Exercises 15 and 16, choose an appropriate viewing window to display two complete periods of each trigonometric function in radian mode.

15. (a) $y = \sec x$ **(b)** $y = \csc x$ **(c)** $y = \cot x$

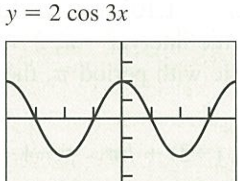
16. (a) $y = \sin x$ **(b)** $y = \cos x$ **(c)** $y = \tan x$

In Exercises 17–22, specify **(a)** the period, **(b)** the amplitude, and **(c)** identify the viewing window that is shown.

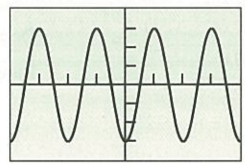
17. $y = 1.5\sin 2x$



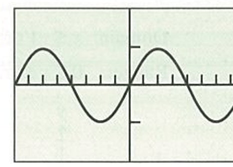
18. $y = 2\cos 3x$



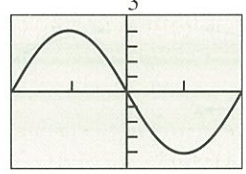
19. $y = -3\cos 2x$



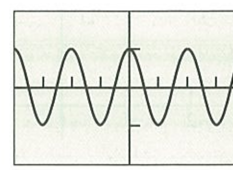
20. $y = 5\sin\frac{x}{2}$



21. $y = -4\sin\frac{\pi}{3}x$



22. $y = \cos\pi x$



23. Group Activity A musical note like that produced with a tuning fork or pitch meter is a pressure wave. Table 1.19 gives frequencies (in Hz) of musical notes on the tempered scale. The pressure versus time tuning fork data in Table 1.20 were collected using a CBL™ and a microphone.

Table 1.19 Frequencies of Notes

Note	Frequency (Hz)
C	262
C♯ or D♭	277
D	294
D♯ or E♭	311
E	330
F	349
F♯ or G♭	370
G	392
G♯ or A♭	415
A	440
A♯ or B♭	466
B	494
C (next octave)	524

Source: CBL™ System Experimental Workbook, Texas Instruments, Inc., 1994.

Table 1.20 Tuning Fork Data

Time (s)	Pressure	Time (s)	Pressure
0.0002368	1.29021	0.0049024	−1.06632
0.0005664	1.50851	0.0051520	0.09235
0.0008256	1.51971	0.0054112	1.44694
0.0010752	1.51411	0.0056608	1.51411
0.0013344	1.47493	0.0059200	1.51971
0.0015840	0.45619	0.0061696	1.51411
0.0018432	−0.89280	0.0064288	1.43015
0.0020928	−1.51412	0.0066784	0.19871
0.0023520	−1.15588	0.0069408	−1.06072
0.0026016	−0.04758	0.0071904	−1.51412
0.0028640	1.36858	0.0074496	−0.97116
0.0031136	1.50851	0.0076992	0.23229
0.0033728	1.51971	0.0079584	1.46933
0.0036224	1.51411	0.0082080	1.51411
0.0038816	1.45813	0.0084672	1.51971
0.0041312	0.32185	0.0087168	1.50851
0.0043904	−0.97676	0.0089792	1.36298
0.0046400	−1.51971		

(a) Find a sinusoidal regression equation for the data in Table 1.20 and superimpose its graph on a scatter plot of the data.

(b) Determine the frequency of and identify the musical note produced by the tuning fork.

24. ***Temperature Data*** Table 1.21 gives the average monthly temperatures for St. Louis for a 12-month period starting with January. Model the monthly temperature with an equation of the form

$$y = a \sin [b(t - h)] + k,$$

y in degrees Fahrenheit, t in months, as follows:

Table 1.21 Temperature Data for St. Louis

Time (months)	Temperature (°F)
1	34
2	30
3	39
4	44
5	58
6	67
7	78
8	80
9	72
10	63
11	51
12	40

(a) Find the value of b assuming that the period is 12 months.

(b) How is the amplitude a related to the difference $80° - 30°$?

(c) Use the information in (b) to find k.

(d) Find h, and write an equation for y.

(e) Superimpose a graph of y on a scatter plot of the data.

In Exercises 25–26, show that the function is one-to-one, and graph its inverse.

25. $y = \cos x, \quad 0 \le x \le \pi$ **26.** $y = \tan x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

In Exercises 27–30, give the measure of the angle in radians and degrees. Give exact answers whenever possible.

27. $\sin^{-1}(0.5)$ **28.** $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

29. $\tan^{-1}(-5)$ **30.** $\cos^{-1}(0.7)$

In Exercises 31–36, solve the equation in the specified interval.

31. $\tan x = 2.5, \quad 0 \le x \le 2\pi$

32. $\cos x = -0.7, \quad 2\pi \le x < 4\pi$

33. $\csc x = 2, \quad 0 < x < 2\pi$ **34.** $\sec x = -3, \quad -\pi \le x < \pi$

35. $\sin x = -0.5, \quad -\infty < x < \infty$ **36.** $\cot x = -1, \quad -\infty < x < \infty$

In Exercises 37–40, use the given information to find the values of the six trigonometric functions at the angle θ. Give exact answers.

37. $\theta = \sin^{-1}\left(\frac{8}{17}\right)$ **38.** $\theta = \tan^{-1}\left(-\frac{5}{12}\right)$

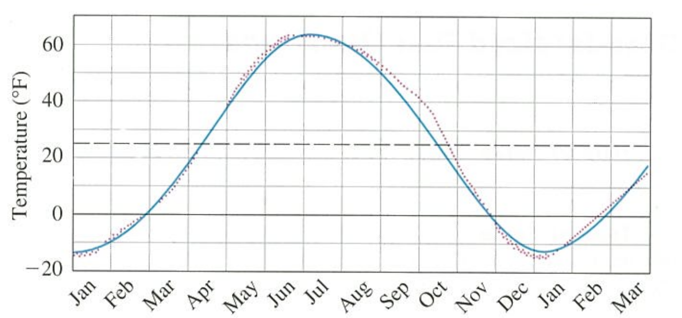
39. The point $P(-3, 4)$ is on the terminal side of θ.

40. The point $P(-2, 2)$ is on the terminal side of θ.

In Exercises 41 and 42, evaluate the expression.

41. $\sin\left(\cos^{-1}\left(\frac{7}{11}\right)\right)$ **42.** $\tan\left(\sin^{-1}\left(\frac{9}{13}\right)\right)$

43. ***Temperatures in Fairbanks, Alaska*** Find the **(a)** amplitude, **(b)** period, **(c)** horizontal shift, and **(d)** vertical shift of the model used in the figure below. **(e)** Then write the equation for the model.



Normal mean air temperature for Fairbanks, Alaska, plotted as data points (red). The approximating sine function $f(x)$ is drawn in blue.
Source: "Is the Curve of Temperature Variation a Sine Curve?" by B. M. Lando and C. A. Lando, *The Mathematics Teacher, 7.6*, Fig. 2, p. 535 (Sept . 1977).

44. ***Temperatures in Fairbanks, Alaska*** Use the equation of Exercise 43 to approximate the answers to the following questions about the temperatures in Fairbanks, Alaska, shown in the figure in Exercise 43. Assume that the year has 365 days.

(a) What are the highest and lowest mean daily temperatures?

(b) What is the average of the highest and lowest mean daily temperatures? Why is this average the vertical shift of the function?

45. ***Even-Odd***

(a) Show that $\cot x$ is an odd function of x.

(b) Show that the quotient of an even function and an odd function is an odd function.

46. ***Even-Odd***

(a) Show that $\csc x$ is an odd function of x.

(b) Show that the reciprocal of an odd function is odd.

47. ***Even-Odd*** Show that the product of an even function and an odd function is an odd function.

48. ***Finding the Period*** Give a convincing argument that the period of $\tan x$ is π.

49. ***Sinusoidal Regression*** Table 1.22 gives the values of the function

$$f(x) = a \sin(bx + c) + d$$

accurate to two decimals.

Table 1.22 Values of a Function

x	$f(x)$
1	3.42
2	0.73
3	0.12
4	2.16
5	4.97
6	5.97

(a) Find a sinusoidal regression equation for the data.

(b) Rewrite the equation with a, b, c, and d rounded to the nearest integer.

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

50. **True or False** The period of $y = \sin(x/2)$ is π. Justify your answer.

51. **True or False** The amplitude of $y = \frac{1}{2}\cos x$ is 1. Justify your answer.

In Exercises 52–54, $f(x) = 2\cos(4x + \pi) - 1$.

52. **Multiple Choice** Which of the following is the domain of f?

(A) $[-\pi, \pi]$ (B) $[-3, 1]$ (C) $[-1, 4]$

(D) $(-\infty, \infty)$ (E) $x \neq 0$

53. **Multiple Choice** Which of the following is the range of f?

(A) $(-3, 1)$ (B) $[-3, 1]$ (C) $(-1, 4)$

(D) $[-1, 4]$ (E) $(-\infty, \infty)$

54. **Multiple Choice** Which of the following is the period of f?

(A) 4π (B) 3π (C) 2π (D) π (E) $\pi/2$

55. **Multiple Choice** Which of the following is the measure of $\tan^{-1}(-\sqrt{3})$ in degrees?

(A) $-60°$ (B) $-30°$ (C) $30°$ (D) $60°$ (E) $120°$

Exploration

56. ***Trigonometric Identities*** Let $f(x) = \sin x + \cos x$.

(a) Graph $y = f(x)$. Describe the graph.

(b) Use the graph to identify the amplitude, period, horizontal shift, and vertical shift.

(c) Use the formula

$$\sin\alpha\cos\beta + \cos\alpha\sin\beta = \sin(\alpha + \beta)$$

for the sine of the sum of two angles to confirm your answers.

Extending the Ideas

57. **Exploration** Let $y = \sin(ax) + \cos(ax)$.

Use the symbolic manipulator of a computer algebra system (CAS) to help you with the following:

(a) Express y as a sinusoid for $a = 2, 3, 4$, and 5.

(b) Conjecture another formula for y for a equal to any positive integer n.

(c) Check your conjecture with a CAS.

(d) Use the formula for the sine of the sum of two angles (see Exercise 56c) to confirm your conjecture.

58. **Exploration** Let $y = a\sin x + b\cos x$.

Use the symbolic manipulator of a computer algebra system (CAS) to help you with the following:

(a) Express y as a sinusoid for the following pairs of values: $a = 2, b = 1$; $a = 1, b = 2$; $a = 5, b = 2$; $a = 2, b = 5$; $a = 3, b = 4$.

(b) Conjecture another formula for y for any pair of positive integers. Try other values if necessary.

(c) Check your conjecture with a CAS.

(d) Use the following formulas for the sine or cosine of a sum or difference of two angles to confirm your conjecture.

$$\sin\alpha\cos\beta \pm \cos\alpha\sin\beta = \sin(\alpha \pm \beta)$$

$$\cos\alpha\cos\beta \pm \sin\alpha\sin\beta = \cos(\alpha \mp \beta)$$

In Exercises 59 and 60, show that the function is periodic and find its period.

59. $y = \sin^3 x$

60. $y = |\tan x|$

In Exercises 61 and 62, graph one period of the function.

61. $f(x) = \sin(60x)$

62. $f(x) = \cos(60\pi x)$

Quick Quiz for AP* Preparation: Sections 1.4–1.6

You should solve the following problems without using a graphing calculator.

1. Multiple Choice Which of the following is the domain of $f(x) = -\log_2(x + 3)$?

(A) $(-\infty, \infty)$ **(B)** $(-\infty, 3)$ **(C)** $(-3, \infty)$

(D) $[-3, \infty)$ **(E)** $(-\infty, 3]$

2. Multiple Choice Which of the following is the range of $f(x) = 5\cos(x + \pi) + 3$?

(A) $(-\infty, \infty)$ **(B)** $[2, 4]$ **(C)** $[-8, 2]$

(D) $[-2, 8]$ **(E)** $\left[-\frac{2}{5}, \frac{8}{5}\right]$

3. Multiple Choice Which of the following gives the solution of $\tan x = -1$ in $\pi < x < \frac{3\pi}{2}$?

(A) $-\frac{\pi}{4}$ **(B)** $\frac{\pi}{4}$ **(C)** $\frac{\pi}{3}$ **(D)** $\frac{3\pi}{4}$ **(E)** $\frac{5\pi}{4}$

4. Free Response Let $f(x) = 5x - 3$.

(a) Find the inverse g of f.

(b) Compute $f \circ g(x)$. Show your work.

(c) Compute $g \circ f(x)$. Show your work.

Chapter 1 Key Terms

absolute value function (p. 17)
base a logarithm function (p. 40)
boundary of an interval (p. 13)
boundary points (p. 13)
change of base formula (p. 42)
closed interval (p. 13)
common logarithm function (p. 41)
composing (p. 18)
composite function (p. 17)
compounded continuously (p. 25)
cosecant function (p. 46)
cosine function (p. 46)
cotangent function (p. 46)
dependent variable (p. 12)
domain (p. 12)
even function (p. 15)
exponential decay (p. 24)
exponential function base a (p. 22)
exponential growth (p. 24)
function (p. 12)
general linear equation (p. 5)
graph of a function (p. 13)
graph of a relation (p. 30)
grapher failure (p. 15)
half-life (p. 24)
half-open interval (p. 13)
identity function (p. 38)
increments (p. 3)
independent variable (p. 12)
initial point of parametrized curve (p. 30)
interior of an interval (p. 13)
interior points of an interval (p. 13)
inverse cosecant function (p. 50)
inverse cosine function (p. 50)
inverse cotangent function (p. 50)
inverse function (p. 38)
inverse properties for a^x and $\log_a x$ (p. 41)
inverse secant function (p. 50)
inverse sine function (p. 50)
inverse tangent function (p. 50)
linear regression (p. 7)
natural domain (p. 13)
natural logarithm function (p. 41)
odd function (p. 15)
one-to-one function (p. 37)
open interval (p. 13)
parallel lines (p. 4)
parameter (p. 30)
parameter interval (p. 30)
parametric curve (p. 30)
parametric equations (p. 30)
parametrization of a curve (p. 30)
parametrize (p. 30)
period of a function (p. 47)
periodic function (p. 47)
perpendicular lines (p. 4)
piecewise defined function (p. 16)
point-slope equation (p. 4)
power rule for logarithms (p. 41)
product rule for logarithms (p. 41)
quotient rule for logarithms (p. 41)
radian measure (p. 46)
range (p. 12)
regression analysis (p. 7)
regression curve (p. 7)
relation (p. 30)
rise (p. 3)
rules for exponents (p. 23)
run (p. 3)
scatter plot (p. 7)
secant function (p. 46)
sine function (p. 46)
sinusoid (p. 48)
sinusoidal regression (p. 49)
slope (p. 4)
slope-intercept equation (p. 5)
symmetry about the origin (p. 15)
symmetry about the y-axis (p. 15)
tangent function (p. 46)
terminal point of parametrized curve (p. 30)
witch of Agnesi (p. 33)
x-intercept (p. 5)
y-intercept (p. 5)

Chapter 1 Review Exercises

The collection of exercises marked in red could be used as a chapter test.

In Exercises 1–14, write an equation for the specified line.

1. through $(1, -6)$ with slope 3

2. through $(-1, 2)$ with slope $-1/2$

3. the vertical line through $(0, -3)$

4. through $(-3, 6)$ and $(1, -2)$

5. the horizontal line through $(0, 2)$

6. through $(3, 3)$ and $(-2, 5)$

7. with slope -3 and y-intercept 3

8. through $(3, 1)$ and parallel to $2x - y = -2$

9. through $(4, -12)$ and parallel to $4x + 3y = 12$

10. through $(-2, -3)$ and perpendicular to $3x - 5y = 1$

11. through $(-1, 2)$ and perpendicular to $\frac{1}{2}x + \frac{1}{3}y = 1$

12. with x-intercept 3 and y-intercept -5

13. the line $y = f(x)$, where f has the following values:

x	-2	2	4
$f(x)$	4	2	1

14. through $(4, -2)$ with x-intercept -3

In Exercises 15–18, determine whether the graph of the function is symmetric about the y-axis, the origin, or neither.

15. $y = x^{1/5}$ **16.** $y = x^{2/5}$

17. $y = x^2 - 2x - 1$ **18.** $y = e^{-x^2}$

In Exercises 19–26, determine whether the function is even, odd, or neither.

19. $y = x^2 + 1$ **20.** $y = x^5 - x^3 - x$

21. $y = 1 - \cos x$ **22.** $y = \sec x \tan x$

23. $y = \dfrac{x^4 + 1}{x^3 - 2x}$ **24.** $y = 1 - \sin x$

25. $y = x + \cos x$ **26.** $y = \sqrt{x^4 - 1}$

In Exercises 27–38, find the **(a)** domain and **(b)** range, and **(c)** graph the function.

27. $y = |x| - 2$ **28.** $y = -2 + \sqrt{1 - x}$

29. $y = \sqrt{16 - x^2}$ **30.** $y = 3^{2-x} + 1$

31. $y = 2e^{-x} - 3$ **32.** $y = \tan(2x - \pi)$

33. $y = 2\sin(3x + \pi) - 1$ **34.** $y = x^{2/5}$

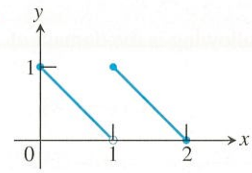
35. $y = \ln(x - 3) + 1$ **36.** $y = -1 + \sqrt[3]{2 - x}$

37. $y = \begin{cases} \sqrt{-x}, & -4 \le x \le 0 \\ \sqrt{x}, & 0 < x \le 4 \end{cases}$

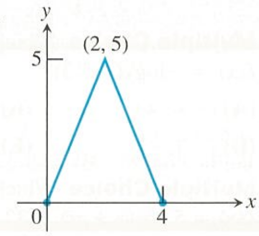
38. $y = \begin{cases} -x - 2, & -2 \le x \le -1 \\ x, & -1 < x \le 1 \\ -x + 2, & 1 < x \le 2 \end{cases}$

In Exercises 39 and 40, write a piecewise formula for the function.

39.



40.



In Exercises 41 and 42, find

(a) $(f \circ g)(-1)$ **(b)** $(g \circ f)(2)$ **(c)** $(f \circ f)(x)$ **(d)** $(g \circ g)(x)$

41. $f(x) = \dfrac{1}{x}$, $g(x) = \dfrac{1}{\sqrt{x + 2}}$

42. $f(x) = 2 - x$, $g(x) = \sqrt[3]{x + 1}$

In Exercises 43 and 44, **(a)** write a formula for $f \circ g$ and $g \circ f$ and find the **(b)** domain and **(c)** range of each.

43. $f(x) = 2 - x^2$, $g(x) = \sqrt{x + 2}$

44. $f(x) = \sqrt{x}$, $g(x) = \sqrt{1 - x}$

In Exercises 45–48, a parametrization is given for a curve.

(a) Graph the curve. Identify the initial and terminal points, if any. Indicate the direction in which the curve is traced.

(b) Find a Cartesian equation for a curve that contains the parametrized curve. What portion of the graph of the Cartesian equation is traced by the parametrized curve?

45. $x = 5\cos t$, $y = 2\sin t$, $0 \le t \le 2\pi$

46. $x = 4\cos t$, $y = 4\sin t$, $\pi/2 \le t < 3\pi/2$

47. $x = 2 - t$, $y = 11 - 2t$, $-2 \le t \le 4$

48. $x = 1 + t$, $y = \sqrt{4 - 2t}$, $t \le 2$

In Exercises 49–52, give a parametrization for the curve.

49. the line segment with endpoints $(-2, 5)$ and $(4, 3)$

50. the line through $(-3, -2)$ and $(4, -1)$

51. the ray with initial point $(2, 5)$ that passes through $(-1, 0)$

52. $y = x(x - 4)$, $x \le 2$

Group Activity In Exercises 53 and 54, do the following.

(a) Find f^{-1} and show that $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$.

(b) Graph f and f^{-1} in the same viewing window.

53. $f(x) = 2 - 3x$ **54.** $f(x) = (x + 2)^2$, $x \ge -2$

In Exercises 55 and 56, find the measure of the angle in radians and degrees.

55. $\sin^{-1}(0.6)$ **56.** $\tan^{-1}(-2.3)$

57. Find the six trigonometric values of $\theta = \cos^{-1}(3/7)$. Give exact answers.

58. Solve the equation $\sin x = -0.2$ in the following intervals.

(a) $0 \le x < 2\pi$ **(b)** $-\infty < x < \infty$

59. Solve for x: $e^{-0.2x} = 4$

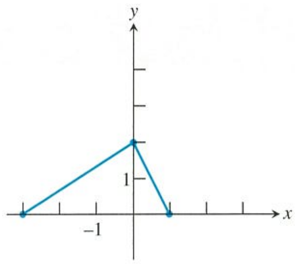
60. The graph of f is shown. Draw the graph of each function.

(a) $y = f(-x)$

(b) $y = -f(x)$

(c) $y = -2f(x + 1) + 1$

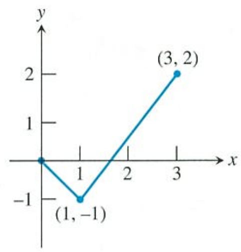
(d) $y = 3f(x - 2) - 2$



61. A portion of the graph of a function defined on $[-3, 3]$ is shown. Complete the graph assuming that the function is

(a) even.

(b) odd.



62. ***Depreciation*** Smith Hauling purchased an 18-wheel truck for \$100,000. The truck depreciates at the constant rate of \$10,000 per year for 10 years.

(a) Write an expression that gives the value y after x years.

(b) When is the value of the truck \$55,000?

63. ***Drug Absorption*** A drug is administered intravenously for pain. The function

$$f(t) = 90 - 52 \ln (1 + t), \quad 0 \le t \le 4$$

gives the number of units of the drug in the body after t hours.

(a) What was the initial number of units of the drug administered?

(b) How much is present after 2 hours? **(c)** Draw the graph of f.

64. ***Finding Time*** If Joenita invests \$1500 in a retirement account that earns 8% compounded annually, how long will it take this single payment to grow to \$5000?

65. ***Guppy Population*** The number of guppies in Susan's aquarium doubles every day. There are four guppies initially.

(a) Write the number of guppies as a function of time t.

(b) How many guppies were present after 4 days? after 1 week?

(c) When will there be 2000 guppies?

(d) **Writing to Learn** Give reasons why this might not be a good model for the growth of Susan's guppy population.

66. ***Doctoral Degrees*** Table 1.23 shows the number of doctoral degrees earned by Hispanic students for several years. Let $x = 0$ represent 1980, $x = 1$ represent 1981, and so forth.

Table 1.23 Doctorates Earned by Hispanic Americans

Year	Number of Degrees
1981	456
1985	677
1990	780
1995	984
2000	1305

Source: Statistical Abstract of the United States, 2004–2005.

(a) Find a linear regression equation for the data and superimpose its graph on a scatter plot of the data.

(b) Use the regression equation to predict the number of doctoral degrees that will be earned by Hispanic Americans in 2002. How close is the estimate to the actual number in 2002 of 1432?

(c) **Writing to Learn** Find the slope of the regression line. What does the slope represent?

67. ***Population of New York*** Table 1.24 shows the population of New York State for several years. Let $x = 0$ represent 1980, $x = 1$ represent 1981, and so forth.

Table 1.24 Population of New York State

Year	Population (thousands)
1980	17,558
1990	17,991
1995	18,524
1998	18,756
1999	18,883
2000	18,977

Source: Statistical Abstract of the United States, 2004–2005.

(a) Find the exponential regression equation for the data and superimpose its graph on a scatter plot of the data.

(b) Use the regression equation to predict the population in 2003. How close is the estimate to the actual number in 2003 of 19,190 thousand?

(c) Use the exponential regression equation to estimate the annual rate of growth of the population of New York State.

AP* Examination Preparation

You may use a graphing calculator to solve the following problems.

68. Consider the point $P(-2, 1)$ and the line L: $x + y = 2$.

(a) Find the slope of L.

(b) Write an equation for the line through P and parallel to L.

(c) Write an equation for the line through P and perpendicular to L.

(d) What is the x-intercept of L?

69. Let $f(x) = 1 - \ln(x - 2)$.

(a) What is the domain of f? **(b)** What is the range of f?

(c) What are the x-intercepts of the graph of f?

(d) Find f^{-1}. **(e)** Confirm your answer algebraically in part (d).

70. Let $f(x) = 1 - 3 \cos(2x)$.

(a) What is the domain of f? **(b)** What is the range of f?

(c) What is the period of f?

(d) Is f an even function, odd function, or neither?

(e) Find all the zeros of f in $\pi/2 \le x \le \pi$.

Chapter 2

Limits and Continuity

An Economic Injury Level (EIL) is a measurement of the fewest number of insect pests that will cause economic damage to a crop or forest. It has been estimated that monitoring pest populations and establishing EILs can reduce pesticide use by 30%–50%.

Accurate population estimates are crucial for determining EILs. A population density of one insect pest can be approximated by

$$D(t) = \frac{t^2}{90} + \frac{t}{3}$$

pests per plant, where t is the number of days since initial infestation. What is the rate of change of this population density when the population density is equal to the EIL of 20 pests per plant? Section 2.4 can help answer this question.