Q101.BC.NOTES: Chapter 2.3, 2.4, 3.1, 3.2

Define $\sqrt{x^2}$:

Definition of $|x|$:

PART 1: IS *f* CONTINUOUS AT *x* = *a*?

Definition of a function continuous at $x = a$:

Theorem:

1. Let $f(x) = \begin{cases} 5-2x; & x \geq 1 \\ x^2+1; & x < 1 \end{cases}$ Prove that *f* is or is not continuous **at** $x = 1$. Support graphically.

2. Let $g(x)=\begin{cases}\dfrac{|x-2|}{x-2}; & x\neq 2\\ 0; & x=2\end{cases}$. Prove that g is or is not continuous **at** $x=2$. Support graphically.

3. Let $p(x)=\begin{cases}\dfrac{x^2-1}{x+1}; & x\neq -1\\ 4; & x=-1\end{cases}$. Prove that p is or is not continuous **at** $x=-1$.

PART 2: DERIVATIVE DEFINITION

DERIVATIVE INTRODUCTION

DEFINITION DEVELOPMENT

Average rate of change of *f* on [a,b]
Slope of the secant line though curve *f* on [a, b]

Standard:

Alternate:

Instantaneous rate of change of *f* at $x = a$
Slope of the line tangent to the curve of *f* at $x = a$
Derivative of *f* at $x = a$

Standard:

Alternate:

1. Consider the function $y = f(x)$ where $f(x) = 3x^2 - 2$

a. Find the average rate of change in *f* on the interval [-1, 2]

b. Using the standard definition of a derivative at $x = a$ …

Find the instantaneous rate of change in *f* at $x = 1$. (*Precede your answer with Lagrange notation).*

c. Write an equation of the tangent line to the graph of $f(x)$ at $x = 1$.

2. Consider the function $y = f(x)$ where $f(x) = \sqrt{x+1}$
a. Find the average rate of change in *f* on the interval [0, 8]

b. Using the alternate definition of a derivative at $x = a$ …
Find the instantaneous rate of change in *f* at $x = 3$. (*Precede your answer with Lagrange notation).*

c. Write an equation of the tangent line to the graph of $f(x)$ at $x = 3$.

3. Consider the function $y = f(x)$ where $f(x) = \begin{cases} 5 - 2x; & x \geq 1 \\ 6 - 4x + x^2; & x < 1 \end{cases}$

a. Find the average rate of change in *f* on the interval [0, 2]

b. Using the standard definition of a derivative at $x = a$ …
Find the instantaneous rate of change in *f* at $x = 1$. (*Precede your answer with Lagrange notation).*

c. Write an equation of the tangent line to the graph of $f(x)$ at $x = 1$.

4. Consider the function $y = g(x)$ where $g(x) = \begin{cases} 5 - x; & x \geq 0 \\ 2x + 5; & x < 0 \end{cases}$

a. Find the average rate of change in g on the interval [-2, 5]

b. Using the standard definition of a derivative at $x = a$ …
Find the instantaneous rate of change in g at $x = 0$. *(Precede your answer with Lagrange notation).*

c. Write an equation of the tangent line to the graph of $g(x)$ at $x = 0$.

LESSON 1 – HW:

1. Prove that $f(x) = x^2 - 2x + 3$ is or is not continuous at $x = 1$.

2. Prove that $g(x) = \begin{cases} 2 + \sqrt{x}; & x > 0 \\ 1 + e^{-x}; & x \le 0 \end{cases}$ is or is not continuous at $x = 0$.

3. Prove that $r(x) = \begin{cases} x + 2; & x \ne -1 \\ 5; & x = -1 \end{cases}$ is or is not continuous at $x = -1$.

4. Prove that $f(x) = \begin{cases} x^2 + 2; & x \ge 0 \\ 1/x; & x < 0 \end{cases}$ is or is not continuous at $x = 0$.

5. A. Prove that $g(x) = \dfrac{x^2 - 9}{x - 3}$ is not continuous at $x = 3$.
   B. Extend $g(x)$, making it a piecewise function that is continuous at $x = 3$.

6. A. Prove that $d(x) = \dfrac{x - 4}{\sqrt{x} - 2}$ is not continuous at $x = 4$.
   B. Extend $d(x)$, making it a piecewise function that is continuous at $x = 4$.

7. Prove that $h(x) = \begin{cases} x \sin \dfrac{1}{x} & x \ne 0 \\ 0 & x = 0 \end{cases}$ is or is not continuous **at** $x = 0$.

Hint: See the Sandwich Theorem of Limits.

8. If $f(2)=3$ and $f^{/}(2)=5$, find an equation of (a) the tangent line, and (b) the normal line to the graph of $y=f(x)$ at the point where $x=2$.

9. Consider the function $y=f(x)$ where $f(x)=x^2-4x$

a. Find the average rate of change in f on the interval [-2, 4]

b. Find the slope of the tangent line to the graph of $f(x)$ at $x=1$, i.e. the derivative of *f* at $x = 1$.

c. Write an equation of the tangent line ℓ to the graph of $f(x)$ at $x=1$.

10. Consider the function $y=f(x)$ where $f(x)=\begin{cases}-x; & x<0\\ x^2-x; & x\geq 0\end{cases}$

a. Find the average rate of change in f on the interval [-2, 4]

b. Find the slope of the tangent line to the graph of $f(x)$ at $x=0$, i.e. the derivative of *f* at $x = 0$.

c. Write an equation of the tangent line ℓ to the graph of $f(x)$ at $x=0$.

11. Consider the function $y=f(x)$ where $f(x)=\begin{cases}-x^2; & x\geq -2\\ 2x; & x<-2\end{cases}$

a. Find the average rate of change in *f* on the interval [-4, 0]

b. Find the instantaneous rate of change in $f(x)$ at $x=-2$.

Classify $f(x)$ as either smooth, a corner, a cusp, a vertical line tangent, or discontinuous at $x=-2$.

c. Write an equation of the tangent line to the graph of $f(x)$ at $x=-2$.

12. Consider the function $g(x)=\frac{1}{x}$.

a. Find $g^{/}(2)$ using the standard definition of the derivative at $x = a$.

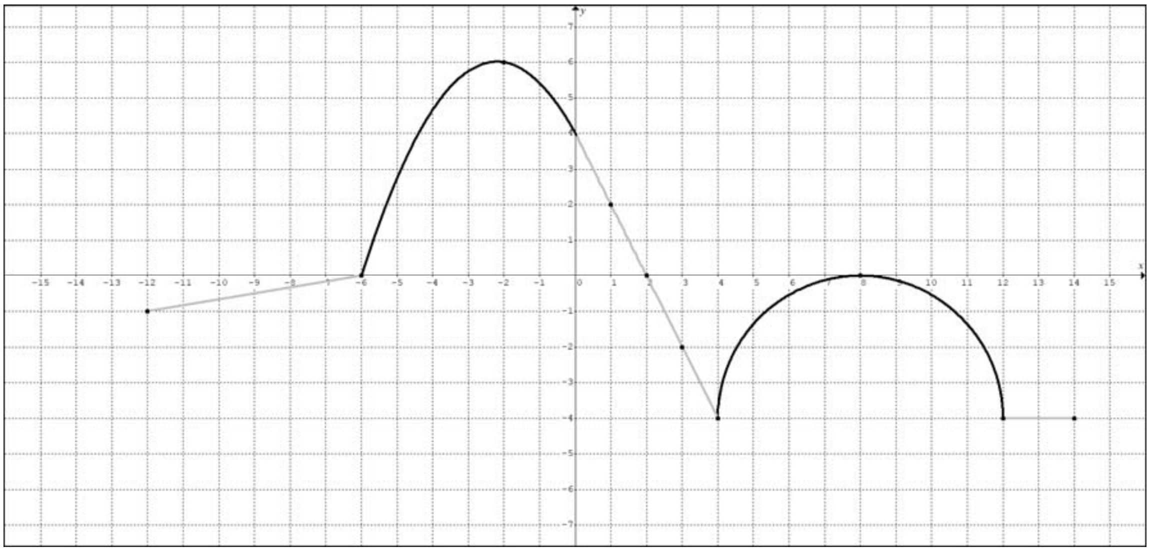
b. Find $g^{/}(2)$ using the alternate definition of the derivative at $x = a$.

BC: Q101 LESSON 2

Review Illustration (What does it mean for a function to be differentiable at $x = a$?)

PART 3: **DERIVATIVE ESTIMATION**

1. The function $y = f(x)$ is shown below. This function consists of line segments (in gray) and other non-linear curves (in black). Integer coordinates have been highlighted.



A) Find or estimate $f'(1)$

B) Find or estimate $f'(-1)$

C) Find or estimate $f'(4)$

D) Find or estimate $f'(0)$

E) Find or estimate $f'(8)$

F) Find or estimate $f'(10)$

G) Find or estimate $f'(-9)$

H) Find or estimate $f'(13)$

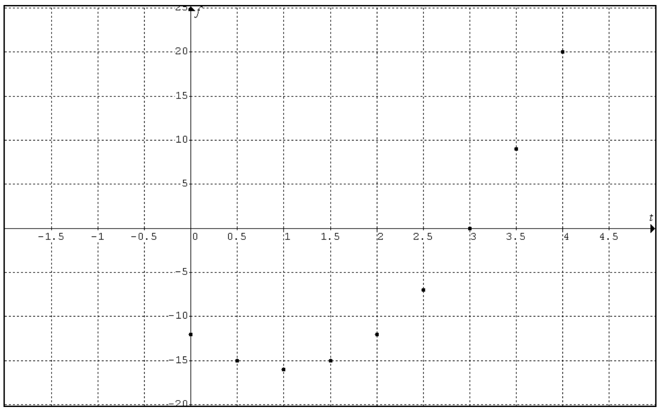
I) Find or estimate $f'(-12)$

J) Find or estimate all values of x where $f(x)$ has a horizontal tangent.

2. **Estimating the derivative with "an average on a small neighborhood."**
The coordinates of f for various values of x are given.

x	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
f	-12	-15	-16	-15	-12	-7	0	9	20

Assuming a smooth curve representation of $f(x)$.



A) Eyeball estimate $f'(1)$

B) Eyeball estimate $f'(2.5)$

C) Use a standard estimation of $f'(1)$. Show your work.

D) Use a standard estimation of $f'(2.5)$. Show your work.

PART 4: SPIRAL REVIW AND THE CONTINUITY AND DIFFERENTIBILITY CONNECTION

3. Consider the function $y = f(x)$ where $f(x) = x^{2/3}$

a. Find the average rate of change in *f* on the interval [-1, 1]

b. Using the alternate definition of a derivative at $x = a$ …
Find the instantaneous rate of change in *f* at $x = 0$. (*Precede your answer with Lagrange notation).*

c. Write an equation of the tangent line to the graph of $f(x)$ at $x = 0$.

4. Consider the function $p(x)=\begin{cases} 2x; & x\geq 1 \\ x+3; & x<1 \end{cases}$

A. Use the standard definition of a derivative at $x=a$ to prove that p is or is not differentiable at $x=1$. In other words, prove that the function p does or does not have a derivative at $x=1$.

B. Classify the behavior of $p(x)$ at $x=1$.

C. Provide a formal proof for whether or not $p(x)$ is continuous at $x=1$.

LOGIC:

THM:

PROOF:

PART 5 DERIVATIVE FUNCTION

DEFINTION: The derivative of the function f with respect to the variable x is the function $f^{/}$ whose value at x is $f^{/}(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$, provided it exists.

(NO ALTERNATE DEFINITION)

<u>DEF: Differentiable on an interval</u>

A function is differentiable on an open interval (a, b) if $f^{/}(x)$ exists for every x in that interval.

A function is differentiable on a closed interval $[a, b]$ if f is differentiable on the open interval (a, b) and if the following limits exist:

$f_{+}^{/}(a) = \lim_{h \to 0^{+}} \dfrac{f(a+h) - f(a)}{h}$ and $f_{-}^{/}(b) = \lim_{h \to 0^{-}} \dfrac{f(b+h) - f(b)}{h}$.

THM: Polynomial functions, unless restricted or broken, are differentiable for all values of x.

Definition: The derivative of the function f with respect to the variable x is the <u>function</u> $f'(x)$ whose value at x is $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$, provided it exists.

NO ALTERNATE DEFINITION FOR $f'(x)$

1. Consider the function $f(x) = 3x^2 - 2$.

A. Find $f'(x)$.

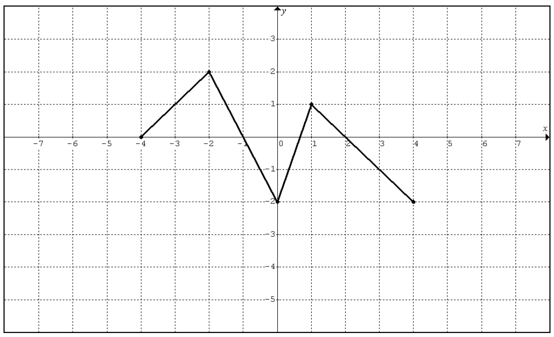
B. Write an equation of the tangent line to the graph of $f(x)$ at $x = -1$ and $x = 5$.

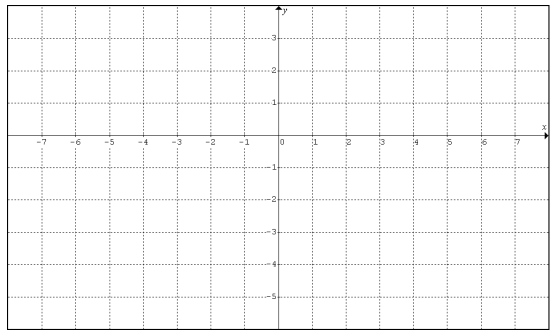
C. Write an equation of the normal to the graph of $f(x)$ at $x = -1$ and $x = 5$.

D. Find the points on the graph of f where the slope of the tangent line is parallel to $y = 4x + 5$.

2. **Graph** $f'(x)$ **from** $f(x)$

The graph of $y = g(x)$ shown here is made of line segments joined end to end. Graph the function's derivative.

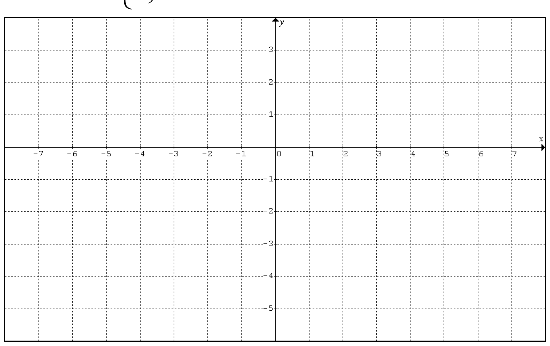




3. **Graph** $f(x)$ **from** $f'(x)$

Sketch a possible graph of a continuous function *f* that has domain [-3, 3], where $f(-1) = -2$ and the equation of $y = f'(x)$ is shown below.

$$f'(x) = \begin{cases} -1; & x < -1 \\ 0; & -1 < x < 2 \\ 3; & x > 2 \end{cases}$$



Property Problem

4. Let f be a function that is differentiable throughout its domain and that has the following properties.

(i) $f(x+y)=\dfrac{f(x)+f(y)}{1-f(x)f(y)}$ for all real numbers x, y, and $x+y$ in the domain of f.

(ii) $\lim\limits_{h\to 0} f(h)=0$ (iii) $\lim\limits_{h\to 0}\dfrac{f(h)}{h}=1$

(a) Show that $f(0)=0$.

(b) Use the definition of the derivative to show that $f'(x)=1+[f(x)]^2$. Indicate where properties (i), (ii), and (iii) are used.

LESSON 2 – HW:

1. Use the table below to estimate a) $f'(1.57)$ and b) $f'(3)$

t	0.00	0.56	0.92	1.19	1.30	1.39	1.57	1.74	1.98	2.18	2.41	2.64	3.24
$f(t)$	1577	1512	1448	1384	1319	1255	1191	1126	1062	998	933	869	805

2. Prove that $f(x) = \begin{cases} x; & x \le 1 \\ 1/x; & x > 1 \end{cases}$ is or is not differentiable at $x = 1$.

Classify $f(x)$ as either smooth, a corner, a cusp, a vertical line tangent, or discontinuous at $x = 1$.

3. Consider the function $b(x) = \begin{cases} x; & x > 1 \\ -2 + x; & x \le 1 \end{cases}$

a. Prove that *b* is or is not continuous at $x = 1$.

b. Using the standard definition of a derivative at $x = a$, prove that *f* is not differentiable $x = 1$.

c. Explain how you could have used part (a) to prove that *f* is not differentiable $x = 1$.

4. Using the definition for a derivative at $x = a$,

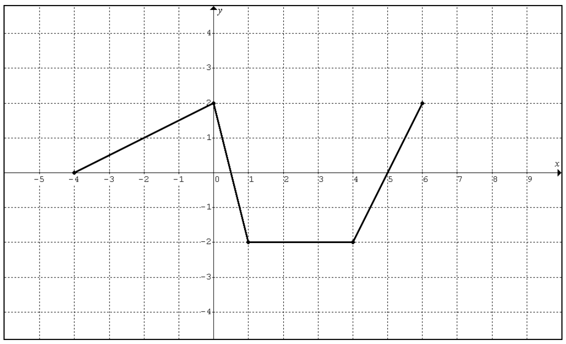
prove that $m(x) = \begin{cases} \sqrt{x}; & x \ge 1 \\ \frac{x}{2} + \frac{1}{2}; & x < 1 \end{cases}$ is or is not <u>differentiable</u> **at** $x = 1$.

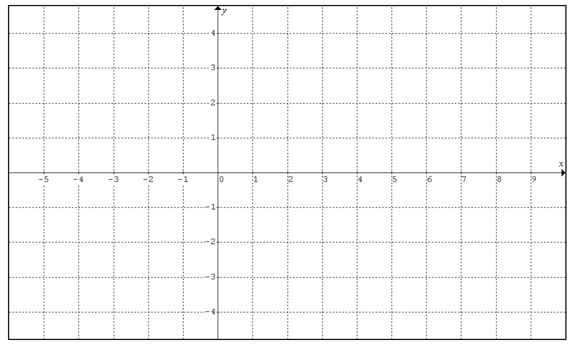
Classify $m(x)$ as either smooth, a corner, a cusp, a vertical line tangent, or discontinuous at $x = 1$.

5. Find the unique values of *a* and *b* that will make *g* both continuous and differentiable.

$$g(x) = \begin{cases} 3 - x; & x < 1 \\ ax^2 + bx; & x \ge 1 \end{cases}$$

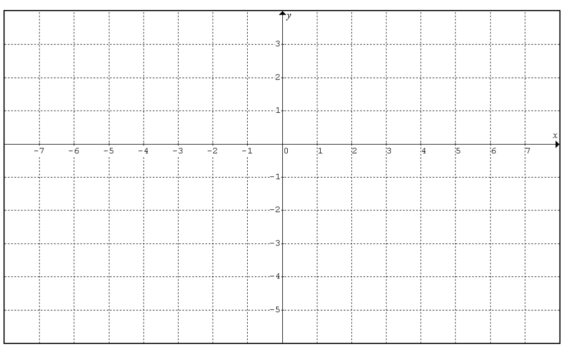
6. The graph of the function $y = f(x)$ shown here is made of line segments joined end to end. Graph $y = f'(x)$ and state its domain.





7. Sketch the graph of a continuous function with domain [-2,2], $f(0) = -1$, and

$$f'(x) = \begin{cases} 1; & x < -1 \\ -2; & x > -1 \end{cases}.$$



8. Using the information from problem 7, write an equation of the line tangent to *f* at $x = 0$.

9. Find $f'(x)$ for $f(x) = 5x^2 - 2x + 1$ using the appropriate definition.

10. Find the value of *x* for which the tangent to $f(x) = 5x^2 - 2x + 1$ is horizontal.

11. Find $f'(x)$ for $f(x) = \dfrac{1}{x+1}$ using the appropriate definition.

12. Find $f'(x)$ for $f(x) = \sqrt{2x+1}$ using the appropriate definition.

13. Suppose that a function f is differentiable at $x = 1$, $f(1) = 0$, and $\lim_{h \to 0} \frac{f(1+h)}{h} = 5$. Find $f^{/}(1)$.

14. Suppose that f is a differentiable function with the property that $f(x+y) = f(x) + f(y) + 5xy$ and $\lim_{h \to 0} \frac{f(h)}{h} = 3$. Find $f(0)$ and $f^{/}(x)$.

Definitions

Is the function continuous at $x = a$?

DEF: A function f is continuous at $x = a$ if and only if $\lim_{x\to a} f(x) = f(a)$.

What is the derivative of f at $x = a$?

DEF (Sandard): The derivative of the function f at the point $x = a$ is

$f^{/}(a) = \lim_{h\to 0} \dfrac{f(a+h) - f(a)}{h}$, provided it exists.

DEF (Alternate): The derivative of the function f at the point $x = a$ is

$f^{/}(a) = \lim_{x\to a} \dfrac{f(x) - f(a)}{x - a}$, provided it exists.

Is the function differentiable at $x = a$?

DEF (Std): A function f is differentiable at $x = a$ if $\lim_{h\to 0} \dfrac{f(a+h) - f(a)}{h}$ exists.

If this limit does not exists then we say that the function is not differentiable at $x = a$.

DEF (alt): A function f is differentiable at $x = a$ if $\lim_{x\to a} \dfrac{f(x) - f(a)}{x - a}$ exists.

If this limit does not exists then we say that the function is not differentiable at $x = a$.

DEF: A function f is differentiable at the left endpoint $x = a$ iff

$f_{+}^{/}(a) = \lim_{h\to 0^{+}} \dfrac{f(a+h) - f(a)}{h}$

DEF: A function f is differentiable at right endpoint $x = b$ iff

$f_{-}^{/}(b) = \lim_{h\to 0^{-}} \dfrac{f(b+h) - f(b)}{h}$.

DEF: A continuous function is a function that is continuous at each point in its domain.

DEF: A differentiable function is a function that is differentiable at each point in its domain.

DEF: A function is continuous on an interval I, if it is continuous at each point in the interval I.

DEF: A function is differentiable on an interval I, if f is differentiable at each point in the interval I.