Q501. Advanced Integration Techniques

Review from the Course
Integration by Parts
Integration by Partial Fractions (Non-Repeating Linear)
Integration by Trigonometric Substitution

Additions (New to the Unit)

I. Integration by Partial Fractions (Repeating and or Non-Linear)
II. Irreducible Quadratic Expression in the Integrand
III. Integration using reduction formulas (Proved with By-Parts)
IV. Trigonometric Integrals using Guidelines
V. Trigonometric Integrals using Guidelines
VI. Miscellaneous Substitution: Integral contains an expression of the $\sqrt[n]{f(x)}$)
VII. Miscellaneous Substitution: Integrand is a rational expression in $\sin x$ and $\cos x$)
VIII. Integration using a Table of Integrals (Table in Supplemental Worksheet)

Lesson 1:

I. (Partial Fraction Decomposition)
Homework:
BC.Q501.LESSON1.HOMEWORK WORKSHEET #16, 17, 21, 22, 23

Lesson 2:

II examples # 1, 2
III examples #1, 2
IV example #1
V example #1
Homework:
IV. Examples #2, 3
V. Example #2
BC.Q501.LESSON2.HOMEWORK WORKSHEET #1, 2, 7, 15-23odd, Extra

Lesson3:

VI examples #1,2
VII examples #1, 2
VIII #1, 2
Homework:
BC.Q501.LESSON3.HOMEWORK WORKSHEET #6, 9, 39, 47, 59, Extra

Review from the Course

Integration by Parts

Example 1: Evaluate $\int xe^{2x}dx.$

Example 2: Evaluate $\int x\sec^2 xdx$

Example 3: Evaluate $\int \ln xdx$

Example 4: Evaluate $\int \tan^{-1} x\, dx$

Example 5: Evaluate $\int x^2 e^{2x}dx.$

Example 6: Evaluate $\int e^x \cos xdx.$

Example 7: Evaluate $\int \sec^3 xdx.$

Integration using Partial Fractions

Example 1: Evaluate $\int \frac{2x+16}{x^2+x-6}dx$

Example 2: Evaluate $\int \frac{1-3x}{3x^2-5x+2}dx$

Example 3: Evaluate $\int \frac{5x+14}{x^2+7x}dx$

Example 4: Evaluate $\int \frac{x^2-6}{x^2-9}dx$

Integration using a Trigonometric Substitution

Expression in Integrand	Trigonometric Substitution
$\sqrt{a^2-x^2}$	$x = a\sin\theta$
$\sqrt{a^2+x^2}$	$x = a\tan\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$

Example 1: Evaluate $\int \frac{dx}{x^2\sqrt{16-x^2}}$

Example 2: Evaluate $\int \frac{dx}{\sqrt{4+x^2}}$

Example 3: Evaluate $\int \frac{\sqrt{x^2-9}dx}{x}$

I. Integration by Partial Fractions (Repeating and or Non-Linear)

Guidelines for Partial Fraction Decompositions of $f(x)/g(x)$

1. If the degree of f(x) is not lower than the degree of g(x), use long division to obtain the proper form.

2. Express g(x) as a product of linear factors $ax+b$ or irreducible quadratic factors ax^2+bx+c, and collect repeated factors so that g(x) is a product of different factors of the form $(ax+b)^n$ or $(ax^2+bx+c)^n$ for a nonnegative integer n.

3. Apply the following rules.

   Rule a: For each factor $(ax+b)^n$ with $n \geq 1$, the partial fraction decomposition contains the sum of n partial fractions of the form

$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\cdots+\frac{A_n}{(ax+b)^n}$, where each numerator A_k is a real number.

   Rule b: For each factor $(ax^2+bx+c)^n$ with $n \geq 1$ and with ax^2+bx+c irreducible, the partial fraction decomposition contains a sum of n partial fractions of the form

$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_nx+B_n}{(ax^2+bx+c)^n}$, where each A_k and B_k is a real number.

Fraction Decomposition Examples: Decompose each fraction showing the appropriate denominators and appropriate numerators (here using A B C, etc.)

1. $\dfrac{4x^2 + 13x - 9}{x^3 + 2x^2 - 3x}$

2. $\dfrac{3x^3 - 18x^2 + 29x - 4}{(x+1)(x-2)^3}$

3. $\dfrac{x^2 - x - 21}{2x^3 - x^2 + 8x - 4}$

4. $\dfrac{5x^3 - 3x^2 + 7x - 3}{(x^2 + 1)^2}$

5. Evaluate $\int \frac{6x+7}{(x+2)^2}dx$

II. Irreducible Quadratic Expression in the Integrand

Example 1: Evaluate $\int \frac{2x-1}{x^2-6x+13}\,dx$

Example 2: Evaluate $\int \frac{1}{\sqrt{x^2+8x+25}}\,dx$

III. Integration using reduction formulas (Proved with By-Parts)

Example 1: Find a reduction formula for $\int \sin^n x\,dx$

Example 2: Use the reduction formula in Example 8 to evaluate $\int \sin^5 x\,dx$

Reduction Formulas:

IV. Trigonometric Integrals $\int \sin^m x \cos^n x\, dx$

Guidelines:

Guidelines for Evaluating $\int \sin^m x \cos^n x\, dx$ 7.2

1 **If m is an odd integer:** Write the integral as

$$\int \sin^m x \cos^n x\, dx = \int \sin^{m-1} x \cos^n x \sin x\, dx$$

and express $\sin^{m-1} x$ in terms of $\cos x$ by using the trigonometric identity $\sin^2 x = 1 - \cos^2 x$. Make the substitution

$$u = \cos x, \quad du = -\sin x\, dx$$

and evaluate the resulting integral.

2 **If n is an odd integer:** Write the integral as

$$\int \sin^m x \cos^n x\, dx = \int \sin^m x \cos^{n-1} x \cos x\, dx$$

and express $\cos^{n-1} x$ in terms of $\sin x$ by using the trigonometric identity $\cos^2 x = 1 - \sin^2 x$. Make the substitution

$$u = \sin x, \qquad du = \cos x\, dx$$

and evaluate the resulting integral.

3 **If m and n are even:** Use half-angle formulas for $\sin^2 x$ and $\cos^2 x$ to reduce the exponents by one-half.

Example 1: Evaluate $\int \sin^5 x\, dx$ without using "by parts."

Example 2: Evaluate $\int \cos^2 x\, dx$ without using "by parts."

Example 3: Evaluate $\int \cos^3 x \sin^4 x\, dx$

V. Trigonometric Integrals $\int \tan^m x \sec^n x \, dx$

Guidelines:

Guidelines for Evaluating $\int \tan^m x \sec^n x \, dx$ **7.3**

1 **If m is an odd integer:** Write the integral as

$$\int \tan^m x \sec^n x \, dx = \int \tan^{m-1} x \sec^{n-1} x \sec x \tan x \, dx$$

and express $\tan^{m-1} x$ in terms of $\sec x$ by using the trigonometric identity $\tan^2 = \sec^2 x - 1$. Make the substitution

$$u = \sec x, \qquad du = \sec x \tan x \, dx$$

and evaluate the resulting integral.

2 **If n is an even integer:** Write the integral as

$$\int \tan^m x \sec^n x \, dx = \int \tan^m x \sec^{n-2} x \sec^2 x \, dx$$

and express $\sec^{n-2} x$ in terms of $\tan x$ by using the trigonometric identity $\sec^2 x = 1 + \tan^2 x$. Make the substitution

$$u = \tan x, \qquad du = \sec^2 x \, dx$$

and evaluate the resulting integral.

3 **If m is even and n is odd:** There is no standard method of evaluation. Possibly use integration by parts.

Example 1: Evaluate $\int \tan^3 x \sec^5 x dx$.

Example 2: Evaluate $\int \tan^2 x \sec^4 x dx$.

VI. Miscellaneous Substitutions (Integral contains an expression of the $\sqrt[n]{f(x)}$)

Example 1: Evaluate $\int \frac{x^3}{\sqrt[3]{x^2+4}}dx$

Example 2: Evaluate $\int \frac{1}{\sqrt{x}+\sqrt[3]{x}}dx$

VII. Miscellaneous Substitution (Integrand is a rational expression in $\sin x$ and $\cos x$)

Example 1: Evaluate $\int \frac{1}{4\sin x - 3\cos x}\,dx$

Example 2: Evaluate $\int \frac{\cos x}{1+\sin^2 x}\,dx$

VIII. Integration using Table of Integration

Example 1: Evaluate $\int \frac{1}{x^2\sqrt{3+5x^2}}\,dx$ for $x>0$ using the brief table of integration.

Example 2: Evaluate $\int x^3 \cos x\, dx$ using the brief table of integration.

BC.Q501.LESSON1.HOMEWORK WORKSHEET

Evaluate each integral.

16. $\int \frac{x^2\,dx}{(x-1)(x^2+2x+1)}$

17. $\int \frac{dx}{(x^2-1)^2}$

21. $\int \frac{x^2-2x-2}{x^3-1}\,dx$

22. $\int \frac{x^2-4x+4}{x^3+1}\,dx$

23. $\int \frac{3x^2-2x+12}{(x^2+4)^2}\,dx$

BC.Q501.LESSON2.HOMEWORK WORKSHEET

1. DO NOT EVALUATE. Determine which approach (u-substitution, integration by parts, partial fractions, reduction formulas, and trigonometric substitution) you would try first to evaluate.

A. $\int x \sin x dx$

B. $\int \cos x \sin x dx$

C. $\int \tan^7 x \, dx$

D. $\int \tan^7 x \sec^2 x dx$

E. $\int \frac{3x^2}{x^3+1} dx$

F. $\int \frac{3x^2}{(x+1)^3} dx$

G. $\int \tan^{-1} x \, dx$

H. $\int \sqrt{4-x^2} dx$

I. $\int x\sqrt{4-x^2} dx$

2. DO NOT EVALUATE. Determine the appropriate trigonometric substitution needed to evaluate each.

A. $\int \sqrt{9+x^2} dx$

B. $\int \sqrt{9-x^2} dx$

C. $\int \sqrt{1-9x^2} dx$

D. $\int \sqrt{x^2-9} dx$

E. $\int \sqrt{9+3x^2} dx$

F. $\int \sqrt{1+(9x)^2}\,dx$

7. In each part, evaluate the integral by making an appropriate substitution and applying a reduction formula.

A. $\int \sin^4 2x dx$

B. $\int x\cos^5\left(x^2\right)dx$

15-23 ODD --- Evaluate Each:

15. $\int \sqrt{\cos\theta}\sin\theta\, d\theta$

17. $\int x\tan^2\left(x^2\right)\sec^2\left(x^2\right)dx$

19. $\int \frac{dx}{\left(3+x^2\right)^{3/2}}$

21. $\int \frac{x+3}{\sqrt{x^2+2x+2}}dx$

23. $\int \frac{dx}{(x-1)(x+2)(x-3)}$

Extra. Evaluate: $\int 1/\left(x^2-4x+8\right)dx$

BC.Q501.LESSON3.HOMEWORK WORKSHEET

6. Evaluate the integral $\int_0^1 \frac{x^3}{\sqrt{x^2+1}}dx$ using …

A. Integration by Parts

B. The substitution $u = \sqrt{x^2+1}$

9. Evaluate the integral $\int \frac{1}{\sqrt{2x-x^2}}dx$ in each of the three ways …

A. $u = \sqrt{x}$ B. $u = \sqrt{2-x}$ C. Completing the Square

39. Make a substitution to express the integrand as a rational function and then evaluate the integral.

$$\int \frac{1}{x\sqrt{x+1}}dx$$

47. Make a substitution to express the integrand as a rational function and then evaluate the integral.

$$\int \frac{e^{2x}}{e^{2x}+3e^x+2}dx$$

59. Use the substitution $\theta = \tan(x/2)$ to evaluate the integral.

$$\int \frac{1}{3\sin x - 4\cos x}dx$$

Extra: Evaluate the integral.

$$\int \frac{1}{x^3-1}dx$$