# BC: Q402.CH9B – Power Series: Writing Power Series (LESSON 1)

Power Series:

An expression of the form

$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$ is a power series centered at $x = 0$.

An expression of the form

$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots + c_n (x-a)^n + \cdots$ is a power series centered at $x = a$.

## I. Write a Power Series:

A. **GEOMETRIC SUM:** Using properties of a Geometric Series Sum
B. **RAW CONSTRUCTION**: Using Raw (*Maclaurin*) Construction – *compile a set of memorized Maclaurin Series*
C. **MANIPULATE KNOWN**: Using Manipulation (integral or derivative) of a known power series
D. **SUBSTITUTE KNOWN**: Using Substitution into a known power series.

## II. Write an nth order polynomial:

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Definition: Taylor Series Generated by *f* at $x = a$

Let *f* be a function with derivatives of all orders throughout some open interval containing *a*. Then the Taylor series generated by *f* at $x = a$ is

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

The partial sum $P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$ is the Taylor polynomial of order *n* for *f* at $x = a$.

Definition: Taylor Series Generated by *f* at $x = 0$ (Maclaurin Series)

Let *f* be a function with derivatives of all orders throughout some open interval containing 0. Then the Taylor series generated by f at $x = 0$ is

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \ldots = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k.$$

This series is also called the Maclaurin Series generated by *f* at $x = 0$.

The partial sum $P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k$ is the Taylor polynomial of order *n* for *f* at $x = 0$.

# I. Write a Power Series:

**A.GEOMETRIC SUM:** Using properties of a Geometric Series Sum
Give a power series representation of …

# I. Write a Power Series:

**B. RAW CONSTRUCTION**: Using Raw (*Maclaurin*) Construction

$$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \cdots + \frac{f^n(0)x^n}{n!} + \cdots$$

Give a power series representation of …

## I. Write a Power Series:

**C. MANIPULATE KNOWN**: Using Manipulation (integral or derivative) of a known
Give a power series representation of …

# I. Write a Power Series:

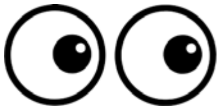
**D. SUBSTITUTE KNOWN**: Using Substitution into a known power series.
Give a power series representation of …

## II. Write an nth order polynomial:

Give a $7^{th}$ order Taylor Polynomial for $\sin x$

Give a $6^{th}$ order Taylor Polynomial for $\cos x$

**HW:**
**PG. 483: #55, 57, 59, 72, 63, 64**
**PG. 492: #3, 5, 7, 10, 12, 22, 24, 27**



**BC: Q402 – CH9B: Lesson 1 continued - A Closer Look**

1. Find a power series for $f(x) = \ln(1+x)$ centered at $x = 0$.

2. Suppose $g'(x) = \dfrac{1}{1+x}$ with $g(0) = 5$ and domain $-1 < x < 1$. Find a power series for the function $g(x)$ centered at $x = 0$.

Connection: In #1 the particular function was given already: $f(x) = \ln(1+x)$ with $f(0) = 0$.
In #2 the particular function was not given, so we needed a condition to find it.

**BC: Q402 – CH9B LESSON2**
Memorize | Substitute into Memorized Maclaurin | Raw Construction

Taylor Series:

1. $f(x) = \cos(x)$ Natural Center:

A. Write a fourth order Taylor polynomial for $f(x) = \cos(x)$ centered at $x = 0$

B. Write a third order Taylor polynomial for $f(x) = \cos(x)$ centered at $x = \frac{\pi}{4}$

2. $f(x) = \cos\left(x - \frac{\pi}{4}\right)$ Natural Center:

A. Write a second order Taylor polynomial for $f(x) = \cos\left(x - \frac{\pi}{4}\right)$ centered at $x = 0$

B. Write a fourth order Taylor polynomial for $f(x) = \cos\left(x - \frac{\pi}{4}\right)$ centered at $x = \frac{\pi}{4}$

3. $f(x) = e^{5x}$ Natural Center:

A. Write a third order Taylor polynomial for $f(x) = e^{5x}$ centered at $x = 0$

B. Write a third order Taylor polynomial for $f(x) = e^{5x}$ centered at $x = 1$

4. $f(x) = e^{(x-2)}$ Natural Center:

A. Write a third order Taylor polynomial for $f(x) = e^{(x-2)}$ centered at $x = 0$

B. Write a third order Taylor polynomial for $f(x) = e^{(x-2)}$ centered at $x = 2$

HW

1. Let $P(x) = 7 - 3(x-4) + 5(x-4)^2 - 2(x-4)^3 + 6(x-4)^4$ be the fourth-degree Taylor polynomial for the function *f* about 4. Assume *f* has derivatives of all orders for all real numbers.

(a) Find $f(4)$ and $f'''(4)$.

(b) Write the second-degree Taylor polynomial for $f'$ about 4 and use it to approximate $f'(4.3)$

(c) Write the fourth-degree Taylor polynomial for $g(x) = \int_4^x f(t)dt$ about 4.

2. Let *f* be a function that has derivatives of all orders for all real numbers. Assume $f(0) = 5, f'(0) = -3, f''(0) = 1, f'''(0) = 4.$

(a) Write the third-degree Taylor polynomial for *f* about $x = 0$ and use it to approximate *f*(0.2).

(b) Write the fourth-degree Taylor polynomial for *g*, where $g(x) = f(x^2)$, about $x = 0$.

(c) Write the third-degree polynomial for *h*, where $h(x) = \int_0^x f(t)dt$, about $x = 0$.

(d) Suppose $r'(x) = f(x)$ with $r(0) = 10$. Write a third-degree polynomial for *r* about $x = 0$.

3. The Maclaurin series $f(x)$ is given by $1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \cdots + \frac{x^n}{(n+1)!} + \cdots$

(a) Find $f'(0)$ and $f^{(17)}(0)$.

(b) Let $g(x) = xf(x)$. Write the Maclaurin series for g(x) , showing the first three nonzero terms and the general term.

(c) Write *g*(x) in terms of a familiar function without using series. Then, write *f*(x) in terms of the same familiar function.

(d) For what values of *x* does the given series for *f*(*x*) converge? Show your reasoning.

**Textbook: Pg. 492 # 4, 14, 19, 21, 23, 33, and 34**

# BC: Q402.CH9B – Taylor Series: Error Analysis (LESSON 3)

THM **E1**: Alternating Series Bound

If a series for $f(x_0)$ is strictly alternating and decreasing in absolute value to zero, the error in using $P_n(x_0)$ to approximate $f(x_0)$ is less than or equal to the first omitted non-zero term:

$$P_n(x_0) \approx f(x_0) = a_0 - a_1 + a_2 - a_3 + \cdots + (-1)^n a_n + \cdots \text{with } |error| = |R_n(x_0)| \le |a_{n+1}|$$

THM **E2**: Taylor's Formula with Remainder

The error in using $P_n(x_0)$ to approximate $f(x_0)$ is equal to $R_n(x_0) = \dfrac{f^{(n+1)}(c)(x_0 - a)^{n+1}}{(n+1)!}$ for some value *c* where *c* is between the center *a* and the value of $x_0$ (inclusive).

$$P_n(x_0) \approx f(x_0) \text{ with } |error| = |R_n(x_0)| = \left| \frac{f^{(n+1)}(c)(x_0 - a)^{n+1}}{(n+1)!} \right| \text{ for } \begin{cases} a \le c \le x_0 & if \quad a < x_0 \\ x_0 \le c \le a & if \quad x_0 < a \end{cases}$$

Consequently: $$|error| = |R_n(x_0)| \le \left| \frac{\left[ \underset{MAX}{f^{(n+1)}(c)} \right](x_0 - a)^{n+1}}{(n+1)!} \right| \text{ for } \begin{cases} a \le c \le x_0 & if \quad a < x_0 \\ x_0 \le c \le a & if \quad x_0 < a \end{cases}$$

**ERROR ANALYSIS (E1: Alternating Series Bound)**

1. $f(x) = \cos x$

A. Use a $5^{\text{th}}$ order Taylor polynomial centered at $x = 0$ to estimate $f(0.5)$.

B. Show that the estimate in part A differs from $f(0.5)$ by no more than $\dfrac{1}{2^6 6!}$.

2. Approximate $\int_0^1 \sin(x^2)dx$ to four decimal places. Estimate the error.

3. Let $f$ be the function defined by $f(x) = \frac{1}{x-1}$.

(a) Write the first four terms and the general term of the Taylor series expansion of $f(x)$ about $x = 2$.

(b) Use the result from part (a) to find the first four terms and the general term of the series expansion about $x = 2$ for $\ln|x-1|$.

(c) Use the series in part (b) to compute a number that differs from $\ln\frac{3}{2}$ by less than 0.05. Justify your answer.

4. The Taylor series about $x = 5$ for a certain function $f$ converges to $f(x)$ for all $x$ in the interval of convergence. The $n$th derivative of $f$ at $x = 5$ is given by $f^{(n)}(5) = \dfrac{(-1)^n n!}{2^n (n+2)}$, and $f(5) = \dfrac{1}{2}$.

(a) Write the third-degree Taylor polynomial for $f$ about $x = 5$.
(b) Find the radius of convergence of the Taylor series for $f$ about $x = 5$.
(c) Show that the sixth-degree Taylor polynomial for $f$ about $x = 5$ approximates $f(6)$ with error less than $\dfrac{1}{1000}$.

5. The function *f* is defined by the power series $f(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n+1)!}$ for all real numbers.

(a). Find $f'(0)$ and $f''(0)$. Determine whether *f* has a local maximum, a local minimum, or neither at $x=0$. Give a reason for your answer.

(b). Show that $1-\frac{1}{3!}$ approximates $f(1)$ with error less than $\frac{1}{100}$.

(c) Show that $y=f(x)$ is a solution to the differential equation $xy'+y=\cos x$.

**ERROR ANALYSIS (E2: Lagrange Error Bound)**

6. $f(x) = \cos x$

A. Use a $5^{\text{th}}$ order Taylor polynomial centered at $x = 0$ to estimate $f(0.5)$.

B. Show that the estimate in part A differs from $f(0.5)$ by no more than $\frac{1}{2^6 6!}$.

7. $f(x) = \cos(2x)$

A. Use a 2nd order Taylor polynomial centered at $x = \frac{\pi}{3}$ to estimate $f\left(\frac{5\pi}{24}\right)$.

B. Show that $\left|f\left(\frac{5\pi}{24}\right) - P_2\left(\frac{5\pi}{24}\right)\right| < \frac{\pi^3}{380}$.

8. $f(x)=\sqrt{1+x}$

A. Use a 1st order Taylor polynomial centered at $x=0$ to estimate $f(0.2)$.

B. Show that $\left|R_1(0.2)\right| \leq .005$

9. $f(x) = e^x$

A. Use a 3rd order Taylor polynomial centered at $x = 0$ to estimate $f(1)$.

B. Show that $\left| f(1) - P_3(1) \right| < \frac{1}{8}$.

**CALCULATOR ACTIVE**

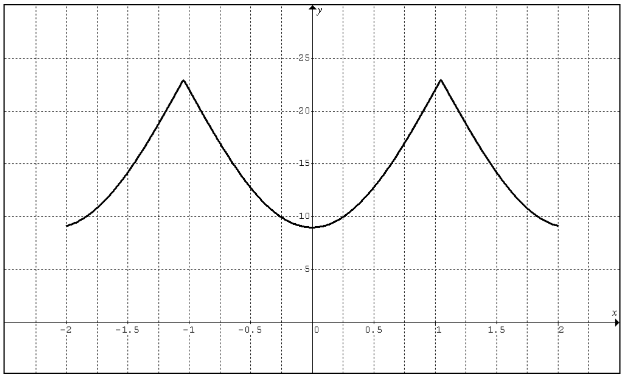
| $x$ | $h(x)$ | $h'(x)$ | $h''(x)$ | $h'''(x)$ | $h^{(4)}(x)$ |
|---|---|---|---|---|---|
| 1 | 11 | 30 | 42 | 99 | 18 |
| 2 | 80 | 128 | $\frac{488}{3}$ | $\frac{448}{3}$ | $\frac{584}{9}$ |
| 3 | 317 | $\frac{753}{2}$ | $\frac{1383}{4}$ | $\frac{3483}{16}$ | $\frac{1125}{16}$ |

10. Let $h$ be a function having derivatives of all orders for $x > 0$. Select values of $h$ and its first four derivatives are indicated in the table above. The function $h$ and these four derivatives are increasing on the interval $1 \le x \le 3$.

A. Write the first-degree Taylor polynomial for $h$ about $x = 2$ and use it to approximate $h(1.9)$. Is this approximation greater than or less than $h(1.9)$? Explain your reasoning.

B. Write the third-degree Taylor polynomial for $h$ about $x = 2$ and use it to approximate $h(1.9)$.

C. Use the Lagrange error bound to show that the third-degree Taylor polynomial for $h$ about $x = 2$ approximates $h(1.9)$ with error less than $3 \times 10^{-4}$.




11. Consider a function $f(x)$ which has non-zero real derivatives of all orders. A graph of $\left|f'''(x)\right|$ on (-2,2) is show above. Show that $\left|f(0.5) - P_2(0.5)\right| < \frac{3}{8}$ where $P_2(x)$ is a Taylor polynomial of second degree centered at zero.

12. Use the first two nonzero terms of the Maclaurin series to approximate sin($x$). Estimate the maximum error if $|x|<1$.