BC: Q401.CH9A – Convergent and Divergent Series (LESSON 1)

INTRODUCTION

Sequence Notation: $a_1, a_2, a_3, \cdots, a_n, \cdots$

Definition: A **sequence** is a function *f* whose domain is the set of positive integers.

Definition:
An **infinite series** (or simply a **series**) is an expression of the form

$$a_1 + a_2 + \cdots + a_n + \cdots = \sum_{k=1}^{\infty} a_k .$$

Each number a_k is a term of the series, and a_n is the ***n*th term**.

POSITIVE TERM SERIES: Observations Test for Convergence or Divergence

Theorem***: n*th-Term test**

(i) If $\lim_{n\to\infty} a_n \neq 0,$ then the series $\sum a_n$ is divergent.

(ii) If $\lim_{n\to\infty} a_n = 0,$ then further investigation is necessary to determine whether the series $\sum a_n$ is convergent or divergent.

Illustration:

Series	nth-Term Test	Conclusion
$\sum_{n=1}^{\infty} \frac{n}{2n+1}$	$\lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2} \neq 0$	Diverges, by ntt
$\sum_{n=1}^{\infty} \frac{1}{n^2}$	$\lim_{n\to\infty} \frac{1}{n^2} = 0$	Further investigation is necessary, by ntt
$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$	$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$	Further investigation is necessary, by ntt
$\sum_{n=1}^{\infty} \frac{e^n}{n}$	$\lim_{n\to\infty} \frac{e^n}{n} = \infty$	Diverges, by ntt

Theorem: **Geometric Series Test**

Let $a \neq 0$. The geometric series $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$

(i) converges and thus has a sum $S = \frac{a}{1-r}$ if $|r| < 1$

(ii) diverges if $|r| \geq 1$

Definition: A ***p*-series** is a series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$

, where *p* is a positive real number.

Theorem: ***p*-series Test**

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ (i) converges if $p > 1$ (ii) diverges if $p \leq 1$

POSITIVE TERM SERIES: Formal Tests for Convergence or Divergence

(These tests will not give the sum S of the series, but instead will tell us whether a sum exists)

INTEGRAL TEST for convergence *(Lesson1)*

If $\sum a_n$ is a series, let $f(n) = a_n$ and let *f* be a function obtained by replacing *n* with *x*. If *f* is a positive-valued, continuous, and decreasing for every real number $x \geq 1$, then the series $\sum a_n$

€ converges if $\int_1^{\infty} f(x)dx$ converges

€ diverges if $\int_1^{\infty} f(x)dx$ diverges

DIRECT COMPARISON TEST (Basic Comparison Test) for convergence *(Lesson1)*

Let $\sum a_n$ and $\sum b_n$ be positive-term series.

€ If $\sum b_n$ converges and $a_n \leq b_n$ for every positive integer *n*, then $\sum a_n$ converges.

€ If $\sum b_n$ diverges and $a_n \geq b_n$ for every positive integer *n*, then $\sum a_n$ diverges.

LIMIT COMPARISON TEST for convergence *(Lesson1)*

Let $\sum a_n$ and $\sum b_n$ be positive-term series. If there is a positive real number *c* such that $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then either both series converge or both series diverge.

RATIO TEST for converges *(Lesson2)*

Let $\sum a_n$ be a positive-term series, and suppose that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$.

€ If L < 1, the series is convergent.

€ If L > 1, or $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \infty$, the series is divergent.

€ If L = 1, apply a different test; the series may be convergent or divergent.

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a_n	**Deleting terms of least magnitude**	**Choice of b_n**
$\frac{3n+1}{4n^3+n^2-2}$	$\frac{3n}{4n^3} = \frac{3}{4n^2}$	$\frac{1}{n^2}$
$\frac{5}{\sqrt{n^2+2n+7}}$	$\frac{5}{\sqrt{n^2}} = \frac{5}{n}$	$\frac{1}{n}$
$\frac{\sqrt[3]{n^2+4}}{6n^2-n-1}$	$\frac{\sqrt[3]{n^2}}{6n^2} = \frac{n^{2/3}}{6n^2} = \frac{1}{6n^{4/3}}$	$\frac{1}{n^{4/3}}$

NOTES I: Determine by observation whether the following series converge or diverge. Justify your answer.

A. $\sum_{n=1}^{\infty} 2(0.1)^{n-1}$

B. $\sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{5}{2}\right)^{n-1}$

C. $\sum_{n=1}^{\infty} \frac{n}{2n+1}$

D. $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$

E. $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n}$

F. $\sum_{n=1}^{\infty} \frac{1}{n^2}$

Notes II.

1. Determine whether **the harmonic series** $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}+\cdots$ converges or diverges.

2: Determine whether the infinite series $\sum_{1}^{\infty} ne^{-n^2}$ converges or diverges.

3: Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{2+5^n}$ converges or diverges using the DCT

4: Determine whether the series $\sum_{n=2}^{\infty} \frac{3}{\sqrt{n}-1}$ converges or diverges using the DCT

5: Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2+1}}$ converges or diverges using the LCT.

6: Determine whether the series $\sum_{n=1}^{\infty} \frac{8n+\sqrt{n}}{5+n^2+n^{7/2}}$ converges or diverges using the LCT.

Let $a_n = \frac{8n+\sqrt{n}}{5+n^2+n^{7/2}}$

CLASS EXPLORATION

Definite Integral versus Sum

Lesson 1 - Homework

Formal Testing

1. Use the **Integral Test** to determine if $\sum_{n=1}^{\infty} \frac{5}{n+1}$ converges or diverges. Pg. 523 #7

2. Use the **Integral Test** to determine if $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ converges or diverges. Pg. 523 #10

3. Use the **DCT** to determine if $\sum_{n=3}^{\infty} \frac{\ln n}{n}$ converges or diverges. Pg. 523 #9

4. Use the **DCT** to determine if $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$ converges or diverges. Pg. 523 #15

5. Use the **LCT** to determine if $\sum_{n=1}^{\infty} \frac{5n^3-3n}{n^2(n+2)(n^2+5)}$ converges or diverges. Pg. 523 #16

Observational Testing: nth term test/p-series/geometric series

6. Determine if $\sum_{n=1}^{\infty} \frac{n}{n+1}$ converges or diverges. Justify. Pg. 511 #29

7. Determine if $\sum_{n=1}^{\infty} \frac{1}{8^n}$ converges or diverges. Justify. Pg. 511 #32

8. Determine if $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n$ converges or diverges. Justify. Pg. 511 #38

9. Determine if $\sum_{n=1}^{\infty} \frac{(2)^n}{3^n}$ converges or diverges. Justify. Pg. 511 #39

10. Determine if $\sum_{n=1}^{\infty} \frac{3}{\sqrt{n}}$ converges or diverges. Justify. Pg. 523 #8

11. Determine if $\sum_{n=1}^{\infty} \frac{1}{(\ln 2)^n}$ converges or diverges. Justify. Pg. 523 #11

12. Determine if $\sum_{n=1}^{\infty} \frac{5}{n+1}$ converges or diverges. Justify. Pg. 523 #7 (yes you did this already)

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POSITIVE TERM SERIES: Formal Tests for Convergence or Divergence

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INTEGRAL TEST for convergence *(Lesson1)*

If $\sum a_n$ is a series, let $f(n) = a_n$ and let *f* be a function obtained by replacing *n* with *x*. If *f* is a positive-valued, continuous, and decreasing for every real number $x \geq 1$, then the series $\sum a_n$

- converges if $\int_1^\infty f(x)dx$ converges
- diverges if $\int_1^\infty f(x)dx$ diverges

DIRECT COMPARISON TEST (Basic Comparison Test) for convergence *(Lesson1)*

Let $\sum a_n$ and $\sum b_n$ be positive-term series.

- If $\sum b_n$ converges and $a_n \leq b_n$ for every positive integer *n*, then $\sum a_n$ converges.
- If $\sum b_n$ diverges and $a_n \geq b_n$ for every positive integer *n*, then $\sum a_n$ diverges.

LIMIT COMPARISON TEST for convergence *(Lesson1)*

Let $\sum a_n$ and $\sum b_n$ be positive-term series. If there is a positive real number *c* such that $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then either both series converge or both series diverge.

RATIO TEST for converges *(Lesson2)*

Let $\sum a_n$ be a positive-term series, and suppose that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$.

- If L < 1, the series is convergent.
- If L > 1, or $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \infty$, the series is divergent.
- If L = 1, apply a different test; the series may be convergent or divergent.

ROOT TEST for convergence (*Not on the AP outline*)

Let $\sum a_n$ be a positive-term series, and suppose that $\lim_{n\to\infty} \sqrt[n]{a_n} = L$.

- If L < 1, the series is convergent.
- If L > 1, or $\lim_{n\to\infty} \sqrt[n]{a_n} = \infty$, the series is divergent.
- If L = 1, apply a different test; the series may be convergent or divergent.

I. Positive Term: Ratio and Root Test Practice

1: Determine whether the infinite series $\sum_{n=1}^{\infty} \frac{3^n}{n!}$ converges or diverges using the Ratio Test.

2: Determine whether the infinite series $\sum_{n=1}^{\infty} \frac{3^n}{n^2}$ converges or diverges using the Ratio Test.

3: Determine whether the infinite series $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ converges or diverges using the Ratio Test.

4: Determine whether the infinite series $\sum_{n=1}^{\infty} \frac{2^{3n+1}}{n^n}$ converges or diverges using the Root Test.

II. POSITIVE TERM SERIES: Formal Tests for Convergence or Divergence <u>DIAGRAM</u>

III. POWER (geometric) SERIES: For what values of *x* does the series converge?

A. Find the interval of convergence (the values of *x* for which the series converges). Justify:

1: $\sum_{n=0}^{\infty} x^n$

2: $\sum_{n=0}^{\infty} (-1)^n x^n$

3: $\sum_{n=0}^{\infty} 3\left(\frac{x-1}{2}\right)^n$

4: $\sum_{n=0}^{\infty} \frac{(x+1)^{2n}}{9^n}$

B. Write an infinite power series to represent the function $f(x)$ and determine for what values of x the infinite power series is true.

4. $f(x) = \dfrac{1}{1-2x}$

5. $f(x) = \dfrac{5}{1+3x}$

CLASS EXPLORATION

Interval of Convergence

Graphing Polynomial Investigation

Lesson 2 - Homework

Formal Testing – Positive Term Series (Ratio Test)

1. Determine whether the infinite series $\sum_{n=0}^{\infty} n^2 e^{-n}$ converges or diverges using the **Ratio Test**.

Pg. 511 #35

2. Determine whether the infinite series $\sum_{n=1}^{\infty} \frac{(n+3)!}{3!n!3^n}$ converges or diverges using the **Ratio Test**.

Pg. 511 #37

3. Determine whether the infinite series $\sum_{n=0}^{\infty} n!e^{-n}$ converges or diverges using the **Ratio Test**.

Pg. 511 #40

4. Determine whether the infinite series $\sum_{n=1}^{\infty} \frac{n!}{(2n+1)!}$ converges or diverges using the **Ratio Test**.

Pg. 511 #43

5. Determine whether the infinite series $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges or diverges using the **Ratio Test**.

Pg. 511 #44

Power Series (Geometric)

Find the interval of convergence of the series and, within this interval, the sum of the series as a function of *x*.

6. $\sum_{n=0}^{\infty} \frac{(x-1)^{2n}}{4^n}$: Pg. 511 #23
7. $\sum_{n=0}^{\infty} (\ln x)^n$: Pg. 511 #26
8. $\sum_{n=0}^{\infty} \left(\frac{x^2-1}{3}\right)^n$: Pg. 511 #27
9. $\sum_{n=0}^{\infty} \left(\frac{\sin x}{2}\right)^n$: Pg. 511 #28

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NON-POSITIVE TERM SERIES: Theorems / Tests for Convergence or Divergence

"CAB" THEOREM

THE CONVERGENCE IN ABSOLUTE (CAB) THEOREM
If $\sum |a_n|$ converges, then $\sum a_n$. Note: If $\sum |a_n|$ diverges, then $\sum a_n$ may or may not converge

ALTERNATING SERIES TEST (AST) (*Observational Test*)

FORMAL ALTERNATING SERIES TEST : SEE PAGE 517

ALTERNATING SERIES TEST
If the terms of the series (i) strictly alternate and (ii) decrease in absolute value to zero, then the series converges

ALTERNATING SERIES TEST as the "Glorified *n*th Term Test"

€If $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ and $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ converges.

€If $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ and $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ diverges as it fails the nth term test.

$$\sum_{n=1}^{\infty}(-1)^{n-1}a_n$$

We see that this series is strictly alternating because of the "alternating indicator" expression: $(-1)^{n-1}$
The series will decrease in absolute value to zero if it passes the nth term test: $\lim_{n\to\infty} a_n = 0$

What do we show: We show $\lim_{n\to\infty} a_n = 0$
What do we say: We say "The series strictly alternates and decreases in absolute value to zero"
What do we conclude: We conclude "Therefore the series $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ converges by the A.S.T"

I. Non – Positive Term: The Converge in Absolute (CAB) Theorem

1: Determine whether the infinite series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^2}$ converges or diverges.

2: Determine whether the infinite series $\frac{1}{2}+\frac{1}{2^2}-\frac{1}{2^3}-\frac{1}{2^4}+\frac{1}{2^5}+\frac{1}{2^6}-\cdots$ converges or diverges.

3: Determine whether the infinite series: $\sin 1 + \frac{\sin 2}{2^2} + \frac{\sin 3}{3^2} + \cdots + \frac{\sin n}{n^2} + \cdots$ converges or diverges.

4: Determine whether the infinite series $\sum_{n=1}^{\infty} (-1)^n \frac{n^2+4}{2^n}$ converges or diverges.

II. Non – Positive Term: The Alternating Series Test (AST)

1: Determine whether the infinite series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{2n}{4n^2 - 3}$ converges or diverges.

2. Determine whether the infinite series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{2n}{4n - 3}$ converges or diverges.

3. Determine whether the infinite series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ converges or diverges.

III. Convergent Series: Absolute and Conditional Convergence

IV. POWER SERIES (Non-Geometric): INTERVAL OF CONVERGENCE

1. Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$.

Also state the center and radius of convergence.

2. Find the interval of convergence of the power series $\sum_{n=0}^{\infty}(-1)^n \frac{(x-3)^n}{n+1}$.

Also state the center and radius of convergence.

COMBINING CONCEPTS

Given the power series: $f(x)=\sum_{n=k}^{\infty} a_n (x-b)^n$

A. Determine the values of x for which the power series converges absolutely.
B. Determine the values of x for which the power series converges conditionally.
C. Determine the values of x for which the power series diverges.

EXAMPLE(S) TO BE EXPLORED IN CLASS

Lesson 3 - Homework

IV. Power Series (Non - Geometric)

Find the interval of convergence of the power series. *Also state the center and radius of convergence.*

1. $\sum_{n=1}^{\infty} \frac{x^n}{n\sqrt{n}3^n}$: Pg. 523 #41
2. $\sum_{n=0}^{\infty} \frac{nx^n}{4^n(n^2+1)}$: Pg. 523 #44
3. $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!}$: Pg. 523 #42
4. $\sum_{n=0}^{\infty} n!(x-4)^n$: Pg. 523 #46

II. Convergence Testing – Non Positive Term Series

5. Determine whether the infinite series $\sum_{n=2}^{\infty} (-1)^{n+1} \frac{1}{n\ln n}$ converges or diverges.

Pg. 523 #25

6. Determine whether the infinite series $\sum_{n=1}^{\infty} \frac{(-1)^n}{1+\sqrt{n}}$ converges or diverges.

Pg. 523 #29

7. Determine whether the infinite series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n!}{2^n}$ converges or diverges.

Pg. 523 #27

8. Determine whether the infinite series $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n}$ converges or diverges.

Pg. 523 #31

III. Absolute Convergence, Conditional Convergence, Divergence
(9 – 12). Classify each series above (5 – 8) as **absolutely convergent, conditionally convergent**, or **divergent**. Show all work.

COMBINING CONCEPTS:

13. Given the power series: $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2 4^n}$

A. Determine the values of *x* for which the power series converges absolutely.
B. Determine the values of *x* for which the power series converges conditionally.
C. Determine the values of *x* for which the power series diverges.

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GRAND FINALE REVIEW

Find the interval of convergence of the power series.

1. $\sum_{n=0}^{\infty}(-1)^n(4x+1)^n$: Pg. 523 #37

Find the interval of convergence of the power series. *Also state the center and radius of convergence.*

2. $\sum_{n=0}^{\infty}\frac{n(x+3)^n}{5^n}$: Pg. 523 #43
3. $\sum_{n=0}^{\infty}(-2)^n(n+1)(x-1)^n$: Pg. 523 #47
4. $\sum_{n=1}^{\infty}\frac{(4x-5)^{2n+1}}{n^{\frac{3}{2}}}$: Pg. 523 #48

Find the interval of convergence of the series and, within this interval, the sum of the series as a function of *x*.

5. $\sum_{n=0}^{\infty}\frac{(x-2)^n}{10^n}$: Pg. 523 #39
6. $\sum_{n=0}^{\infty}(\ln x)^n$: Pg. 523 #50

Given the power series …

A. Determine the values of *x* for which the power series converges absolutely.
B. Determine the values of *x* for which the power series converges conditionally.
C. Determine the values of *x* for which the power series diverges.

7. $\sum_{n=1}^{\infty}\frac{(-1)^n(x-1)^n}{2n+3}$

** Positive Term Series Review **

Determine whether each <u>positive-term-series</u> converges or diverges.
There may be more than one test that yields conclusive results.

1. $\sum_{n=1}^{\infty} \frac{1}{n^4 + n^2 + 1}$

2. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+4}$

3. $\sum_{n=1}^{\infty} \frac{1+2^n}{1+3^n}$

4. $\sum_{n=1}^{\infty} ne^{-n}$

5. $\sum_{n=1}^{\infty} \frac{3n+1}{2^n}$

6. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$

7. $\sum_{n=1}^{\infty} \frac{100^n}{n!}$

8. $\sum_{n=1}^{\infty} \frac{1}{n^n}$