0303 PRACTICE EXAM Solutions

1. $\lim\limits_{x\to 0} \dfrac{e^{2x}-1}{\sin x}$ $\qquad \lim\limits_{x\to 0} e^{2x}-1 = 0$ and $\lim\limits_{x\to 0} \sin x = 0$

L'HOPITAL'S RULE

$$= \lim_{x\to 0} \frac{2e^{2x}}{\cos x} = \frac{2}{1} = \boxed{2}$$

2. $\lim\limits_{x\to 1^-} (2-x)^{\tan(\frac{\pi}{2}x)}$ $\quad \{1^\infty\}$ $\quad \lim\limits_{x\to 1^-} (2-x) = 1 \quad \lim\limits_{x\to 1^-} \tan(\frac{\pi}{2}x) = \infty$

Let $y = (2-x)^{(\tan \frac{\pi}{2}x)}$ $\quad$ Find $\lim\limits_{x\to 1^-} y$

BUT FIRST: $\lim\limits_{x\to 1^-} \ln y = \lim\limits_{x\to 1^-} \tan(\frac{\pi}{2}x)\cdot \ln(2-x) \qquad \{\infty\cdot 0\}$

$$= \lim_{x\to 1^-} \frac{\ln(2-x)}{\cot(\frac{\pi}{2}x)} \qquad \left\{\frac{0}{0}\right\}$$

$$= \lim_{x\to 1^-} \frac{\frac{1}{2-x}\overset{\text{chain Rule}}{(-1)}}{-\csc^2(\frac{\pi}{2}x)\cdot \underset{\text{chain Rule}}{\frac{\pi}{2}}} = \frac{-1}{-\frac{\pi}{2}} = \frac{2}{\pi}$$

AND NOW: $\lim\limits_{x\to 1^-} y = \lim\limits_{x\to 1^-} e^{\ln y} = e^{\lim_{x\to 1^-} \ln y} = \boxed{e^{2/\pi}}$

3. $\lim\limits_{x\to 0} \dfrac{\int_5^{5-x} \frac{\sin(t/5)}{2t}\,dt}{x} \qquad \left\{\frac{0}{0}\right\}$

$$= \lim_{x\to 0} \frac{\frac{\sin((5-x)/5)\overset{\text{chain Rule}}{(-1)}}{2(5-x)}}{1} = \frac{-\sin(1)}{10} = \boxed{-\frac{\sin(1)}{10}}$$

4. $$\int_1^4 \frac{t-2}{(t+1)(t-4)}\,dt = \lim_{b\to 4^-} \int_1^b \frac{t-2}{(t+1)(t-4)}\,dt$$

5. $$\int_0^2 \ln x\,dx = \lim_{a\to 0^+} \int_a^2 \ln x\,dx$$

6. $$\int_{-2}^1 \frac{dx}{x^{2/3}} = \lim_{b\to 0^-} \int_{-2}^b x^{-2/3}\,dx + \lim_{a\to 0^+} \int_a^1 x^{-2/3}\,dx$$

7. $$\int_1^\infty 12x^2 f(x)\,dx = \int_1^\infty (-3)(-4x^2 f(x))\,dx = -3\int_1^\infty f'(x)\,dx$$

$$= -3 \lim_{b\to\infty} \int_1^b f'(x)\,dx = -3 \lim_{b\to\infty} \left[f(b) - f(1)\right]$$

$$= -3 \lim_{b\to\infty} f(b) + 3f(1)$$

$$= -3(0) + 3(-2) = \boxed{-6}$$

8. Let $f(x) = \frac{1}{2e^x - 3}$ and $g(x) = \frac{1}{e^x}$

i) f and g are positive and continuous on $[1, \infty)$

ii) $\int_1^\infty g(x)\,dx = \int_1^\infty \frac{1}{e^x}\,dx$ converges by the "e integral"

iii) $$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\frac{1}{2e^x-3}}{\frac{1}{e^x}} = \lim_{x\to\infty} \frac{e^x}{2e^x - 3} \quad \left\{\frac{\infty}{\infty}\right\}$$

$$= \lim_{x\to\infty} \frac{e^x}{2e^x} = \lim_{x\to\infty} \frac{1}{2} = \frac{1}{2} \in (0, \infty)$$

$\therefore \int_1^\infty f(x)\,dx = \int_1^\infty \frac{1}{2e^x - 3}\,dx$ converges also by the limit comparison test.

BEYOND

A. $$\int x e^{-x^2}\,dx = -\frac{1}{2}\int e^{u}\,du = -\frac{1}{2}e^{-x^2} + c$$

$u = -x^2$

$du = -2x\,dx$

$dx = \dfrac{du}{-2x}$

B. $$\int_0^{\infty} x^2 e^{-x^2}\,dx = \lim_{b\to\infty}\int_0^{b} x^2 e^{-x^2}\,dx = \lim_{b\to\infty}\left[\left[-\frac{1}{2}x e^{-x^2}\right]_0^b + \frac{1}{2}\int_0^b e^{-x^2}\,dx\right]$$

lim as b goes to infinity

$$= \lim_{b\to\infty}\left[-\frac{1}{2}\,b e^{-b^2} - 0\right] + \frac{1}{2}\int_0^b e^{-x^2}\,dx$$

$$= 0 + \frac{1}{2}\int_0^b e^{-x^2}\,dx$$

limit as b goes to infinity

$$= \frac{1}{2}\,\frac{\sqrt{\pi}}{2} = \boxed{\frac{\sqrt{\pi}}{4}}$$

□ $\int x^2 e^{-x^2}\,dx$

$u = x \qquad dv = x e^{-x^2}\,dx$

$du = dx \qquad v = -\frac{1}{2}e^{-x^2}$

□ $\lim\limits_{b\to\infty} b e^{-b^2} \qquad \{\infty \cdot 0\}$

$= \lim\limits_{b\to\infty} \dfrac{b}{e^{b^2}} \qquad \left\{\dfrac{\infty}{\infty}\right\}$

$= \lim\limits_{b\to\infty} \dfrac{1}{e^{b^2}\cdot 2b} = 0$

EVAL # 4: $\int \frac{t-2}{(t+1)(t-4)}\,dt = \int \frac{\frac{3}{5}}{t+1}\,dt + \int \frac{\frac{2}{5}}{t-4}\,dt$

$$\frac{A}{t+1} + \frac{B}{t-4} = \frac{t-2}{(t+1)(t-4)}$$

$$A(t-4) + B(t+1) = t-2$$

$$A + B = 1$$
$$-4A + B = -2$$
$$5A = 3$$
$$A = \frac{3}{5}$$
$$B = \frac{2}{5}$$

$$\frac{3}{5}\ln|t+1| + \frac{2}{5}\ln|t-4| + C$$

$$\int_1^b \frac{t-2}{(t+1)(t-4)}\,dt = \frac{3}{5}\ln|b+1| + \frac{2}{5}\ln|b-4|$$
$$-\frac{3}{5}\ln|2| + \frac{2}{5}\ln|3|$$

$$\lim_{b\to 4^-} \frac{2}{5}\ln|b-4| \to -\infty$$

THE INTEGRAL DOES NOT EXIST

EVAL # 5 $\int \ln x\,dx = x\ln x - x + C$

u $\ln x$ $\frac{dv}{dx}$

$\frac{dx}{x}$ x

$$\lim_{a\to 0^+} \int_a^2 \ln x\,dx = 2\ln(2) - 2 - \lim_{a\to 0^+}\left[\cancel{a\ln a}^{\,0} - \cancel{a}^{\,0}\right]$$

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$$\lim_{a\to 0^+} a\ln a \quad \{0\cdot\infty\}$$

$$= \lim_{a\to 0^+} \frac{\ln a}{\frac{1}{a}} \quad \left\{\frac{\infty}{\infty}\right\}$$

$$= \lim_{a\to 0^+} \frac{\frac{1}{a}}{\frac{-1}{a^2}} = \lim_{a\to 0^+} -a = 0$$

$$= \boxed{2\ln(2) - 2}$$

$$\int_{-2}^{0} x^{-2/3}\,dx = \dots = 3(2)^{1/3}$$

$$\int_0^1 x^{-2/3}\,dx = \dots = 3$$

$$\int_{-2}^{1} \frac{dx}{x^{2/3}} = \dots = 3(2)^{1/3} + 3$$