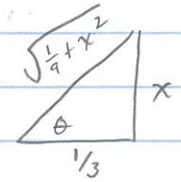
[Solutions]

Review Solutions

1. $$\int \frac{dx}{1+9x^2} = \int \frac{dx}{1+(3x)^2} = \frac{1}{3}\int \frac{du}{1+u^2} = \frac{1}{3}\tan^{-1}(u) + c = \frac{1}{3}\tan^{-1}(3x) + c$$

$u = 3x \quad du = 3dx$

1. $$\int \frac{dx}{1+9x^2} = \frac{1}{9}\int \frac{dx}{\frac{1}{9}+x^2} = \frac{1}{9}\int \frac{\frac{1}{3}\sec^2\theta\, d\theta}{\frac{1}{9}+\frac{1}{9}\tan^2 x} = \frac{1}{3}\int \frac{\sec^2\theta\, d\theta}{1+\tan^2\theta} = \frac{1}{3}\int d\theta$$



$x = \frac{1}{3}\tan\theta$

$dx = \frac{1}{3}\sec^2\theta\, d\theta$

$\tan\theta = \frac{x}{1/3}$

$$= \frac{1}{3}\theta + c = \frac{1}{3}\tan^{-1}\left(\frac{x}{1/3}\right) + c$$
$$= \frac{1}{3}\tan^{-1}(3x) + c$$

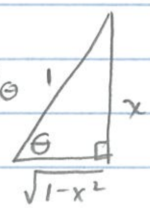
2A $$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}(x) + c$$ BOOK OF MEMORIES :)

2B $$\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{\cos\theta\, d\theta}{\sqrt{1-\sin^2\theta}} = \int \frac{\cos\theta}{\cos\theta}\, d\theta = \int d\theta = \theta + c$$
$$= \sin^{-1}(x) + c$$

$x = \sin\theta$

$dx = \cos\theta\, d\theta$

$\sin\theta = x$



3A $$\int \frac{dx}{1-x^2} = \int \frac{\frac{1}{2}dx}{1-x} + \int \frac{\frac{1}{2}dx}{1+x} = -\frac{1}{2}\ln|1-x| + \frac{1}{2}\ln|1+x| + c$$
$$= \ln\sqrt{\frac{|1+x|}{|1-x|}} + c \quad (\text{OR}) \quad \frac{\ln\left(\frac{|1+x|}{|1-x|}\right)}{2} + c$$

$$\frac{1}{1-x^2} = \frac{A}{1-x} + \frac{B}{1+x}$$

$A(1+x) + B(1-x) = 1$

$(A-B)x + A + B = 1$

$A - B = 0 \qquad A = 1/2$

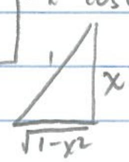
$A + B = 1 \qquad B = 1/2$

3B $$\int \frac{dx}{1-x^2} = \int \frac{\cos\theta\, d\theta}{1-\sin^2\theta} = \int \frac{d\theta}{\cos\theta} = \int \sec\theta\, d\theta$$

$x = \sin\theta$

$dx = \cos\theta\, d\theta$

$$= \ln|\sec\theta + \tan\theta| + c$$
$$= \ln\left|\frac{1}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}}\right| + c$$



$$= \ln\left|\frac{1+x}{\sqrt{1-x^2}}\right| + c = \ln|1+x| - \left[\frac{1}{2}\ln|1-x^2|\right] + c$$
$$= \ln|1+x| - \frac{1}{2}\ln(1-x) - \frac{1}{2}\ln(1+x) + c$$
$$= \frac{1}{2}\ln|1+x| - \frac{1}{2}\ln|1-x| + c$$

BY PARTS

(4) $\int \frac{(\ln x)^2}{x^3}\,dx \quad = -\frac{(\ln x)^2}{2x^2} + \int \frac{(\ln x)\,dx}{x^3}$

$$\boxed{\begin{array}{ll} u = (\ln x)^2 & dv = \frac{dx}{x^3} \\ du = \frac{2(\ln x)\,dx}{x} & v = \frac{x^{-2}}{-2} \end{array}} \qquad \boxed{\begin{array}{ll} u = \ln x & dv = \frac{dx}{x^3} \\ du = \frac{dx}{x} & v = \frac{x^{-2}}{-2} \end{array}}$$

$$= -\frac{(\ln x)^2}{2x^2} - \frac{(\ln x)}{2x^2} + \int \frac{dx}{2x^3}$$

$$= -\frac{(\ln x)^2}{2x^2} - \frac{(\ln x)}{2x^2} - \frac{1}{4x^2} + C$$

(5) $\int \sin(\ln x)\,dx$

$$\boxed{\begin{array}{ll} u = \sin(\ln x) & dv = dx \\ du = \dfrac{\cos(\ln x)}{x} & v = x \end{array}}$$

$= x\sin(\ln x) - \int \cos(\ln x)\,dx$

$$\boxed{\begin{array}{ll} u = \cos(\ln x) & dv = dx \\ du = \dfrac{-\sin(\ln x)}{x} & v = x \end{array}}$$

$= x\sin(\ln x) - \left[x\cos(\ln x) + \int \sin(\ln x)\,dx\right]$

$2\int \sin(\ln x)\,dx = x\sin(\ln x) - x\cos(\ln x) + C$

$\int \sin(\ln x)\,dx = \frac{1}{2}x\sin(\ln x) - \frac{1}{2}x\cos(\ln x) + C$

(6) $$\int \frac{\sqrt{x^2-9}}{x^3}\,dx = \int \frac{\sqrt{9\sec^2\theta - 9}}{27\sec^3\theta}\, 3\sec\theta\tan\theta\, d\theta$$

$x = 3\sec\theta$

$dx = 3\sec\theta\tan\theta\, d\theta$

$$= \frac{9}{27}\int \frac{\tan\theta\tan\theta}{\sec^2\theta}\, d\theta$$

$$= \frac{9}{27}\int \frac{\tan^2\theta}{\sec^2\theta}\, d\theta = \frac{1}{3}\int \frac{\sin^2\theta}{\cos^2\theta}\cdot \cos^2\theta\, d\theta$$

$$= \frac{1}{3}\int \sin^2\theta\, d\theta$$

OPTION WITH IDENTITY

$\cos 2u = 1 - 2\sin^2 u$

$$= \frac{1}{3}\int \frac{1}{2}(1-\cos 2\theta)\, d\theta$$

$$= \frac{1}{6}\int (1-\cos 2\theta)\, d\theta$$

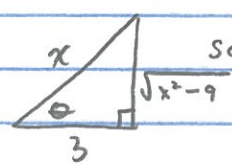
$$= \frac{1}{6}\theta - \frac{1}{12}\sin 2\theta + C$$

IDENTITY

$\sin 2u = 2\sin u \cos u$

$$= \frac{1}{6}\theta - \frac{1}{6}\sin\theta\cos\theta + C$$

$$= \frac{1}{6}\sec^{-1}\left(\frac{x}{3}\right) - \frac{1}{6}\cdot\frac{\sqrt{x^2-9}}{x}\cdot\frac{3}{x}$$



$\sec\theta = \frac{x}{3}$

$$= \frac{1}{6}\sec^{-1}\left(\frac{x}{3}\right) - \frac{1}{2}\frac{\sqrt{x^2-9}}{x^2}$$

(7) $\int x 2^x \, dx = \dfrac{x 2^x}{\ln 2} - \int \dfrac{2^x}{\ln 2} \, dx = \dfrac{x 2^x}{\ln 2} - \dfrac{2^x}{(\ln 2)^2} + C$

$u = x \quad dv = 2^x$

$du = dx \quad v = \dfrac{2^x}{\ln 2}$

BC.Q203 Review

⑥ $$\int \frac{\sqrt{x^2-9}}{x^3}\,dx$$

Let $$f(x) = \int \frac{\sqrt{x^2-9}}{x^3}\,dx$$

Let $x = 3\sec\theta \Rightarrow dx = 3\sec\theta\tan\theta\,d\theta$ — Trig sub

$$f(x) = \int \frac{\sqrt{9\sec^2\theta - 9}}{3^3\sec^3\theta}\,dx = \int \frac{3\sqrt{\sec^2\theta - 1}}{3^3\sec^3\theta}$$ — Substitute | simplify

$$f(x) = \int \frac{\cancel{3}\tan\theta\,\cancel{\sec\theta}\tan\theta\,d\theta\,\cancel{3}}{\cancel{3^3}\sec^{\cancel{3}\,2}\theta} = \frac{1}{3}\int \frac{\tan^2\theta}{\sec^2\theta}\,d\theta$$ — $\tan^2\theta + 1 = \sec^2\theta$ | substitute dx

$$3f(x) = \int \frac{(\sin\theta/\cancel{\cos\theta})^2}{(1/\cancel{\cos\theta})^2} = \underbrace{\int \sin^2\theta\,d\theta}_{\text{By parts:}}$$ — Simplify by rewriting using sin and cos + Multiply by 3 so no need to worry about the 1/3 in front of integral

$u = \sin\theta \qquad dv = \sin\theta\,d\theta$

$du = \cos\theta\,d\theta \qquad v = -\cos\theta$

$$3f(x) = \int \sin^2\theta\,d\theta = -\sin\theta\cos\theta - \int -\cos^2\theta\,d\theta$$ — Integrate by parts

$$3f(x) = -\sin\theta\cos\theta + \int (1-\sin^2\theta)\,d\theta$$ — $\sin^2\theta + \cos^2\theta = 1$

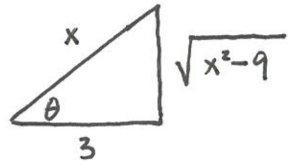
$$3f(x) = -\sin\theta\cos\theta + \int d\theta - \underbrace{\int \sin^2\theta\,d\theta}_{3f(x)}$$ — Split integral

$\int \sin^2\theta\,d\theta = 3f(x)$

$$6f(x) = -\sin\theta\cos\theta + \theta$$ — Add to both sides

$$f(x) = \frac{1}{6}(\theta - \sin\theta\cos\theta) = \frac{1}{6}\operatorname{arcsec}\frac{x}{3} - \frac{\sqrt{x^2-9}}{2x^2}$$ — Substitute (see triangles)

$$\therefore \boxed{\int \frac{\sqrt{x^2-9}}{x^3} = \frac{1}{6}\operatorname{arcsec}\frac{x}{3} - \frac{\sqrt{x^2-9}}{2x^2} + c}$$



$\theta = \operatorname{arcsec}\frac{x}{3}$

$\sin\theta = \frac{\sqrt{x^2-9}}{x}$

$\cos\theta = \frac{3}{x}$

$x = 3\sec\theta$

$\sec\theta = x/3$