Lesson 3

Supplemtal Exercises Solutions Pg #3

6.

a) $\int_0^1 \frac{x^3}{\sqrt{x^2+1}}\,dx$

$$\frac{x^3}{\sqrt{x^2+1}} = \frac{x}{\sqrt{x^2+1}} \cdot x^2$$

$u = x^2 \quad dv = \frac{x}{\sqrt{x^2+1}}\,dx$ $\quad w = x^2+1$, $dw = 2x\,dx$, $\frac{1}{2}\int w^{-1/2}\,dw$

$du = 2x\,dx \quad v = (x^2+1)^{1/2}$

$$x^2(x^2+1)^{1/2} - \int 2x(x^2+1)^{1/2}\,dx$$ $\quad w = x^2+1$, $dw = 2x\,dx$

$$x^2(x^2+1)^{1/2} - \frac{2}{3}(x^2+1)^{3/2} + C$$

$$\int_0^1 \frac{x^3}{\sqrt{x^2+1}}\,dx = x^2(x^2+1)^{1/2} - \frac{2}{3}(x^2+1)^{3/2}\Big]_0^1 = \left(\sqrt{2} - \frac{2}{3}(2)^{3/2}\right) - \left(-\frac{2}{3}\right)$$

$$= (2)^{1/2}\left(1 - \frac{2}{3}(2)\right) + \frac{2}{3}$$

$$(2)^{1/2}\left(1 - \frac{4}{3}\right) + \frac{2}{3}$$

$$(2)^{1/2}\left(-\frac{1}{3}\right) + \frac{2}{3}$$

$$= \boxed{\frac{2}{3} - \frac{\sqrt{2}}{3}}$$

b) $\int \frac{x^3}{\sqrt{x^2+1}}\,dx \quad u = \sqrt{x^2+1} \quad du = \frac{x}{\sqrt{x^2+1}}\,dx$

$u^2 = x^2+1 \rightarrow u^2 - 1 = x^2$

$2u\,du = 2x\,dx$

$u\,du = x\,dx$

$$= \int \frac{x^2 \cdot x\,dx}{\sqrt{x^2+1}} \quad \int \frac{(u^2-1)\,u\,du}{\cancel{u}}$$

$$= \int (u^2 - 1)\,du = \frac{u^3}{3} - u\Big]_{u=1}^{u=\sqrt{2}} = \frac{2^{3/2}}{3} - \sqrt{2} - \left(\frac{1}{3} - 1\right)$$

$$= \boxed{\frac{2}{3} - \frac{\sqrt{2}}{3}}$$

9. $\int \frac{dx}{\sqrt{2x-x^2}}$ (i) $u=\sqrt{x}$ $\quad u^2=x$ $\quad 2u\,du=dx$

$u^4=x^2$

$$\int 2u\,du \cdot \frac{1}{\sqrt{2u^2-u^4}} = \int \frac{2\cancel{u}\,du}{\cancel{u}\sqrt{2-u^2}} = \int \frac{2\,du}{\sqrt{2-u^2}}$$

$u=\sqrt{2}\sin\theta \quad du=\sqrt{2}\cos\theta\,d\theta$

$$=\int \frac{2\sqrt{2}\cos\theta\,d\theta}{\sqrt{2-2\sin^2\theta}} = 2\int \frac{\cos\theta\,d\theta}{\sqrt{1-\sin^2\theta}} = 2\int d\theta = 2\theta + c$$

$$= 2\sin^{-1}\left(\frac{u}{\sqrt{2}}\right) + c$$

$$= \boxed{2\sin^{-1}\left(\sqrt{\frac{x}{2}}\right) + c}$$

$\int \frac{dx}{\sqrt{2x-x^2}}$ (ii) $u=\sqrt{2-x}$ $\quad u^2=2-x$ $\quad x=2-u^2$

$2u\,du=-dx$

$$-\int \frac{2\cancel{u}\,du}{\sqrt{2-u^2}\,\cancel{u}} = -\int \frac{2\,du}{\sqrt{2-u^2}} = -2\sin^{-1}\left(\frac{u}{\sqrt{2}}\right)+c$$

$u=\sqrt{2}\sin\theta$ $\quad \sin\theta = \frac{u}{\sqrt{2}}$

$$= \boxed{-2\sin^{-1}\left(\frac{\sqrt{2-x}}{\sqrt{2}}\right)+c}$$

(iii) $$\frac{dx}{\sqrt{-(x^2-2x)}} = \frac{dx}{\sqrt{-(x^2-2x+1-1)}} = \frac{dx}{\sqrt{-(x^2-2x+1)+1}}$$

$$= \frac{dx}{\sqrt{-(x-1)^2+1}}$$

$u=x-1 \quad x=u+1$

$dx=du$

$$\int \frac{du}{\sqrt{-u^2+1}} = \int \frac{du}{\sqrt{1-u^2}} \quad u=\sin\theta \quad du=\cos\theta$$

$$= \int \frac{\cos\theta\,d\theta}{\sqrt{1-\sin^2\theta}} = \int d\theta = \theta + C = \sin^{-1}(u)+c$$

b) derivative of each is original. $\quad = \boxed{\sin^{-1}(x-1)+c}$

Lesson 3

Supp W/S Solutions Pg. #1

Pg 482 # 39 $\int 1/(x\sqrt{x+1})\,dx$

$u=\sqrt{x+1} \qquad u^2 = x+1 \qquad u^2-1 = x \qquad 2u\,du = dx$

$$= \int \frac{2\cancel{u}\,du}{(u^2-1)\cancel{u}} = 2\int \frac{du}{u^2-1} = 2\int \frac{A}{u+1}\,du + 2\int \frac{B}{u-1}\,du$$

$A(u-1)+B(u+1)=1$

$(A+B)u - A + B = 1$

$A+B=0 \qquad B = 1/2$

$-A+B = 1 \qquad A = -1/2$

$$= -\int \frac{1}{u+1}\,du + \int \frac{1}{u-1}\,du$$

$$= -\ln|u+1| + \ln|u-1| + C$$

$$= \ln\left|\frac{u-1}{u+1}\right| + C$$

$$= \boxed{\ln\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right| + C}$$

Pg 482 # 47 $\int e^{2x}/(e^{2x}+3e^x+2)\,dx$

$u = e^x$

$du = e^x dx$

$$\int \frac{u^2\,du}{u}\cdot\frac{1}{u^2+3u+2} = \int \frac{u}{u^2+3u+2}\,du$$

$$\frac{u}{u^2+3u+2} = \frac{A}{u+2} + \frac{B}{u+1} \qquad A(u+1)+B(u+2) = (A+B)u + A + 2B = u$$

$(u+2)(u+1)$

$-A - B = -1$

$A + 2B = 0$

$B=-1 \qquad A = 2$

$$\int \frac{2}{u+2}\,du - \int \frac{du}{u+1} = 2\ln|u+2| - \ln|u+1| + c$$

$$= \ln(u+2)^2 - \ln(u+1) + c$$

$$= \ln\frac{(u+2)^2}{u+1} + c = \boxed{\ln\frac{(e^x+2)^2}{e^x+1} + c}$$

482 # 59. $\int 1/(3\sin x - 4\cos x)\,dx$

$u = \tan x/2 \quad \int \frac{2/(1+u^2)}{3\left(\frac{2u}{1+u^2}\right) - 4\left(\frac{1-u^2}{1+u^2}\right)}\,du$

$= \int \frac{2\,du}{6u - 4 + 4u^2} = \int \frac{du}{2u^2 + 3u - 2}$

$(2u-1)(u+2)$

$\frac{A}{(2u-1)} + \frac{B}{(u+2)} \qquad A(u+2) + B(2u-1) = (A+2B)u + 2A - B = 1$

$A + 2B = 0 \qquad A + 2B = 0$

$2A - B = 1 \qquad 4A - 2B = 2$

$5A = 2$

$A = 2/5$

$B = -1/5$

$\frac{4}{5} - B = 1$

$B = \frac{4}{5} - 1 = -1/5$

$\frac{2}{5}\int \frac{du}{2u-1} - \frac{1}{5}\int \frac{du}{u+2}$

$v = 2u - 1$

$dv = 2\,du$

$\frac{1}{5}\ln|2u-1| - \frac{1}{5}\ln|u+2|$

$$\boxed{\frac{1}{5}\ln\left|\frac{2\tan(x/2) - 1}{\tan(x/2) + 2}\right| + c}$$

Lesson 3
extra

$$\int \frac{dx}{x^3-1} = \underbrace{\int \frac{dx}{3(x-1)}}_{*} - \underbrace{\int \frac{(x+2)\,dx}{3(x^2+x+1)}}_{\triangle} = \frac{1}{3}\ln|x-1| \overbrace{- \frac{1}{6}|x^2+x+1| - \frac{\sqrt{3}}{3}\tan^{-1}\left(\frac{2(x+\frac{1}{2})}{\sqrt{3}}\right)}^{\square} + C$$

$$= \frac{1}{6}\left(2\ln|x-1| - \ln|x^2+x+1|\right) - \frac{\sqrt{3}}{2}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + c$$

$$= \frac{1}{6}\ln\left(\frac{(x-1)^2}{|x^2+x+1|}\right) - \frac{\sqrt{3}}{3}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

* $\frac{1}{x^3-1} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1} \dots$ $A = 1/3 \quad B = -1/3 \quad C = -2/3$

$$= \frac{1}{3(x-1)} - \frac{1}{3}\left(\frac{x+2}{x^2+x+1}\right)$$

$\triangle$ $\int \frac{dx}{3(x-1)} = \frac{1}{3}\ln|x-1| + C$

$\square$ $-\frac{1}{3}\int \frac{x+2}{x^2+x+1}dx = -\frac{1}{3}\int\left(\frac{x+2}{x^2+x+1/4-1/4+1}\right)dx = -\frac{1}{3}\int\frac{x+2}{(x+\frac{1}{2})^2+3/4}dx = -\frac{1}{3}\int\frac{u-1/2+2}{u^2+3/4}$

$u = x + 1/2 \quad du = dx$

$u - 1/2 = x$

$$= -\frac{1}{3}\int\frac{u+3/2}{u^2+3/4} = -1/3\int\frac{u\,du}{u^2+3/4} - 1/3\int\frac{3/2}{u^2+3/4}du$$

$v = u^2 + 3/4$ $\qquad v = \frac{2u}{\sqrt{3}}$

$$= -1/3\left(\frac{1}{2}\ln|u^2+3/4| + \frac{3/2}{3/4}\int\frac{du}{\left(\frac{2u}{\sqrt{3}}\right)^2+1}\right)$$

$$= -\frac{1}{6}\ln|x^2+x+1| - \frac{\sqrt{3}}{3}\tan^{-1}\left(\frac{2u}{\sqrt{3}}\right) + C$$