Lesson 3 (E1) - HW Solutions

Hw Solutions

[From Notes]

3

a) $f(x) = \frac{1}{x-1} = \frac{1}{1+(x-2)} = 1-(x-2)+(x-2)^2-(x-2)^3+\cdots+(-1)^n(x-2)^n+\cdots$ for $1<x<3$

[OR] → (RAW CONSTRUCT)

b) $\ln|x-1| = \int_2^x \frac{1}{t-1}\,dt = \int_2^x [1-(t-2)+(t-2)^2-(t-2)^3+\cdots]\,dt$

$= (x-2) - \frac{(x-2)^2}{2} + \frac{(x-2)^3}{3} + \cdots + \frac{(-1)^n(x-2)^{n+1}}{(n+1)} + \cdots$ for $1 \le x \le 3$

harmonic   alt. harmonic

c) $\ln\frac{3}{2} = \ln\left|\frac{5}{2}-1\right|$

$= \left(\frac{5}{2}-2\right) - \frac{\left(\frac{5}{2}-2\right)^2}{2} + \frac{\left(\frac{5}{2}-2\right)^3}{3} - \cdots$

$= \frac{1}{2} - \frac{1}{8} + \frac{1}{24} - \cdots$

$\ln\frac{3}{2} \approx \frac{1}{2} - \frac{1}{8}$ with $|\text{error}| \le \frac{1}{24} < 0.05$

This is true b/c the series for $\ln(\frac{3}{2})$ is strictly alternating and decreasing in absolute value to zero. (EI)

4 [HW Solution]

a) $P_3(x) = \frac{1}{2} - \frac{1}{6}(x-5) + \frac{1}{16}(x-5)^2 - \frac{1}{40}(x-5)^3$

b) ♡ RATIO Test $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\cancel{(n+1)!}}{2^{n+1}(n+3)} \frac{(x-5)^{n+1}}{\cancel{(n+1)!}} \cdot \frac{2^n(n+2)\cancel{n!}}{\cancel{n!}(x-5)^n}\right|$

$= \lim_{n\to\infty} \frac{1}{2}\,\frac{n+2}{n+3}\,|x-5| = \frac{1}{2}|x-5| < 1$

(question just calls for radius) $\boxed{\rho = 2}$

(prelim interval of convergence without checking end points) $3 < x < 7$

c) 6 is within the interval of convergence!

$f(6) \approx P_6(6)$ with $|\text{error}| = |R_6(6)| \le |a_{n+1}| = \left|\frac{f^7(5)(6-5)^7}{7!}\right|$

$= \left|\frac{7!}{2^7(9)} \cdot \frac{(1)^7}{7!}\right| = \frac{1}{2^7(9)} = \frac{1}{1152} < \frac{1}{1000}$

This is true b/c the series for $f(6)$ is strictly alternating and decreasing in absolute value to zero. (EI)

5. $f(x) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$

$f'(x) = 0 - \frac{2x}{3!} + \frac{4x^3}{5!} - \frac{6x^5}{7!} + \cdots$

a) $f'(0) = 0$

$f''(x) = \frac{-2}{3!} + \frac{12x^2}{5!} - \cdots$

$f''(0) = \frac{-2}{3!} = \frac{-2}{6} = -1/3$

f has a local max at $x=0$ b/c $f'(0)=0$ and $f''(0)<0$ [2nd derivative Test]

b) $f(1) \approx 1 - \frac{1}{3!}$ with $|\text{error}| \le \left|\frac{1}{5!}\right| = \frac{1}{120} < \frac{1}{100}$ ✓

This is true because the series for $f(1)$ is strictly alternating and decreasing in absolute value to zero. [E1]

c)

$x(y') + y = x\left(-\frac{2x}{3!} + \frac{4x^3}{5!} - \frac{6x^5}{7!} + \cdots\right) + \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots\right)$

$= \left(-\frac{2x^2}{3!} + \frac{4x^4}{5!} - \frac{6x^6}{7!} + \cdots\right) + \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots\right)$

$= 1 - \frac{(2+1)x^2}{3!} + \frac{(4+1)x^4}{5!} - \frac{(6+1)}{7!}x^6 + \cdots$

$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \cos(x)$ ✓

Lesson 3 (E2) - HW Solutions

HW 9. $f(x)=e^x$

A. Use a 3rd order Taylor polynomial centered at $x=0$ to estimate $f(1)$.

B. Show that $|f(1)-P_3(1)|<\frac{1}{8}$.

$$f(x) \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

$$f(1) = 1 + 1 + \frac{1}{2} + \frac{1}{6}$$

$$|\text{error}(1)| \le \left| \frac{f^4_{max}(c)\,(1-0)^4}{4!} \right| < \frac{3\,(1)^4}{4!} = \frac{1}{8}$$

$0 \le c \le 1$

$$f^4(x) = e^x$$

$$|f^4(c)| = |e^c| \le \left[e^c_{max}\right] < 3$$

$0 \le c \le 1$ $\quad 0 \le c \le 1$

$$e^1 \approx 2.718 < 3$$

$$e^1 \approx 1 + 1 + \frac{1}{2} + \frac{1}{6} = \frac{8}{3} = 2\tfrac{2}{3} \text{ with } |\text{error}| < \frac{1}{8}$$

CALCULATOR ACTIVE

x	$h(x)$	$h'(x)$	$h''(x)$	$h'''(x)$	$h^{(4)}(x)$
1	11	30	42	99	18
2	80	128	$\frac{488}{3}$	$\frac{448}{3}$	$\frac{584}{9}$
3	317	$\frac{753}{2}$	$\frac{1383}{4}$	$\frac{3483}{16}$	$\frac{1125}{16}$

HW 10. Let h be a function having derivatives of all orders for $x > 0$. Select values of h and its first four derivatives are indicated in the table above. The function h and these four derivatives are increasing on the interval $1 \le x \le 3$.

A. Write the first-degree Taylor polynomial for h about $x = 2$ and use it to approximate $h(1.9)$. Is this approximation greater than or less than $h(1.9)$? Explain your reasoning.

B. Write the third-degree Taylor polynomial for h about $x = 2$ and use it to approximate $h(1.9)$.

C. Use the Lagrange error bound to show that the third-degree Taylor polynomial for h about $x = 2$ approximates $h(1.9)$ with error less than 3×10^{-4}.

A] $h(x) \approx h(2) + h'(2)(x-2)$

$h(x) \approx 80 + 128(x-2)$

$h(1.9) \approx 80 + 128(-0.1) = 67.2$

$h''(x) > 0$ on $1 \le x \le 3$
$\therefore$ h is concave up on $(1,3)$
$\therefore$ All linear approximations made about $x=2$ on $(1,3)$ will underestimate the value of h.
$h(1.9)$ is one such estimation.

OR

$h'(x)$ is increasing on $(1.9, 2)$
$\therefore$ The linear approximation for $h(1.9)$ about $x=2$ will be an under-estimate

B] $$h(x) \approx 80 + 128(x-2) + \frac{488}{3}\frac{(x-2)^2}{2!} + \frac{448}{3}\frac{(x-2)^3}{3!}$$

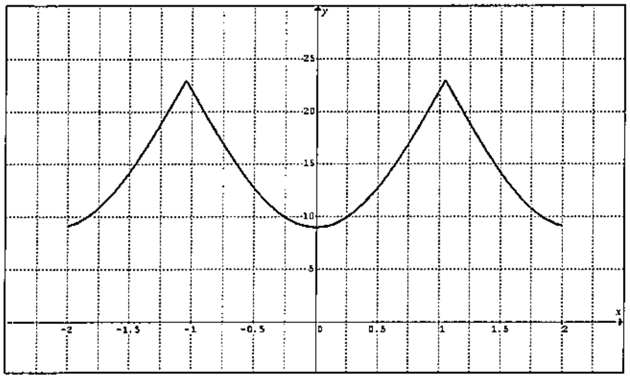
$$h(1.9) \approx 80 + 128(-0.1) + \frac{488}{3}\frac{(-0.1)^2}{2!} + \frac{448}{3}\frac{(-0.1)^3}{3!}$$ (no need to simplify)

C] $h(x)$ is centered at $x = 2$; $a = 2$ $x_0 = 1.9$

$$|\text{error}| = |R_3(1.9)| = \left|\frac{f'''_{max}(c)(1.9-2)^4}{4!}\right| \le \frac{\frac{584}{9}(-0.1)^4}{4!} < 0.0003$$

$\to 0.00027\ldots$

$1.9 \le c \le 2$



HW 11. Consider a function $f(x)$ which has non-zero real derivatives of all orders. A graph of $|f'''(x)|$ on (-2,2) is show above. Show that $|f(0.5)-P_2(0.5)|<\frac{3}{8}$ where $P_2(x)$ is a Taylor polynomial of second degree centered at zero.

$$|\text{error}(0.5)| = |R_3(0.5)| = \left|\frac{f'''_{max}(c)\,(0.5-0)^3}{3!}\right| \le \frac{15\,(0.5)^3}{3!} = \frac{5}{2^3\cdot 2} = \frac{5}{2^4}$$

$$= \frac{5}{16} < \frac{6}{16} = \frac{3}{8} \checkmark$$

* Carefull ... Do not pick $f'''(c)\ max = 12.5$ ... You cannot say for sure this is true. You can say $f'''(c)\ max = 15$ as a ceiling with certainty!

12. Use the first two nonzero terms of the Maclaurin series to approximate sin(x). Estimate the maximum error if $|x|<1$.

- Each x_0 must be restricted to the interval $-1 < x_0 < 1$
- Since the center is $x=0$ and c must be between zero and x_0, it concludes that c must also be restricted to the interval $-1 < c < 1$.

0 $\sin x$	4 $\sin x$
1 $\cos x$	5. $\cos x \leftarrow f^5(x)$
2 $-\sin x$	
3 $-\cos x$	

- Now $\sin x \approx x - \frac{x^3}{3!} = P_3(x) = P_4(x)$

$$|\text{error}| = |R_4(x)| = \left|\frac{f^5_{max}(c)\,(x-0)^5}{5!}\right| < \left|\frac{\cos(c)_{max}\,x^5}{5!}\right| \le \left|\frac{(1)x^5}{5!}\right| < \frac{(1)(1)^5}{5!} = \frac{1}{120}$$

$-1 < c < 1$ $\quad$ $-1 < c < 1$ $\quad$ $-1 < x < 1$

$-1 < x < 1$ $\quad$ $-1 < x < 1$

* $|\cos c| < 1$