$\boxed{\text{Pg. } 483}$ HW Solutions 9B Lesson 1

#55 $\dfrac{x}{1-2x}$

$$\frac{x}{1-2x} = x + x(2x) + x(2x)^2 + \dots + x(2x)^3 + \dots x(2x)^n + \dots$$

Geom: $a_1 = x$ $r = 2x$

$$\frac{x}{1-2x} = \sum_0^\infty 2^n x^{n+1} \quad \text{for } -\tfrac{1}{2} < x < \tfrac{1}{2}$$

#57. $$\frac{1}{1+(x-4)} = 1 - (x-4) + (x-4)^2 - \dots = \sum_0^\infty (-1)^n (x-4)^n \text{ for } 3<x<5$$

Geom: $a_1 = 1$ $r = -(x-4)$

#59. $$\frac{1}{2-x} = \frac{1}{2(1-\frac{x}{2})} = \frac{1}{2} \cdot \frac{1}{1-\frac{x}{2}} = \frac{1}{2}\left(1 + \frac{x}{2} + \frac{x^2}{4} + \dots + \frac{x^n}{2^n} + \dots\right)$$

$$= \frac{1}{2}\sum_0^\infty \left(\frac{x}{2}\right)^n = \sum_0^\infty \left(\frac{1}{2}\right)\left(\frac{x}{2}\right)^n \text{ for } -2<x<2$$

$\boxed{* \text{ or } \frac{1}{1-(x-1)} \quad a_1 = 1,\ r = (x-1)}$

$\boxed{\text{or } \sum_0^\infty (x-1)^n \text{ for } 0<x<2}$

#72. $$\frac{4}{1+t^2} = 4\left(\frac{1}{1+t^2}\right) = 4\left(\overbrace{1 - t^2 + t^4 - \dots + (-1)^n t^{2n} + \dots}^{-1<t<1}\right)$$

$$= 4 - 4t^2 + 4t^4 - \dots + (-1)^n(4)t^{2n} + \dots$$

$$G(x) = \int_0^x f(t)\,dt = \int_0^x (4 - 4t^2 + 4t^4 - \dots + (-1)^n 4t^{2n} + \dots)\,dt$$

$$= \left[4t - \frac{4t^3}{3} + \frac{4t^5}{5} - \dots + (-1)^n \frac{4t^{2n+1}}{2n+1} + \dots\right]_0^x$$

$$= 4\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \dots + (-1)^n \frac{x^{2n+1}}{2n+1} + \dots\right)$$

$x = -1$: $$4\sum \frac{(-1)^n(-1)^{2n+1}}{2n+1} = 4\sum \frac{(-1)^n \cancel{(-1)^{2n}}^{\,1} \cdot (-1)}{2n+1} = -4\sum \frac{(-1)^n}{2n+1}$$ converges A.S.T

$x = 1$: $4\sum \frac{(-1)^n}{2n+1}$ converges A.S.T

Terms strictly alternate and decrease in absolute value to zero

$[-1 \le x \le 1]$

also note $G(x) = 4\tan^{-1}(x)$:)

\#63 $\quad \dfrac{1}{x} = \dfrac{1}{1+(x-1)} = 1-(x-1)+(x-1)^2 - \cdots$

for $0<x<2$ center $=1$

$$\ln x = \int \frac{1}{x}\,dx \qquad \ln x = \int_{1 \leftarrow \text{center}}^{x} \frac{1}{t}\,dt$$

$x>0$

$$= \int_1^x 1-(t-1)+(t-1)^2 - \cdots$$

for Interval of Convergence

$$\boxed{0 < x \le 2}$$

$$= \left[t - \frac{(t-1)^2}{2} + \frac{(t-1)^3}{3} - \cdots \right]_1^x$$

$$= \left(x - \frac{(x-1)^2}{2} - \frac{(x-1)^3}{3} + \cdots \right) - \left(1 - 0 + 0 \cdots \right)$$

$$= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots + \frac{(-1)^{n-1}(x-1)^n}{n} + \cdots$$

\#64 $\quad \dfrac{2}{(1-x)^3} = \dfrac{d}{dx}\left(\dfrac{1}{(1-x)^2}\right) = \dfrac{d}{dx}\left(\dfrac{d}{dx}\left(\dfrac{1}{1-x}\right)\right) \quad -1<x<1$

$$= \frac{d}{dx}\left(\frac{d}{dx}\left(\overbrace{1+x+x^2+x^3+\cdots}\right)\right)$$

$$= \frac{d}{dx}\left(1+2x+3x^2+\overset{4x^3}{\cdots}+nx^{n-1}+\cdots\right)$$

$$= 2+6x+12x^2+\cdots+(n)(n-1)x^{n-2}+\cdots$$

$$= \sum_{2}^{\infty}(n)(n-1)x^{n-2} \quad \text{or} \quad \sum_{1}^{\infty}(n+1)(n)x^{n-1}$$

Interval of convergence

$-1 < x < 1$

$$\text{or} \quad \sum_{0}^{\infty}(n+2)(n+1)x^{n}$$

492 : #3 $f(x) = \frac{1}{x+2}$

$f(x) = (x+2)^{-1}$ $f(0) = 1 \cdot \frac{1}{2}$ $f''''(x) = 24(x+2)^{-5}$ $f^4(0) = 24 \cdot \frac{1}{32}$

$f'(x) = -(x+2)^{-2}$ $f'(0) = -1 \cdot \frac{1}{4}$ $f^5(x) = 120(x+2)^{-6}$ $f^5(0) = 120 \cdot \frac{1}{64}$

$f''(x) = 2(x+2)^{-3}$ $f''(0) = 2 \cdot \frac{1}{8}$

$f'''(x) = -6(x+2)^{-4}$ $f'''(0) = -6 \cdot \frac{1}{16}$

$$f(x) = \frac{1}{2} - \frac{x}{4} + \frac{2x^2}{8 \cdot 2!} - \frac{6x^3}{16 \cdot 3!} + \dots$$

$$= \frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} - \frac{x^3}{16} + \dots \frac{(-1)^n x^n}{2^{n+1}} + \dots \quad \text{for } \boxed{-2 < x < 2}$$

$$f(x) \approx P_5(x) = \frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} - \frac{x^3}{16} + \frac{x^4}{32} - \frac{x^5}{64}.$$

492 : #5 $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \dots$

$$\sin(2x) = (2x) - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} + \dots + \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!} + \dots \quad \boxed{\forall x}$$

492 : #7 $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots \frac{(-1)^n (x)^{2n+1}}{(2n+1)} + \dots \quad -1 \le x \le 1$

$$\tan^{-1}(x^2) = x^2 - \frac{(x^2)^3}{3} + \frac{(x^2)^5}{5} - \dots \frac{(-1)^n (x^2)^{2n+1}}{2n+1} + \dots$$

$$= \sum_0^\infty \frac{(-1)^n x^{4n+2}}{2n+1} \quad \boxed{-1 \le x \le 1}$$

492 : #10 $x^2 \cos x = x^2 \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \right)$

$$= x^2 - \frac{x^4}{2!} + \frac{x^6}{4!} - \dots (-1)^n \frac{x^2 (x)^{2n}}{(2n)!} + \dots$$

$$= x^2 - \frac{x^4}{2!} + \frac{x^6}{4!} - \dots + \frac{(-1)^n x^{2n+2}}{(2n)!} + \dots \quad \boxed{\forall x}$$

#12 $e^{-2x} = 1 + (-2x) + \frac{(-2x)^2}{2!} + \frac{(-2x)^3}{3!} + \dots + \frac{(-2x)^n}{n!} + \dots$

$= 1 - 2x + \frac{4x^2}{2!} - \frac{8x^3}{3!} + \dots + \frac{(-1)^n 2^n x^n}{n!} + \dots$ $\boxed{\forall x}$

#22 $f(0) = 4 \quad f'(0) = 5 \quad f''(0) = -8 \quad f'''(0) = 6$

a) $f(x) \approx 4 + 5x - \frac{8x^2}{2!} + \frac{6x^3}{3!}$

$= 4 + 5x - 4x^2 + x^3$

$f(0.2) \approx 4 + 5(0.2) - 4(0.2)^2 + (0.2)^3 = 4.848$

#24 $f(x) = 1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \dots + \frac{x^n}{(n+1)!} + \dots$

$f'(0)x = \frac{x}{2!} \quad \therefore f'(0) = \frac{1}{2!} = \frac{1}{2}$

$\frac{f^{10}(0)x^{10}}{10!} = \frac{x^{10}}{11!} \quad \therefore f^{10}(0) = \frac{1}{11}$

$g(x) = x f(x) = \underbrace{x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots + \frac{x^n}{n!} + \dots}_{\text{like } e^x \text{ but missing a } 1}$

$g(x) = e^x - 1$

#27 $f(0) = (1+0)^{1/2} = 1$

$f'(0) = \frac{1}{2}(1+0)^{-1/2} = \frac{1}{2}$

$f''(0) = -\frac{1}{4}(1+0)^{-3/2} = -\frac{1}{4}$

$f'''(0) = \frac{3}{8}(1+0)^{-5/2} = \frac{3}{8}$

RAW CONSTRUCT

a) $\sqrt{1+x} \approx 1 + \frac{1}{2}x - \frac{x^2}{4 \cdot 2!} + \frac{3x^3}{8 \cdot 3!}$

$= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} = P_4(x)$

DIFFEQ

c) $h(x) = \int g(x)dx = x + \frac{x^3}{6} - \frac{x^5}{40} + \dots + C$

ANTIDERIVATIVE

$h(0) = 5 \quad \therefore C = 5$

$\therefore h(x) \approx 5 + x + \frac{x^3}{6} - \frac{x^5}{40}$

Substitute known

b) $g(x) = 1 + \frac{(x^2)}{2} - \frac{(x^2)^2}{8} + \frac{(x^2)^3}{16} + \dots$

$= 1 + \frac{x^2}{2} - \frac{x^4}{8} + \frac{x^6}{16} + \dots$