Lesson 3 — HW Solutions

1. ♡ RATIO TEST $|a_n| = \left|\frac{x^n}{(n)\sqrt{n}\ 3^n}\right|$ $\quad |a_{n+1}| = \left|\frac{x^{n+1}}{(n+1)\sqrt{n+1}\ 3^{n+1}}\right|$

ii) $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)\sqrt{n+1}\ 3^{n+1}} \cdot \frac{n\sqrt{n}\ 3^n}{x^n}\right|$$

$$= \lim_{n\to\infty} \frac{n\sqrt{n}}{3(n+1)\sqrt{n+1}}\left|x\right| = \frac{1}{3}|x|$$

$$\frac{1}{3}|x| < 1 \rightarrow |x| < 3 \longrightarrow -3 < x < 3$$

Test Endpoints

$x = 3$: $\sum_1^\infty \frac{(3)^n}{n\sqrt{n}\ 3^n} = \sum_1^\infty \frac{1}{n^{3/2}}$ converges as p-series $p = 3/2$

$x = -3$: $\sum_1^\infty \frac{(-3)^n}{n\sqrt{n}\ 3^n} = \sum_1^\infty \frac{(-1)^n}{n\sqrt{n}}$ converges by the A.S.T

terms strictly alt, and dec. in abs. to zero

$$\boxed{-3 \le x \le 3}$$

Center $x = 0$

radius $\rho = 3$

2. ♡ RATIO TEST $|a_n| = \left|\frac{n x^n}{4^n(n^2+1)}\right|$ $\quad |a_{n+1}| = \left|\frac{(n+1)\, x^{n+1}}{4^{n+1}((n+1)^2+1)}\right|$

ii) $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)\, x^{n+1}}{4^{n+1}((n+1)^2+1)} \cdot \frac{4^n(n^2+1)}{n\ x^n}\right|$$

$$= \lim_{n\to\infty} \frac{1}{4}\frac{(n+1)(n^2+1)}{((n+1)^2+1)n}\left|x\right| = \frac{1}{4}|x|$$

$$|x| < 4 \rightarrow -4 < x < 4$$

Test Endpoints

$x = -4$: $\sum_0^\infty \frac{n(-4)^n}{4^n(n^2+1)} = \sum_1^\infty \frac{(-1)^n n}{n^2+1}$ converges by A.S.T

Terms strictly alt and dec in abs. value to zero.

$x = 4$: $\sum_0^\infty \frac{n(4)^n}{4^n(n^2+1)} = \sum_1^\infty \frac{n}{n^2+1}$ ... diverges by L.C.T

(SHOW DETAILS)

yes true

$a_n = \frac{n}{n^2+1}$ $\quad b_n = \frac{1}{n}$

i) a_n and b_n are pos, dec and cont on 1, 2, 3...

ii) $\sum_1^\infty b_n = \sum_1^\infty \frac{1}{n}$ diverges as the harmonic Series

iii) $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{n^2+1} \cdot \frac{n}{1} = 1 \in (0, \infty)$

$\therefore \sum a_n = \sum_1^\infty \frac{n}{n^2+1}$ diverges by the L.C.T

$$\boxed{-4 \le x < 4}$$

Center $x = 0$

radius $\rho = 4$

3. ♡ RATIO TEST $|a_n| = \left|\frac{x^{2n+1}}{n!}\right|$ $|a_{n+1}| = \left|\frac{x^{2(n+1)+1}}{(n+1)!}\right|$

ii) $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+3}}{(n+1)!}\cdot\frac{n!}{x^{2n+1}}\right| = \lim_{n\to\infty}\frac{1}{n+1}\left|x^2\right| = 0|x^2|$

$0 \cdot |x^2| < 1$ always!

$\therefore$ The series will converge for all values of x.

Interval $\boxed{-\infty < x < \infty}$

Center $x = 0$

radius $\rho = \infty$

4. ♡ RATIO TEST $|a_n| = |n!(x-4)^n|$ $|a_{n+1}| = |(n+1)!(x-4)^{n+1}|$

ii) $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!(x-4)^{n+1}}{n!(x-4)^n}\right| = \lim_{n\to\infty} n+1\left|x-4\right| \to \infty|x-4|$

$\lim_{A\to\infty} A|x-4|$ only converges when $x = 4$

Interval of convergence: $x = 4$

Center of convergence: $x = 4$

radius of convergence: $\rho = 0$

NOTE: $x = 4$; $\sum_{1}^{\infty} 0 = 0 + 0 + \dots = \underline{\underline{0}}$

5, 6 Convergences A.L.T

The terms strictly alternate and decrease in absolute value to zero

7. Diverges as the series fails the nth term test (factorial will always out-grow exponential)

8. $\sum_{1}^{\infty}\frac{\cos(n\pi)}{n} = \frac{\cos(\pi)}{1} + \frac{\cos(2\pi)}{2} + \frac{\cos(3\pi)}{3} + \dots$

$= -\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots\right)$

Converges as this Series is the ALTernating Harmonic.

9. $\sum_{n=2}^{\infty} (-1)^{n-1} \frac{1}{n \ln n}$ Converges by the A.S.T

$\sum_{2}^{\infty} \left| (-1)^{n-1} \frac{1}{n \ln n} \right| = \sum_{2}^{\infty} \frac{1}{n \ln n}$

$a_n = \frac{1}{n \ln n} \quad f(x) = \frac{1}{x \ln x}$

$$\int_2^{\infty} \frac{dx}{x \ln x} = \lim_{b \to \infty} \int_2^{b} \frac{dx}{x \ln x} = \lim_{b \to \infty} \ln|\ln x| \Big]_2^b = \lim_{n \to \infty} \left[\ln|\ln b| - \ln|\ln 2| \right] = \infty$$

Diverges

$\int \frac{dx}{x \ln x} = \int \frac{du}{u} = \ln|u| + c = \ln|\ln x| + c$

$u = \ln x$

$du = \frac{dx}{x}$

$dx = x\, du$

$\therefore \sum_{2}^{\infty} a_n = \sum_{2}^{\infty} \frac{1}{n \ln n}$ diverges by the Integral Test.

$\sum_{n=2}^{\infty} (-1)^{n-1} \frac{1}{n \ln n}$ is classified as a conditionally convergent.

10. $\sum_{n=1}^{\infty} \frac{(-1)^n}{1+\sqrt{n}}$ converges by the A.S.T

$\sum_{1}^{\infty} \left| \frac{(-1)^n}{1+\sqrt{n}} \right| = \sum_{1}^{\infty} \frac{1}{1+\sqrt{n}}$ diverges $\longrightarrow$ L.C.T $b_n = \frac{1}{\sqrt{n}} \quad a_n = \frac{1}{1+\sqrt{n}}$

ii) $\sum b_n = \sum \frac{1}{\sqrt{n}}$ diverges p-series $p = 1/2$ iii) $\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{\frac{1}{1+\sqrt{n}}}{\frac{1}{\sqrt{n}}} = \lim_{n \to \infty} \frac{\sqrt{n}}{1+\sqrt{n}} = 1 \in (0, \infty)$

$\therefore \sum_{1}^{\infty} a_n = \sum_{1}^{\infty} \frac{1}{1+\sqrt{n}}$ diverges by the L.C.T

$\sum_{1}^{\infty} \frac{(-1)^n}{1+\sqrt{n}}$ is classified as a condinally convergent series

11. Diverges : Fails the nth term test: $\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{n!}{2^n} = \infty \neq 0$

12. $\sum_{1}^{\infty} \frac{\cos(n\pi)}{n}$ Converges as the alt. harmonic

$\sum_{1}^{\infty} \left| \frac{\cos(n\pi)}{n} \right|$ diverges as the harmonic

$\therefore \sum_{1}^{\infty} \frac{\cos(n\pi)}{n}$ is classified as conditionally convergent.