Lesson 1 – Homework Solutions

#1 7) $\sum_{1}^{\infty} \frac{5}{n+1}$ $a_n = \frac{5}{n+1}$ $f(x) = \frac{5}{x+1}$

i) $f(x)$ is continous, decreasing, and positive on $[1, \infty)$

ii) $\int_{1}^{\infty} f(x)\,dx = \int_{1}^{\infty} \frac{5}{x+1}\,dx = \lim_{b \to \infty} \int_{1}^{b} \frac{5}{x+1}\,dx$

$= 5 \lim_{b \to \infty} \int_{1}^{b} \frac{dx}{x+1} = 5 \lim_{b \to \infty} \ln(x+1) \Big]_{1}^{b}$

$= 5 \lim_{b \to \infty} \underbrace{\ln(b+1)}_{\to \infty} - \ln(2) = \infty$ (diverges)

$\therefore \sum_{1}^{\infty} a_n = \sum_{1}^{\infty} \frac{5}{n+1}$ diverges by the I. T.

#2 10) $\sum_{1}^{\infty} \frac{1}{2n-1}$ $a_n = \frac{1}{2n-1}$ $f(x) = \frac{1}{2x-1}$

i) $f(x)$ is continous, decreasing, and positive on $[1, \infty)$

ii) $\int_{1}^{\infty} f(x)\,dx = \int_{1}^{\infty} \frac{1}{2x-1}\,dx = \lim_{b \to \infty} \int_{1}^{b} \frac{1}{2x-1}\,dx$

$= \frac{1}{2} \lim_{b \to \infty} \int_{1}^{b} \frac{1}{x - 1/2} = \frac{1}{2} \lim_{b \to \infty} \ln(x - 1/2) \Big]_{1}^{b}$

$= 1/2 \lim_{b \to \infty} \underbrace{\ln(b - 1/2)}_{\to \infty} - \ln(1/2) = \infty$ diverges

$\therefore \sum_{1}^{\infty} a_n = \sum_{1}^{\infty} \frac{1}{2n-1}$ diverges by the I.T

#3 a) $\sum_{3}^{\infty} \frac{\ln(n)}{n}$ Let $a_n = \frac{\ln(n)}{n}$ $b_n = \frac{1}{n}$

i) a_n, b_n, are positive and continuous on $3, 4, 5, \ldots$

ii) $\sum_{3}^{\infty} b_n = \sum_{3}^{\infty} 1/n$ which diverges by p-series ($p=1$)
(or diverges as the harmonic series)

iii)

$a_n = \frac{\ln(n)}{n} > \frac{1}{n} = b_n$ on $3, 4, 5, \ldots$

$\therefore \sum_{3}^{\infty} a_n = \sum_{3}^{\infty} \frac{\ln(n)}{n}$ diverges by D.C.T

#4 15) $\sum_{1}^{\infty} \frac{\sqrt{n}}{n^2+1}$ Let $a_n = \frac{\sqrt{n}}{n^2+1}$ $b_n = \frac{1}{n^{3/2}}$

i) a_n, b_n are positive, and continuous on $1, 2, 3, \ldots$

ii) $\sum_{1}^{\infty} b_n = \sum_{1}^{\infty} 1/n^{3/2}$ converges by p-series ($p = 3/2$) > 1

iii)

$a_n = \frac{\sqrt{n}}{n^2+1} = \frac{\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}} \cdot \frac{\sqrt{n}}{(n^2+1)} = \frac{1}{n^{3/2} + n^{-1/2}} < \frac{1}{n^{3/2}} = b_n$

$\therefore \sum_{1}^{\infty} a_n = \sum_{1}^{\infty} \frac{\sqrt{n}}{n^2+1}$ converges by D.C.T

OR * OR

$\frac{\sqrt{n}}{n^2+1} - \frac{1}{n^{3/2}} = \frac{\sqrt{n}\, n^{3/2} - (n^2+1)}{(n^2+1)(n^{3/2})} = \frac{-1}{(n^2+1)(n^{3/2})} < 0$ for $n = 1, 2, 3, \ldots$

This means that $\frac{1}{n^{3/2}} > \frac{\sqrt{n}}{n^2+1}$ on $1, 2, 3, \ldots$

#5 16) $\sum_{1}^{\infty} \frac{5n^3-3n}{n^2(n+2)(n^2+5)}$ Let $a_n = \frac{5n^3-3n}{n^2(n+2)(n^2+5)}$

$b_n = n^3/n^5 = 1/n^2$

i) a_n, b_n are positive and continuous on $1, 2, 3, \ldots$

ii) $\sum_{1}^{\infty} b_n = \sum_{1}^{\infty} 1/n^2$ converges by p-series ($p=2$)

iii) $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{5n^3-3n}{n^2(n+2)(n^2+5)}}{1/n^2} = \lim_{n\to\infty} \frac{5n^3-3n}{\cancel{n^2}(n+2)(n^2+5)} \times \frac{\cancel{n^2}}{1}$

$= \lim_{n\to\infty} \frac{5n^3-3n}{(n+2)(n^2+5)} = \lim_{n\to\infty} \frac{5n^3-3n}{n^3+2n^2+5n+10} = 5 \in (0,\infty)$

$\therefore \sum_{1}^{\infty} a_n = \sum_{1}^{\infty} \frac{5n^3-3n}{n^2(n+2)(n^2+5)}$ converges by L.C.T

#10 8) $\sum_{1}^{\infty} \frac{3}{\sqrt{n}} = 3\sum_{1}^{\infty} 1/\sqrt{n}$ diverges by p-series ($p = 1/2$)

#11 11) $\sum_{1}^{\infty} 1/(\ln(2))^n = \sum_{1}^{\infty} \left(\frac{1}{\ln(2)}\right)^n$ diverges as geometric $r = \frac{1}{\ln 2} > 1$

$\because \ln 2 < 1 \therefore \frac{1}{\ln 2} > 1$

#12 7) $\sum_{n=1}^{\infty} \frac{5}{n+1} = 5\sum_{n=1}^{\infty} \frac{1}{n+1} = 5\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\right)$

$= 5$ (the harmonic series) diverges

or $= 5$ (p-series $p = 1$) diverges

#6 2a) $\sum_{1}^{\infty} \frac{n}{n+1}$ $a_n = \frac{n}{n+1}$ (using n^{th} term test)

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0 \therefore$ diverges

FAILS THE nth Term Test

#7 32) $\sum_{1}^{\infty} 1/8^n$ (using geometric sequence)

$= \sum_{1}^{\infty} (1/8)^n$ Converges as geometric $r = \frac{1}{8} < 1$

$S = \frac{a_1}{1-r} = \frac{\frac{1}{8}}{1-\frac{1}{8}} = \frac{1/8}{7/8} = \boxed{\frac{1}{7}}$ ← converges to $\frac{1}{7}$

#8 38) $\sum_{1}^{\infty} (1 + 1/n)^n$ (n^{th} term test) $a_n = (1 + 1/n)^n$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (1 + 1/n)^n = e \neq 0$ Fails the nth term test

Proof: $\lim_{n\to\infty} e^{\ln(y)} = e^{\lim_{n\to\infty} \ln(y)} = \boxed{e^1}$

$y = (1 + 1/n)^n$

$\ln(y) = n \ln(1 + 1/n)$

$\lim_{n\to\infty} \ln(y) = \lim_{n\to\infty} n \ln(1 + 1/n)$ $\{0 \cdot \infty\}$

$\lim_{n\to\infty} \ln(y) = \lim_{n\to\infty} \frac{\ln(1 + 1/n)}{1/n}$ $\{0/0\}$

$\lim_{n\to\infty} \ln(y) = \lim_{n\to\infty} \frac{\left(\frac{1}{1+1/n}\right)(-1/n^2)}{-1/n^2} = \lim_{n\to\infty} \left(\frac{1}{1+1/n}\right) = 1$

#9 39) $\sum_{1}^{\infty} \frac{(2)^n}{(3)^n} = \sum_{1}^{\infty} \left(\frac{2}{3}\right)^n$ converges as a geometric

with $r = 2/3 < 1$

$S = \frac{2/3}{1-2/3} = \frac{2/3}{1/3} = \boxed{2}$ ← converges to