BC Lesson 2 Hw Solutions

#2. A] $\int_1^\infty x^{-1/3}dx$ diverges as p-integral $p = 1/3 \le 1$

B] $\lim_{b\to\infty} \int_1^b x^{-1/3}dx$

C] Does not apply

11. A] CANNOT BE DETERMINED BY p-integral observation (not a p-integral)

B, C] $\lim_{a\to-\infty} \int_a^{-2} \frac{2dx}{x^2-1} = \lim_{a\to-\infty} \left[\ln|x-1| - \ln|x+1|\right]_a^{-2} = \lim_{a\to-\infty} \left[\ln(3) - \ln(1) - \ln|a-1| + \ln|a+1|\right]$

$= \lim_{a\to-\infty}\left[\ln(3) + \ln\left|\frac{a+1}{a-1}\right|\right] = \ln(3) + \lim_{a\to-\infty}\ln\left|\frac{a+1}{a-1}\right| = \ln(3) + \ln\left|\lim_{a\to-\infty}\frac{a+1}{a-1}\right|$

$= \ln(3) + 0$

$= \boxed{\ln(3)}$

* NOTE $\frac{2}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1} = \frac{1}{x-1} - \frac{1}{x+1}$

* $\lim_{a\to-\infty} \frac{a+1}{a-1} = \lim_{a\to-\infty} 1 = 1$

$|x| = \begin{cases} x & ; x \ge 0 \\ -x & ; x < 0 \end{cases}$

#21. A] NOT A P-INTEGRAL

B, C] $\int_{-\infty}^{\infty} e^{-|x|}dx = \int_{-\infty}^{0} e^{x}dx + \int_0^\infty e^{-x}dx = 2\int_0^\infty e^{-x}dx = 2(1) = \boxed{2}$

□ $\int_0^\infty e^{-x}dx = \lim_{b\to\infty}\int_0^b e^{-x}dx = \lim_{b\to\infty}\left[-e^{-x}\right]_0^b = \lim_{b\to\infty}\left[\cancel{-e^{-b}} + e^0\right] = 1$

* $e^{-|x|} = \begin{cases} e^{x} & x \ge 0 \\ e^{-x} & x < 0 \end{cases}$

* $y = e^{-|x|}$ is symmetric about the y-axis

#25. A] NOT A P-INTEGRAL

B, C] $\int_0^2 \frac{dx}{1-x^2} = \int_0^1 \frac{dx}{1-x^2} + \int_1^2 \frac{dx}{1-x^2} = \lim_{b\to1^-}\int_0^b \frac{dx}{1-x^2} + \lim_{a\to1^+}\int_a^2 \frac{dx}{1-x^2}$ [Diverges]

□ $\lim_{a\to1^+}\int_a^2 \frac{dx}{1-x^2} = \lim_{a\to1^+}\left[-\frac{1}{2}\ln|1-x| + \frac{1}{2}\ln|1+x|\right]_a^2 = \lim_{a\to1^+}\left[\frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|\right]_a^2 = \lim_{a\to1^+}\left[\frac{1}{2}\ln|3| - \frac{1}{2}\ln\left|\frac{1+a}{1-a}\right|\right] = -\infty$

(OR) $\equiv \lim_{a\to1^+}\left[\frac{1}{2}\ln\left|\frac{x+1}{x-1}\right|\right]_a^2$

$= \frac{1}{2}\ln(3) - \lim_{a\to1^+}\frac{1}{2}\ln\left|\frac{a+1}{a-1}\right| = -\infty$

* $\frac{1}{1-x^2} = \frac{A}{1-x} + \frac{B}{1+x} = \frac{1/2}{1-x} + \frac{1/2}{1+x}$

(OR) $\frac{-1}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1} = \frac{-1/2}{x-1} + \frac{1/2}{x+1}$

#35. A] NOT A P-INTEGRAL

B, C] $\int_0^{\ln 2} y^{-2}e^{1/y}dy = \lim_{a\to0^+}\int_a^{\ln 2} y^{-2}e^{1/y}dy = \lim_{a\to0^+}\left[-e^{1/y}\right]_a^{\ln 2} = \lim_{a\to0^+}\left[-e^{(\ln 2)^{-1}} + e^{1/a}\right]$

* $u = 1/y \quad du = \frac{-1}{y^2}dy$

$= -e^{(\ln 2)^{-1}} + \lim_{a\to0^+} e^{1/a} = \infty$ [Diverges]

#40. A] NOT A P. INTEGRAL

B] $\int_{-\infty}^{0} xe^x\,dx = \lim\limits_{a\to-\infty}\int_a^0 xe^x\,dx = \lim\limits_{a\to-\infty}\left[xe^x - e^x\right]_a^0$

NOTE: $x \quad e^x$
$1 \quad e^x$
$0 \quad e^x$
(TABULAR)

$= \lim\limits_{a\to-\infty}\left[-1 - \overset{\to 0}{\cancel{ae^a}} - \overset{\to 0}{\cancel{e^a}}\right] = -1$

See note

NOTE: $\lim\limits_{a\to-\infty} ae^a \quad \{-\infty \cdot 0\}$

$= \lim\limits_{a\to-\infty} \frac{a}{e^{-a}} \left\{\frac{\infty}{\infty}\right\} = \lim\limits_{a\to-\infty} \frac{1}{-e^{-a}} = 0$ (L'HOPITAL'S Rule)

#43. $\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim\limits_{b\to\infty}\int_1^b \frac{\ln x}{x^2}\,dx = \lim\limits_{b\to\infty}\left[-\frac{\ln x}{x} - \frac{1}{x}\right]_1^b = \lim\limits_{b\to\infty}\left[\overset{\to 0}{\cancel{-\frac{\ln b}{b}}} - \overset{\to 0}{\cancel{\frac{1}{b}}} + 1\right] = \boxed{1}$

See note

$u = \ln x \quad dv = \frac{1}{x^2}dx$

$du = \frac{1}{x}dx \quad v = \frac{-1}{x}$

$\int \frac{\ln x}{x^2}dx = \frac{-\ln x}{x} - \frac{1}{x} + c$

* NOTE: $\lim\limits_{b\to\infty} \frac{\ln b}{b} \left\{\frac{\infty}{\infty}\right\} = \lim\limits_{b\to\infty} \frac{1/b}{1} = \frac{0}{1} = 0$

#32. let $f(x) = \frac{1}{x^3+1}$ and $g(x) = \frac{1}{x^3}$

i) $f(x)$ and $g(x)$ are positive and continuous on $[1, \infty)$

ii) $\int_1^\infty g(x)\,dx = \int_1^\infty \frac{1}{x^3}dx$ converges by p-integral $p = 3 > 1$

iii) $f(x) = \frac{1}{x^3+1} < \frac{1}{x^3} = g(x)$ on $[1, \infty)$

$\therefore \int_1^\infty f(x)\,dx = \int_1^\infty \frac{1}{x^3+1}dx$ converges also by the D.C.T

#33. let $f(x) = \frac{2+\cos x}{x} \quad g(x) = \frac{1}{x}$

i) $f(x)$ and $g(x)$ are positive & continuous on $[\pi, \infty)$

ii) $\int_\pi^\infty g(x)\,dx = \int_\pi^\infty \frac{1}{x}dx$ diverges by p-integral $p = 1$

iii $f(x) = \frac{2+\cos x}{x} \geq \frac{2-1}{x} = \frac{1}{x} = g(x)$ on $[\pi, \infty)$

$\therefore \int_1^\infty f(x)\,dx = \int_1^\infty \frac{2+\cos x}{x}dx$ diverges also by the D.C.T

SUPP: let $f(x) = \frac{1}{\sqrt{x^2-1}} \quad g(x) = \frac{1}{x}$

i) $f(x)$ and $g(x)$ are positive and continuous on $[2, \infty)$

ii) $\int_2^\infty g(x)\,dx = \int_2^\infty \frac{1}{x}dx$ diverges by p. integral

iii) $f(x) = \frac{1}{\sqrt{x^2-1}} > \frac{1}{\sqrt{x^2}} = \frac{1}{x} = g(x)$ on $[2, \infty)$

$\therefore \int_1^\infty f(x)\,dx = \int_1^\infty \frac{1}{\sqrt{x^2-1}}dx$ diverges also by the D.C.T