BC. Q204 LESSON 2

1. $\dfrac{dy}{dx} = \dfrac{3x^2+1}{2y}$ $\qquad f'(x,y) = \dfrac{3x^2+1}{2y}$

a) $x=1 \rightarrow y=4$ b/c $f(1)=4$

$$f'(1,4) = \frac{3(1)^2+1}{2(4)} = \frac{1}{2}$$

b) $L(x) = f(1) + f'(1,4)(x-1)$

$L(x) = 4 + \frac{1}{2}(x-1)$

$f(1.2) \approx L(1.2) = 4 + \frac{1}{2}(0.2) = \boxed{4.1}$

c) $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{3x^2+1}{2y}\right) = \frac{2y(6x) - (3x^2+1)(2)\frac{dy}{dx}}{(2y)^2}$$

$$\left.\frac{dy}{dx}\right|_{x=1,4} = \frac{1}{2}$$

$$f''\left(x=1, y=4, \frac{dy}{dx}=\frac{1}{2}\right) = \frac{2(4)(6)(1) - (3+1)(2)(\frac{1}{2})}{(8)^2} = \frac{48-4}{64} = \frac{44}{64} = \frac{11}{16} > 0$$

The graph of f is concave up at the point (1,4). Linear approximations, centered at (1,4), will underestimate f for x-values near x = 1. The approximation found in part b is an under-estimation.

D) $(x_0\ y_0)$

x_0	y_0
1	4
0	7/2

$$\hat{y}_1 = 4 + \left[\frac{3(1)^2+1}{2(4)}\right](-1.0) = 4 - \frac{1}{2} = \frac{7}{2}$$

$$\hat{y}_2 = \frac{7}{2} + \left[\frac{3(0)^2+1}{2(7/2)}\right](-1.0) = \frac{7}{2} - \frac{1}{7} = \frac{47}{14} \rightarrow \left(-1, \frac{47}{14}\right)$$

$f(-1) \approx E(-1) = \frac{47}{14}$

E] $\int 2y\,dy = \int (3x^2+1)\,dx$

$y^2 = x^3 + x + C$

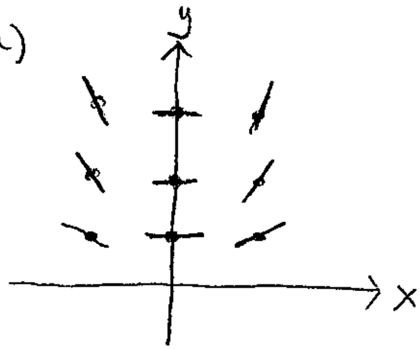
$y = \pm\sqrt{x^3+x+C}$ $\rightarrow$ $4 = \sqrt{1+1+C}$

$\therefore C = 14$

$$\boxed{y = \sqrt{x^3+x+14}}$$

2. $\frac{dy}{dx} = \frac{xy}{2}$

a)



b) $(x_0 \;\; y_0)$

$(0 \;\; 3) \quad \hat{y}_1 = 3 + \left[\frac{0(3)}{2}\right] 0.1 = 3$

$(0.1, 3) \quad \hat{y}_2 = 3 + \left[\frac{(0.1)(3)}{2}\right] 0.1 = 3.015 \rightarrow (0.2, 3.015)$

$f(0.2) \approx E(0.2) = 3.015$

c)

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{xy}{2}\right) = \frac{1}{2}\left(x\frac{dy}{dx} + y\right)$

$\left.\frac{dy}{dx}\right|_{x=0,\, y=3} = 0$

$\left.\frac{d^2y}{dx^2}\right|_{x=0,\, y=3,\, \frac{dy}{dx}=0} = \frac{1}{2}(0 \cdot 0 + 3) = 3/2 > 0$

The approximation in part b is an underestimation to f(0.2).

D) $(x_0 \;\; y_0)$

$(0, 3) \quad \hat{y}_1 = 3 + \left(\frac{0(3)}{2}\right)\frac{1}{3} = 3$

$(1/3, 3) \quad \hat{y}_2 = 3 + \left(\frac{1/3 \cdot 3}{2}\right) \cdot \frac{1}{3} = 3 + \frac{1}{6} = \frac{19}{6}$

$(2/3, \frac{19}{6}) \quad \hat{y}_3 = \frac{19}{6} + \left(\frac{\frac{2}{3} \cdot \frac{19}{6}}{2}\right) \cdot \frac{1}{3} = \frac{19}{6} + \frac{19}{54} = \frac{95}{27} \rightarrow (1, \frac{95}{27})$

$\boxed{f(1) \approx E(1) = \frac{95}{27}}$

E) $\int \frac{dy}{y} = \int \frac{x}{2}\, dx$

$\ln|y| = \frac{x^2}{4} + c$

$|y| = ce^{x^2/4}$

$y = c^* e^{x^2/4}$

$3 = c^* e^0 \quad \therefore c^* = 3$

$\boxed{y = 3e^{x^2/4}}$

#3 $\frac{dy}{dx} = y + x \qquad y(0) = 1$

A] $L(x) = f(0) + f'(0,1)(x)$

$L(x) = 1 + [1+0]\,x$

$L(x) = 1 + x$

$y(1.2) \approx L(1.2) = 1 + 1.2 = \boxed{2.2}$

B] $(0,1): \quad \hat{y}_1 = 1 + [1+0]\,0.4 = 1.4 \rightarrow (0.4, 1.4)$

$(0.4, 1.4): \quad \hat{y}_2 = 1.4 + [1.4+0.4]\,0.4 = 2.12 \rightarrow (0.8, 2.12)$

$(0.8, 2.12): \quad \hat{y}_3 = 2.12 + [2.12+0.8]\,0.4 = 3.288 \rightarrow (1.2, 3.288)$

$y(1.2) \approx E(1.2) = \boxed{3.288}$

Additional HW

#4 $\frac{dy}{dx} = e^{y}(3x^2 - 6x)$ $\qquad f(1) = 0$

A] $L(x) = f(1) + f'(1,0)(x-1)$

$L(x) = 0 + [e^0(3-6)](x-1)$

$L(x) = -3(x-1)$

$f(1.2) \approx L(1.2) = -3(1.2-1) = -3(0.2) = -0.6$

$\frac{dy}{e^y} = (3x^2 - 6x)dx$

C] $\int e^{-y} dy = \int (3x^2 - 6x)dx$

$-e^{-y} = x^3 - 3x^2 + c$

$e^{-y} = -x^3 + 3x^2 + c$

$-y = \ln(-x^3 + 3x^2 + c)$

$y = -\ln(-x^3 + 3x^2 + c)$

$0 = -\ln(-1 + 3 + c)$

$-1 + 3 + c = 1$

$c = -1$

$\therefore \boxed{y = -\ln(-x^3 + 3x^2 - 1)}$

(OR) $-\frac{1}{e^y} = x^3 - 3x^2 + c$

$e^y = \frac{-1}{x^3 - 3x^2 + c}$

$y = \ln\left(\frac{-1}{x^3 - 3x^2 + c}\right)$

$y = \ln\left(\frac{1}{-x^3 + 3x^2 + c}\right)$

$c = -1$

$y = \ln\left(\frac{1}{-x^3 + 3x^2 - 1}\right)$

$y = \ln(1) - \ln(-x^3 + 3x^2 - 1)$

$\boxed{y = -\ln(-x^3 + 3x^2 - 1)}$

B] $f''(x,y) = \frac{d}{dx}[e^y(3x^2 - 6x)] = \frac{d}{dx}[3x^2e^y - 6xe^y]$

$\frac{d^2y}{dx^2} = 3x^2e^y\frac{dy}{dx} + 6xe^y - 6xe^y\frac{dy}{dx} - 6e^y$

$\left.\frac{d^2y}{dx^2}\right|_{x=1,\, y=0,\, \frac{dy}{dx}=-3} = 3(1)(1)(-3) + 6(1)(1) - 6(1)(1)(-3) - 6(1)$

$= \boxed{9} > 0$

$\therefore$ The approx in part (b) is an under-estimation

#5 $\frac{dy}{dx} = x(2-y)$

(a) $\int \frac{dy}{2-y} = \int x\,dx \rightarrow -\ln|2-y| = \frac{x^2}{2} + c \rightarrow \ln|2-y| = \frac{-x^2}{2} + c$

$|2-y| = Ce^{-x^2/2} \rightarrow 2-y = c^* e^{-x^2/2} \rightarrow y = 2 - c^* e^{-x^2/2}$

$5 = 2 - c^* e^{0}$ ∴ $c^* = -3$

$\boxed{y = 2 + 3e^{-x^2/2}}$

(b) $\lim\limits_{x\to\infty} 2 + 3e^{-x^2/2} = \lim\limits_{x\to\infty} 2 + \cancel{\frac{3}{e^{x^2/2}}}^{\,0} = 2$

(c) $y(0.5) = 2.5$

$y'(0.5, 2.5) = 0.5(2-2.5) = 0.5(-0.5) = -0.25$

$L(x) = 2.5 - 0.25(x-0.5)$

$y(1.5) \approx L(1.5) = 2.5 - 0.25(1.5-0.5) = 2.25$

distribute makes easier

(d) $\frac{d}{dx}\left(x(2-y)\right) = \frac{d}{dx}\left(2x - xy\right)$

$= 2 - \left(x\frac{dy}{dx} + y\right)\Big|_{x=0.5\;\; y=2.5\;\; \frac{dy}{dx}=-0.25}$

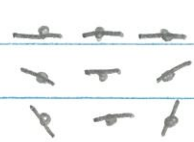
$= 2 - \left(\frac{1}{2}\left(-\frac{1}{4}\right) + \frac{5}{2}\right)$

$= 2 + \frac{1}{8} - \frac{5}{2} = \frac{16}{8} + \frac{1}{8} - \frac{20}{8} = -3/8 < 0$

∴ y is concave down at $(0.5, 2.5)$

∴ The approximation found in part (c) is an over estimation of y at $x = 1.5$

(e)



$f'(1,0) = 2 \quad f'(0,y) = 0$

$f'(1,2) = 1$

#6

A] $\frac{dy}{dx} = \frac{x}{y^2}$ $\quad \int y^2\,dy = \int x\,dx \quad \frac{y^3}{3} = \frac{x^2}{2} + c$

$$y^3 = \frac{3}{2}x^2 + c$$

$$y = \sqrt[3]{\frac{3}{2}x^2 + c}$$

$u = 1 + x^2$
$du = 2x\,dx$
$dx = \frac{du}{2x}$

B] $\frac{dy}{dx} = \frac{xy^2}{1+x^2}$ $\quad \int y^{-2}\,dy = \int \frac{x}{1+x^2}\,dx$

$-y^{-1} = \frac{1}{2}\ln|1+x^2| + c \qquad y = \frac{-2}{\ln|1+x^2| + c}$

C] $\frac{dy}{dx} = \frac{8y}{x(x-2)}$ $\quad \int \frac{dy}{8y} = \int \frac{dx}{x(x-2)} \Rightarrow \int \frac{dy}{8y} = \int \frac{-dx}{2x} + \int \frac{dx}{2(x-2)}$

$\frac{1}{x(x-2)} = \frac{A}{x} + \frac{B}{x-2} = \frac{-1}{2x} + \frac{1}{2(x-2)}$

$A(x-2) + B(x) = 1$

$(A+B) = 0$

$-2A = 1$

$A = -\frac{1}{2} \quad B = \frac{1}{2}$

$$\frac{\ln|y|}{8} = \frac{-\ln|x|}{2} + \frac{\ln|x-2|}{2} + C$$

$$\ln|y| = -4\ln|x| + 4\ln|x-2| + c$$

$$\ln|y| = 4\left(\ln|x-2| - \ln|x|\right) + C$$

$$\ln|y| = 4\ln\left|\frac{x-2}{x}\right| + c$$

$$\ln|y| = \ln\left(\frac{x-2}{x}\right)^4 + c$$

$$|y| = C\left(\frac{x-2}{x}\right)^4$$

$$y = c^*\left(\frac{x-2}{x}\right)^4$$