BC . Q101 . LESSON 1 HW SOLUTIONS

① (i) $f(1) = (1)^2 - 2(1) + 3 = 2$

(ii) $\lim_{x \to 1} f(x) = \lim_{x \to 1} x^2 - 2x + 3 = 2$

(iii) $\lim_{x \to 1} f(x) = f(1)$

$\therefore$ f is continuous at $x = 1$

② (i) $g(0) = 1 + e^0 = 2$

(ii) $\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} 2 + \sqrt{x} = 2$

$\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} 1 + e^{-x} = 2$

$\therefore \lim_{x \to 0} g(x) = 2$

(iii) $\lim_{x \to 0} g(x) = g(0)$

$\therefore$ g is continuous at $x = 0$

③ (i) $r(-1) = 5$

(ii) $\lim_{x \to -1} r(x) = \lim_{x \to -1} x + 2 = 1$

(iii) $\lim_{x \to -1} r(x) \neq r(-1)$

$\therefore$ r is not continuous at $x = -1$

④ (i) $f(0) = 0^2 + 2 = 2$

(ii) $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x^2 + 2 = 2$

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{1}{x} \to -\infty$ "DNE"

$\therefore \lim_{x \to 0} f(x)$ DNE

(iii) $\lim_{x \to 0} f(x) \neq f(0)$ $\therefore$ f is not continuous at $x = 0$

* NOTE: ④ STEP (i) could have been skipped in this problem

⑤ (i) $g(3)$ DNE

(iii) $\lim_{x \to 3} g(x) \neq g(3)$ "nothing equals $g(3)$"

$\therefore$ g is not continuous at $x = 3$

B] $g(x) = \begin{cases} \frac{x^2-9}{x-3} & x \neq 3 \\ 6 & x = 3 \end{cases}$ ← "6" plugs in the hole

⑥ (i) $d(4)$ DNE

(iii) $\lim_{x \to 4} d(x) \neq d(4)$

$\therefore$ d is not continuous at $x = 4$

B] $d(x) = \begin{cases} \frac{x-4}{\sqrt{x}-2} & x \neq 4 \\ 4 & x = 4 \end{cases}$ ← math shows us that $\lim_{x \to 4} d(x) = 4$

* $\lim_{x \to 4} d(x) = \lim_{x \to 4} \frac{(x-4)(\sqrt{x}+2)}{(\sqrt{x}-2)(\sqrt{x}+2)} = \lim_{x \to 4} \frac{\cancel{(x-4)}(\sqrt{x}+2)}{\cancel{x-4}} = \lim_{x \to 4} \sqrt{x}+2 = 4$ *

⑦ (i) $h(0) = 0$

(ii) $-1 \leq \sin \frac{1}{x} \leq 1$ ← Given $\sin \theta$ is alway between -1 and 1

$-x \leq x \sin \frac{1}{x} \leq x$ ← multiply all sides by x

$\lim_{x \to 0} -x = 0$

$\lim_{x \to 0} x = 0$

$\therefore$ $\lim_{x \to 0} x \sin \frac{1}{x} = 0$ (Sandwich Thm)

(iii) $\lim_{x \to 0} h(x) = h(0)$

$\therefore$ h is continuous at $x = 0$

(8) (a) $y - 3 = 5(x-2)$

(b) $y - 3 = -\frac{1}{5}(x-2)$

} Same point $(3,2)$

$m_{\text{Normal}} = -\text{reciprocal}$

(9) (a) $\underset{[-2,4]}{\text{Ave rate}\,\Delta} = \frac{f(4)-f(-2)}{4-(-2)} = \frac{(0)-(12)}{6} = \boxed{-2}$

(b) $f'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{[(1+h)^2-4(1+h)]-[-3]}{h}$

$= \lim_{h\to 0} \frac{1+2h+h^2-4-4h+3}{h} = \lim_{h\to 0} \frac{h(h-2)}{h} = \lim_{h\to 0} h-2 = \boxed{-2}$

(c) $f(1) = -3 \quad f'(1) = -2 \rightarrow \boxed{y+3 = -2(x-1)}$

(10) (a) $\underset{[-2,4]}{\text{Ave rate}\,\Delta} = \frac{f(4)-f(-2)}{4-(-2)} = \frac{[12]-[2]}{6} = \boxed{\frac{5}{3}}$

(b) $f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = ?$

- $f'_+(0) = \lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{[(0+h)^2-(0+h)]-[0]}{h} = \lim_{h\to 0^+} \frac{h^2-h}{h} = \lim_{h\to 0^+} h-1 = -1$

- $f'_-(0) = \lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{-(0+h)-0}{h} = \lim_{h\to 0^-} \frac{-h}{h} = \lim_{h\to 0^-} -1 = -1$

$\therefore \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = -1$; i.e $f'(0) = -1$

The graph of $f(x)$ is smooth at $x = 0$

(c) $f(0) = 0 \quad f'(0) = -1$; $\boxed{y = -(x)}$

(11) a) $\underset{[-4,0]}{\text{average rate}\,\Delta} = \dfrac{f(0)-f(-4)}{0-(-4)} = \dfrac{(0)-(-8)}{4} = \boxed{2}$

b) $f'(-2) = \lim\limits_{h\to 0} \dfrac{f(-2+h)-f(-2)}{h} = ?$

□ $f'_+(-2) = \lim\limits_{h\to 0^+} \dfrac{f(-2+h)-f(-2)}{h} = \lim\limits_{h\to 0^+} \dfrac{-(-2+h)^2-[-4]}{h}$

$= \lim\limits_{h\to 0^+} \dfrac{-4+4h-h^2+4}{h} = \lim\limits_{h\to 0^+} \dfrac{h(4-h)}{h}$

$= \lim\limits_{h\to 0^+} 4-h = \underline{4}$

□ $f'_-(-2) = \lim\limits_{h\to 0^-} \dfrac{f(-2+h)-f(-2)}{h} = \lim\limits_{h\to 0^-} \dfrac{2(-2+h)-[-4]}{h}$

$= \lim\limits_{h\to 0^-} \dfrac{-4+2h+4}{h} = \lim\limits_{h\to 0^-} \dfrac{2h}{h} = \lim\limits_{h\to 0^-} 2 = \underline{2}$

$\therefore \lim\limits_{h\to 0} \dfrac{f(-2+h)-f(-2)}{h}$ DNE $\quad f'(-2)$ DNE

$\boxed{\text{There is a corner at } x=-2}$

c) The graph of f does not have a tangent at $x=-2$

12 a) $g'(2) = \lim_{h \to 0} \frac{g(2+h) - g(2)}{h} = \lim_{h \to 0} \frac{\frac{1}{2+h} - \frac{1}{2}}{h}$

$= \lim_{h \to 0} \frac{\frac{2}{2(2+h)} - \frac{2+h}{2(2+h)}}{h} = \lim_{h \to 0} \frac{\frac{-h}{2(2+h)}}{h}$

Common denominator

$\cancel{h}$

$= \lim_{h \to 0} \frac{-1}{2(2+h)} = \boxed{\frac{-1}{4}}$

b) $g'(2) = \lim_{x \to 2} \frac{g(x) - g(2)}{x - 2} = \lim_{x \to 2} \frac{\frac{1}{x} - \frac{1}{2}}{x - 2}$

$= \lim_{x \to 2} \frac{\frac{2}{2x} - \frac{x}{2x}}{x - 2} = \lim_{x \to 2} \frac{\frac{2-x}{2x}}{x-2}$

$= \lim_{x \to 2} \frac{-(x-2)}{2x} \cdot \frac{1}{(x-2)} = \lim_{x \to 2} \frac{-1}{2x} = \boxed{-\frac{1}{4}}$