Solve the differential equation.

1. \( \frac{dy}{dx} = 3\sqrt{xy} \) where \( y = 256 \) when \( x = 4 \)

2. \( y' = \frac{xy}{8} \)

3. \( f'(x, y) = e^{x-y} \)
A slope field is a lattice of line segments on the Cartesian plane that indicate the slope of a function or other curve at the designated points if the curve were to go through the point.

Ex: 1 \[ \frac{dy}{dx} = 3x^2 \] .
Sketch the slope field on the given points and find the general solution to the differential equation.

Ex: 2. \[ \frac{dy}{dx} = \frac{2x}{3y} \] .
Sketch the slope field on the given points and find the general solution to the differential equation.
AP CALCULUS AB: Q302: DIFFERENTIAL EQUATIONS and SLOPE FIELDS

Derive a formula for a **linear approximation** (a tangent to a function or curve used to estimate the value of the function or curve)

Derive a formula for **Euler’s approximation** (a series of line segments that use the slope of the line or function used to estimate the value of the function or curve)
1A. Suppose $\frac{dy}{dx} = 2x$ with $y(1.5) = 4$. Write an equation of a line tangent to the graph of $y$ at $x = 1.5$ and use it to approximate $y(3)$.

1B. Use Euler’s Method to approximate $y(3)$ starting with $y(1.5) = 4$ and using $\Delta x = 0.5$.

1C. Find the exact solution to the differential equation $\frac{dy}{dx} = 2x$ with $y(1.5) = 4$. 
1A. Suppose \( \frac{dy}{dx} = \frac{2x}{3y} \) where the point (0.5, 1) lies on the curve. Write an equation of a line tangent to the graph of \( y \) at \( x = 0.5 \) and use it to approximate the positive value of \( y \) when \( x = 1.5 \).

1B. Use Euler’s Method to approximate the positive value of \( y \) when \( x = 1.5 \) starting with the point (0.5, 1) and using \( \Delta x = 0.5 \).

1C. Find the exact solution (the positive value of \( y \) when \( x = 1.5 \)) to the differential equation \( \frac{dy}{dx} = \frac{2x}{3y} \) where point (0.5, 1) lies on the curve.
Practice 1: Let $f$ be a function with $f(1) = 4$ such that for all points $(x, y)$ on the graph of $f$ the slope is given by $\frac{3x^2 + 1}{2y}$.

(a) Find the slope of the graph of $f$ at the point where $x = 1$.
(b) Write an equation for the line tangent to the graph of $f$ at $x = 1$ and use it to approximate $f(1.2)$.
(c) Find $f(x)$ by solving the separable differential equation $\frac{dy}{dx} = \frac{3x^2 + 1}{2y}$ with initial condition $f(1) = 4$.
(d) Use your solution from part (c) to find the exact solution to $f(1.2)$. 
Practice 2: Consider the differential equation given by \( \frac{dy}{dx} = \frac{xy}{2} \).

(a) On the axes provided below, sketch a slope field for the given differential equation at the nine points indicated.

(b) Let \( y = f(x) \) be a particular solution to the given differential equation with initial condition \( f(0) = 3 \). Use a linear approximation centered at \( x = 0 \) to approximate \( f(0.2) \). Show the work that leads to your answer.

(c) Find the particular solution \( y = f(x) \) to the given differential equation with initial condition \( f(0) = 3 \). Use the solution to find \( f(0.2) \).
Practice 3: Let $f$ be a function whose graph goes through the point $(3, 6)$ and whose derivative is given by $f'(x) = \frac{1+e^x}{x^2}$.

(a) Write an equation of the line tangent to the graph of $f$ at $x = 3$ and use it to approximate $f(3.1)$. Use $f''$ to explain why this approximation is less than $f(3.1)$.

(b) Use Euler’s method, starting at $x = 3$ with step size 0.05, to approximate $f(3.1)$. 
Chapter 6.4, 6.5

Warm Up:

Solve the differential equation \( \frac{dy}{dt} = ky \) subject to the initial condition \( y(0) = y_0 \). \( k \) is a constant.

Comment:

Solve the differential equation \( \frac{dy}{dt} = k(L - y) \) subject to the initial condition \( y = y_0 \) when \( t = 0 \). \( L \) and \( k \) are constants.

Comment:
Exponential Growth and Decay

Differential equation \( \frac{dy}{dt} = ky \) subject to \( y = y_0 \) at \( t = 0 \) is \( y = y_0 e^{kt} \)

Example 1: Solve the differential equation \( \frac{dy}{dt} = 0.04y \) given that \( y_0 = 15 \). Find \( y(6) \).

Example 2: (Biological Growth) History indicates that the population \( y \) of the world (during the last 200 years) has been growing at a rate proportional to the population \( y \) with the growth constant \( k = 0.0198 \). The population of the world was about 4 billion on January 1, 1975. When will the world’s population reach 10 billion?

Example 3: (Biological Growth) The cells in a certain bacterial culture divide (double) on average every 2.5 hours. If there were 500 cells initially, how many cells would we expect to find after 12 hours?
Example 4: (Radioactive Decay) Radium has a half-life of 1690 years. Determine the decay constant $k$ for the radium and then calculate how much of 10 grams of radium will be left after 2400 years (Assume that radium decays at a rate proportional to the amount of radium present with some decay constant $k$). When will the amount remaining be 1 gram?

Example 5: (Carbon Dating) Human hair from a grave in Africa proved to have only 74% of the carbon 14 found in living tissue. Given that carbon 14 has a half-life of 5570 years, determine when the body died. (Assume the exponential decay model)
Simple Bounded Growth (Newton’s Law of Cooling)

Definition: The solution to the differential equation \( \frac{dy}{dt} = k(L - y) \) subject to \( y = y_0 \) at \( t = 0 \) is 
\[
y = L - (L - y_0)e^{-kt}.
\]

Example 6: Solve the differential equation \( \frac{dy}{dt} = 0.4(10 - y) \) subject to \( y_0 = 3 \).

Example 7: The news that the mayor of a certain city had been killed was announced at noon, and in 3 hours it was thought that 75% of the people in the city had heard it. How long will it take for 99% of the people to hear it?

Example 8: The classic model for fish growth assumes that the rate of change in fish length is proportional to the difference between theoretical maximum length and actual length. A certain variety of fish is hatched at length 0.5 inches and never grows beyond 12 inches. If a typical such fish has a length of 6 inches in 20 weeks, how long will it be after 50 weeks?
Example 9: A hard-boiled egg at 98°C is put in a pan under running 18°C water to cool. After 5 minutes, the egg’s temperature is found to be 38°C. How much longer will it take the egg to reach 20°C?

Example 10: A cup of water with a temperature of 95°C is placed in a room with constant temperature of 21°C. (a) Assuming that Newton’s Law of cooling applies, set up and solve an initial-value problem whose solution is the temperature t minutes after it is placed in the room. (b) How many minutes will it take for the water to reach a temperature of 51°C if it cools to 85°C in one minute?

Example 11: Let $P(t)$ represent the number of wolves in a population at time $t$ years, when $t \geq 0$. The population $P(t)$ is increasing at a rate directly proportional to $800 - P(t)$ where the constant of proportionality is $k$. (a) if $P(0)=500$, find $P(t)$ in terms of $t$ and $k$. (b) If $P(2) = 700$, find k. (c) find $\lim_{t \to \infty} P(t)$.